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GAME CHROMATIC NUMBER OF SOME NETWORK GRAPHS[1]

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In this paper, we determine the exact values of the game chromatic number of some interconnection network graph families such as shuffle exchange network, cube-connected cycles and wrapped around butterflies

Key words : Graphs; game chromatic number; shuffle exchange network; cube-connected cycles; butterfly graphs.

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1. Introduction

We consider the well known graph coloring game which was introduced by Bodlaender [2]. Let $G = (V, E)$ be a finite graph and X be a set of colors. The game chromatic number of G is defined through a two person game. Two players, say Alice and Bob, with Alice starting first, alternately color a vertex of G with a color from the color set X so that no two adjacent vertices receive the same color. Alice wins the game if all the vertices of G are colored. Bob wins the game if at any stage of the game, there is an uncolored vertex which is adjacent to vertices of all colors from X. The *game chromatic number*, $\chi_g(G)$, of G is the least number of colors in the color set X for which Alice has a winning strategy in the coloring game on G. This parameter is well defined since Alice always wins if $|X| = |V|$. It is obvious that $\chi(G) \leq \chi_g(G) \leq \triangle(G) + 1$, where $\chi(G)$ is the usual chromatic number of G and $\triangle(G)$ is the maximum degree of G.

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Let $\mathcal{H}$ be a family of graphs. We define the game chromatic number of $\mathcal{H}$ as $\chi_g(\mathcal{H}) = max\{\chi_g(G) : G \in \mathcal{H}\}$.

An important class of network graphs is hypercube network. Hypercubes are computationally powerful. But their drawback is that vertex degree increases with the size of the network. Whereas, butterfly network, cube connected cycle network, shuffle-exchange and Benes network graphs are variations of the hypercubes with constant degree. It can be easily checked using case by case analysis that the game chromatic number of hypercubes of dimensions 3 and 4 is 4 and 5 respectively. But, the game chromatic number of hypercube of dimension k, Q_k, $k > 4$, is unknown. In this paper, we discuss about the game chromatic number of shuffle exchange network, cube connected cycles and wrapped around butterfly graphs.

We say a color i is an available color for an uncolored vertex x if no neighbors of x have been colored by color i. For a graph G, let $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. For a vertex v, let $N_c(v)$ denotes the set of distinctly colored neighbors of v and $N_{uc}(v)$ denotes the set of uncolored neighbors of v. An uncolored vertex v is called color-i critical [4], if the following holds:

(i) color i is the only color available for v

(ii) v has an uncolored neighbor v' such that color i is the one of the available color for v'.

Note that if a vertex x is color i-critical and

$\diamond$ if it is Bob's turn, then he will win the game.

$\diamond$ If it is Alice's turn, then she has to save the vertex x either by coloring x with color i or make color i unavailable for all neighbors of x.

The main idea behind Bob's strategy is to create two critical vertices so that Alice cannot save both of them in a single move.

2. Shuffle Exchange Network

Definition 2.1 — The n dimensional shuffle exchange network denoted by $SE(n)$ has vertex set $V = V(Q_n)$ and two vertices $x = x_1x_2...x_n$ and $y = y_1y_2...y_n$ are adjacent if and only if either

(i) x and y differ in precisely the n^{th} bit or

(ii) x is a left or right cyclic shift of y.

The edge defined by the condition (i) is called an exchange edge, (ii) is called shuffle edge. The condition (ii) means that either $y_1y_2...y_n = x_2x_3...x_nx_1$ or $y_1y_2...y_n = x_nx_1x_2...x_{n-2}x_{n-1}$.

Viewing the edges of shuffle exchange network of higher dimensions is quite challenging. Kishore and Albert Williams [1] have redefined the shuffle exchange network by considering the power set of $X = \{2^0, 2^1, 2^2, ..., 2^{n-1}\}$ as vertices. This enables a good view of edges of shuffle exchange network for any dimension. Their representation of shuffle exchange network is as follows.

Definition 2.2 — Let $X = \{2^0, 2^1, 2^2, ..., 2^{n-1}\}$ and let $P(X)$ denote the power set of X. $SE(n)$ is the graph whose vertex set is $P(X)$ and the edges are given by

i) two vertices S and S' are adjacent if $S \triangle S' = \{2^0\}$ (or)

ii) if $|S| = |S'| = k$, where $S = \{2^{x_1}, 2^{x_2}, ..., 2^{x_k}\}$ and $S' = \{2^{y_1}, 2^{y_2}, ..., 2^{y_k}\}$ then S and S' are adjacent if $y_i = (x_i + 1)\text{mod } n$ for all i, $1 \leq i \leq k$.

Note that the edges defined by the condition (i) is called an exchange edge, (ii) is called shuffle edge. Observe that, in this new representation, graph is drawn by excluding the loops and parallel edges so that the graph is simple. In this paper, we will be using the definition 2.2 as the definition of shuffle exchange network.

Properties :

It is apparent that the shuffle edges occur in cycles which we call necklaces. The removal of the exchange edges partitions the graph into a set of necklaces. The necklaces which have n vertices are called full necklaces. The necklaces which have fewer than n vertices are called degenerate necklaces. Thomson *et al*. [3] have proved that the number of degenerate necklaces in $SE(n)$ is quite smaller compared to the number of full necklaces.

Lemma 2.3 — Let G be an even cycle with each vertex having a pendant vertex. Consider the coloring game on G and assume that three colors are given. If Alice starts the game with coloring one of the vertices in the cycle then Bob can win this game.

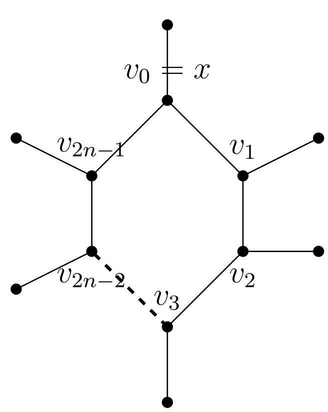

Figure 2.1 : Graph G

PROOF : The winning strategy for Bob is as follows.

Step 1 : Alice colors any vertex of the cycle, say x, in the first move.

Let us label the vertices of the cycle starting with x, say, $x = v_0, v_1, v_2, ..., v_{2n-1}$.

Step 2 : Let $i = 1$

Step 3 : If $i < n - 1$ then goto step 4 else goto step 6

Step 4 : Bob colors v_{2i} with a color different from that of v_{2i-2}. This forces Alice to color v_{2i-1} or the pendant vertex of v_{2i-1}

Step 5 : Assign $i = i + 1$, go to step 3

Step 6 : Bob colors v_{2n-2} with a color different from that of v_{2n-4} and v_0. This makes the vertices v_{2n-3} and v_{2n-1} critical. Hence Bob wins. □

Note:

Consider a necklace N composed of m vertices, say, $S_1, S_2, S_3, ..., S_m$. When $|S_i| = k$, $1 \leq i \leq m$, we call N as k element subset necklace and it is denoted by N_k. If m is even(odd), we call N to be *even necklace (odd necklace). Edge necklace* is a necklace composed of two vertices with an edge between them and it is denoted by N_e.

In an odd necklace, consider any two vertices Y and Y'. It divides the necklace into two parts such that one part contains odd number of vertices between Y and Y'. We call such a part to be *odd part from* Y *to* Y'. The other part which contains even number of vertices between Y and Y' called *even part from* Y *to* Y'.

Observation 2.5 — Let n be odd. In this case, there is exactly one N_1 necklace and $\frac{n-1}{2}$ disjoint N_2 necklaces of length n. Each of the N_2 necklaces is joined to N_1 by exactly two exchange edges. Also, the end points of these two exchange edges will not form a cycle of length four.

When n is odd, each of N_{n-2} necklaces is joined to N_{n-1} necklace by exactly two exchange edges since corresponding to each vertex in N_{n-2}, there is a vertex in N_2 and corresponding to each vertex in N_{n-1}, there is a vertex in N_1.

Observation 2.4 — If there is exactly one vertex in the necklace N_k which does not contain 2^0, then $k = n - 1$.

Let $S = \{2^{x_1}, 2^{x_2}, 2^{x_3}, ..., 2^{x_k}\}$, where $x_1 > x_2 > x_3 > ... > x_k$ be the only vertex of N_k which does not contain 2^0. Since all other vertices of N_k contain 2^0, $x_i = n - i$, $1 \leq i \leq k$. This implies

$k = n - 1$.

Lemma 2.6 — If two odd necklaces (either full or degenerate) of $SE(n)$ are connected by two exchange edges then there are two even cycles formed from these two odd necklaces.

PROOF : Let the two odd necklaces be C_1 and C_2. Let ZZ' and YY' be the two exchange edges such that $Z, Y \in V(C_1)$ and $Z', Y' \in V(C_2)$. Since C_1 and C_2 are odd necklaces, we will have an odd and even parts between Z and Y in C_1 and between Z' and Y' in C_2. Let us denote the odd and even parts in C_i as P_o^i and P_e^i respectively, $i = 1, 2$.

It is easy to see that, the path P_o^1 from Z to Y in C_1 followed by the edge YY' and the path P_o^2 from Y' to Z', the edge $Z'Z$ forms an even cycle. Similarly a cycle which is formed by P_e^1 from Z to Y in C_1 followed by the edge YY' and P_e^2 from Y' to Z' followed by $Z'Z$ is an even cycle. □

In the coloring game of $SE(n)$, we are giving a strategy for Bob to win with three colors. According to his strategy, Bob sometimes keeps playing in the same necklace and at times he jumps from one necklace to another. The conditions under which Bob jumps from one necklace to another is described below and we call this as "Jumping Rule".

Jumping Rule:

Let C_1 be a necklace and let V be the colored vertex of C_1 and it is Bob's turn to color now. Then Bob will choose an uncolored vertex U of C_1 such that $2^0 \notin U$. By Observation 2.5, this is always possible. Since $2^0 \notin U$, there exists an exchange edge UU' between C_1 and another necklace C_2.

(i) If UV is an edge of C_1, then Bob colors U' of C_2 and this makes U critical. Alice has to defend it. Bob jumps to the necklace C_2.

(ii) If UV is not an edge of C_1, then Bob plays in C_1 itself and he will color a vertex which lies on the even part from V to U and at a distance two from V. Call it as Y. Now he keeps playing on the even part from V to U and at vertices which are at a distance two from the previously colored vertices. Once he reaches vertex which is adjacent to U in the even part, Bob will color U' of C_2. Thus he jumps to the necklace C_2. Note that throughout this jumping rule, Bob forces Alice to color particular vertices. □

Note :

Observe that in the jumping rule, if C_1 is a N_k necklace then Bob jumps to C_2 which is a N_{k+1} necklace. At times he may needs to jump to a N_{k-1} necklace from C_1. The jumping rule will be same for this case also except in choosing an uncolored vertex U of C_1. Bob will choose an uncolored vertex U of C_1 such that $2^0 \in U$.

***Theorem* 2.7** — $\chi_g(SE(n)) = 4$, $n \geq 3$.

PROOF : As the maximum degree of $SE(n)$ is 3, $\chi_g(SE(n)) \leq 4$. For $n = 3, 4$, it can be easily checked that $\chi_g(SE(n)) = 4$. For $n \geq 5$, we will show that Bob has a winning strategy using three colors and thus $\chi_g(SE(n)) = 4$. The strategy of Bob is as follows. At each step of the game, Bob colors a vertex U, a neighbor of uncolored vertex W, with a color which is not present in the neighborhood of W. Alice starts the game by coloring any vertex.

Case 1 : Suppose in the first move, Alice colors a vertex say V, such that $V \neq \emptyset$ and $V \neq X$. We know that V lies on some odd or even or edge necklace.

Case 1.1 : If V lies on an even necklace then Bob wins the game using Lemma 2.3.

Case 1.2 : Suppose V lies on an odd necklace such that it is joined to another odd necklace by exactly two exchange edges.

Let the two exchange edges be UU' and WW', where $U, W \in V(C_1)$ and $U', W' \in V(C_2)$(W not necessarily distinct from V) between C_1 and C_2. Now Bob's response will be according to the situations which are discussed below.

Subcase 1.2.1 : $UW \notin E(C_1)$ and $U'W' \notin E(C_2)$

Bob can find an even cycle containing the vertex V by Lemma 2.6 and he wins the game using Lemma 2.3.

Subcase 1.2.2 : $UW \in E(C_1)$ and $U'W' \notin E(C_2)$

By Lemma 2.6, we get an even cycle containing V, say C. In C, U and W may be adjacent or not. If U and W are adjacent in C, we call UW as an edge of C. Otherwise we call UW as a chord of C.

(i) If UW is an edge of C, then W has to be V which is already colored. Now using Lemma 2.3, Bob wins.

(ii) If UW is a chord of C then Bob will consider the odd cycle C' consisting of the odd part of C_1 starting from U followed by the edge WW', even part of C_2 starting from W' and the edge $U'U$.

Let us relabel the vertices of C' as follows. The vertices of odd part of C_1 as $U = V_0, V_1, V_2, ..., V_{2k} = W$ and the vertices of even part of C_2 as $W' = V_{2k+1}, V_{2k+2}, V_{2k+3}, ..., V_{2k+2l} = U'$. In this labelling, assume that the label of V be V_j, $1 \leq j \leq 2k - 1$. Now in C', Bob's move will be as follows.

Intialize : Let $i = j$, $m = 2k + 2l$ and $n = 1$

If $(i < 2k)$

Bob colors V_{i+2} // Alice colors V_{i+1} or a vertex from $N_{uc}(V_{i+1})$

$i = i + 2$

End If $(i > 2k)$

If (V_0 is colored) // Alice due to criticality may color V_0 or V_{i-1}

While ($m > i + 4$)

Bob colors V_{m-1}

$m = m - 2$

End While

Bob colors V_{i+2} // This makes the vertices V_{i+1} and V_{i+3} critical.

End If

Else

While($n < j - 2$)

Bob colors V_n

$n = n + 2$

End While

Bob colors V_{j-2} // This makes the vertices V_{j-3} and V_{j-1} critical.

End Else

End Else If

Else (that is $i = 2k$)

Bob wins.

Note that in the stage when $i = 2k$, the vertex V_{2k} was colored by Bob and makes V_{2k-1} critical. So Alice has to defend it. Now the vertices $V_{2k}, V_{2k+1}, V_{2k+2}, ..., V_{2k+2l}, V_0, V_{2k}$ forms an even cycle with V_{2k} is colored and it is Bob's turn to play in this even cycle. Hence by Lemma 2.3 Bob wins.

Subcase 1.2.3 : $UW \notin E(C_1)$ and $U'W' \in E(C_2)$

This case can be prove in a similar way as subcase 1.2.2.

Subcase 1.2.4 : $UW \in E(C_1)$ and $U'W' \in E(C_2)$

In this case, using Jumping Rule, Bob will jump from C_1 to C_2 and then from C_2 to another necklace, say C_3. If C_3 happens to be $n - 1$ element subset necklace then he finds a $n - 2$ element subset necklace , say C_4, such that C_4 is joined to C_3 by two exchange edges and also ensure that C_4

is other than C_2 (by Observation 2.4 this is always possible). Note that between C_3 and C_4 these two exchange edges will not forms a cycle of length four. Hence the situation will be same as discussed in subcase 1.2.1, subcase 1.2.2 or subcase 1.2.3. Therefore Bob will play accordingly and wins.

If not, that is C_3 is not a $n-1$ element subset necklace then the situation will be one of the case as discussed in case 1.1, case 1.2 or case 1.3. Hence Bob will play accordingly and wins.

Observe that, while using jumping rule, if Bob happens to color W of C_1 then he wins. Because in the next move, Bob will color U' of C_2 and this makes U of C_1 and W' of C_2 critical.

Case 1.3 : Now let us consider the case when V lies on an odd necklace, say C_1 and it is joined to another necklace, say C_2 and the number of exchange edges between C_1 and C_2 is not equal to two.

Since the number of exchange edges between C_1 and C_2 is not two, by Observation 2.4, C_1 cannot be N_{n-1}. Now Bob will be applying Jumping Rule. Bob will be continuously applying Jumping Rule until the necklace where he jumped in is of even length and all its vertices are uncolored except one or the necklace where he jumped in is of odd length and it is joined to another odd necklace by exactly two exchange edges. Note that either one of this stage is always attainable by Bob. Because, in the worst case of the game, Bob will keep using the jumping rule until he jumps to N_{n-2} or N_{n-1}. When n is odd, by Observation 2.4, N_{n-2} is joined to N_{n-1} by exactly two exchange edges and when n is even, the N_{n-1} is an even necklace.

Case 1.4 : Suppose V lies on an edge necklace, N_e, then Bob plays in the other end point of that edge. Now the rest of the game is same as discussed in case 1.1, case 1.2 or case 1.3 except if Alice plays in a vertex of necklace C_1 which is joined to N_e. If Alice plays in C_1, then Bob jumps to a immediate necklace C_2 which may be of either higher or lesser element subset necklace(compared to C_1) from C_1 and then the game will be same as discussed above. Because, observe that when C_1 is even, Alice will be safe if Bob continues to play in C_1.

Case 2 : Suppose in the first move, if Alice colors a vertex $\emptyset$ or X then Bob colors X or $\emptyset$ respectively. In the next move, Alice has to color some vertex V and now the rest of the game is same as discussed in case 1. Hence Bob wins. $\square$

3. CUBE-CONNECTED CYCLES

Definition 3.1 — The cube-connected cycle of order n, denoted by CCC_n, is defined as a graph formed from a set of $n2^n$ vertices, indexed by pairs of numbers (x, y) where $0 \leq x < 2^n$ and $0 \leq y < n$. Each such vertex is adjacent to the three vertices $(x, (y+1) \bmod n)$, $(x, (y-1) \bmod n)$ and $(x \oplus 2^y, y)$ where $\oplus$ denotes the bitwise exclusive OR operation on binary numbers.

Observation 3.2 — The cube-connected cycle graph CCC_n is obtained by replacing each vertex of the n-cube by an n-cycle. It is a regular graph of degree three.

Observation 3.3 — In CCC_n, any point (x, y) lies on a cycle of length eight, denoted by C_8. Using the vertex adjacency of CCC_n, this C_8 has vertices (x, y), $(x \oplus 2^y, y)$, $(x \oplus 2^y, (y+1)\text{mod } n)$, $((x \oplus 2^y) \oplus 2^{(y+1)\text{mod } n}, (y+1)\text{mod } n)$, $((x \oplus 2^y) \oplus 2^{(y+1)\text{mod } n}, y)$, $(x \oplus 2^{(y+1)\text{mod } n}, y)$, $(x \oplus 2^{(y+1)\text{mod } n}, (y+1)\text{mod } n)$, $(x, (y+1)\text{mod } n)$.

The following theorem implies that $\chi_g(\mathcal{C}) = 4$ for the family of cube connected cycles $\mathcal{C}$.

***Theorem* 3.4** — *For any positive integer* $n \geq 2$, $\chi_g(CCC_n) = 4$.

PROOF : Since CCC_n is a regular graph of degree three, we have $\chi_g(CCC_n) \leq 4$. Let us show that Bob has a winning strategy using three colors in the coloring game played on CCC_n and thus $\chi_g(CCC_n) = 4$. The winning strategy of Bob is as follows.

Step 1 : Alice colors any vertex, say (x, y), in the first move.

By Observation 3.3, (x, y) lies on the cycle C_8. Relabel the vertices (x, y), $(x \oplus 2^y, y)$, $(x \oplus 2^y, (y+1)\text{mod } n)$, $((x \oplus 2^y) \oplus 2^{(y+1)\text{mod } n}, (y+1)\text{mod } n)$, $((x \oplus 2^y) \oplus 2^{(y+1)\text{mod } n}, y)$, $(x \oplus 2^{(y+1)\text{mod } n}, y)$, $(x \oplus 2^{(y+1)\text{mod } n}, (y+1)\text{mod } n)$, $(x, (y+1)\text{mod } n)$ of C_8 by $v_0,\ v_1,\ v_2, \ldots,\ v_7$ respectively.

Step 2 : Let $i = 1$

Step 3 : if $i < 3$ then Bob colors v_{2i} with a color different from v_{2i-2}. This forces Alice to color v_{2i-1} or a vertex from $N_{uc}(v_{2i-1})$. Assign $i = i + 1$, go to step 3.

Step 4 : else Bob colors v_6 with a color different from the colors of v_0 and v_4. This makes the vertices v_5 and v_7 critical. Hence Bob wins.

4. BUTTERFLY GRAPHS

An n-dimensional butterfly, $BF(n)$, has $(n+1)2^n$ vertices corresponding to pairs (w, i) where w is an n-bit binary number and i is the level $0 \leq i \leq n$. Two vertices (w, i), (w', i') are adjacent if and only if $i' = i + 1$ and $w = w'$ or w and w' differ only in the i'^{th} bit. The n-dimensional wrapped-around butterfly $W - BF(n)$ is defined by taking the $BF(n)$ and identifying level n with level 0. It is not difficult to check that combining the vertex set $\{(w, i) | 0 \leq i \leq n\}$ into a single vertex results in the hypercube.

Definition 4.1 — The Wrapped around Butterfly graph of dimension n, denoted by $W - BF(n)$, is defined as follows. The vertex set V of $W-BF(n)$ is the set of ordered pairs (α, v) where $\alpha = 0, 1, 2, ..., n-1$ and $v = x_{n-1}...x_1x_0$, a binary word of length n. Two vertices (α, v) and $(\alpha^{'}, v^{'})$ are adjacent if $\alpha^{'} \equiv (\alpha+1)(mod\, n)$ and $x_j = x_j^{'}\ \forall j \neq \alpha^{'}$.

Thus there are three types of edges in the edge set E of $W-BF(n)$

(i) straight edges joining (α, v) to $(\alpha+1, v)$

(ii) slanting edges joining (α, v) to $(\alpha+1, v^{'})$

(iii) winged edges joining $(0, v)$ to $(n-1, v)$

Observation 4.2 — In $W-BF(4)$, for each vertex (α, v) there exists a vertex (α, v') such that (α, v) and (α, v') have two common neighbors.

Observation 4.3 — Similar to the above observation, for each vertex (α, v) in $W-BF(4)$, there exists a vertex (α', v) such that (α, v) and (α', v) have two common neighbors.

$W-BF(n)$ is a 4-regular graph with $n2^n$ vertices. We know that $W-BF(n)$ is a Cayley graph and every Cayley graph is vertex transitive. Since $W-BF(n)$ is a regular graph of degree four, $\chi_g(W-BF(n)) \leq 5$. The following theorem implies that $\chi_g(\mathcal{B}) = 5$ for the family of wrapped around butterfly graphs $\mathcal{B}$.

***Theorem* 4.4** — $\chi_g(W-BF(4)) = 5$.

PROOF : It can be easily checked that $\chi_g(W-BF(4)) \geq 4$. If Alice and Bob play the coloring game on $W-BF(4)$ with four colors then we show that Bob has a winning strategy using four colors. The winning strategy of Bob is as follows. In this strategy, at each step, Bob colors a vertex u, a neighbor of uncolored vertex w, with a color which is not present in the neighborhood of w.

Step 1 : Alice colors $x = (\alpha, v)$ in her first move.

Step 2 : Bob colors $y = (\alpha, v')$ such that $xy \notin E(W-BF(4))$ and $\exists$ vertices w and z such that $xw, xz, yw, yz \in E(W-BF(4))$.

Step 3 : Alice colors anywhere.

Now there exists one uncolored vertex, say u_1, such that $u_1 \in \{w, z\}$ and there exists an uncolored vertex u_2 such that $|N_{uc}(u_1) \cap N_{uc}(u_2)| = 1$, $N_c(u_1) \subseteq \{x, y\}$ and $N_c(u_2)$ is a non empty subset of $\{x, y\}$.

Step 4 : Let $a \leftarrow u_1$

Step 5 : Bob colors a vertex from $N_{uc}(a) \cap N_{uc}(b)$, where b is any uncolored vertex other than a, such that $|N_c(a) \cap N_c(b)| = 1$.

Step 6 : if $|N_c(a)| = 3$ and $|N_c(b)| = 2$ then a becomes critical and Alice is forced to color either a or a vertex from $N_{uc}(a)$.

Assign $a \leftarrow b$, go to step 5.

Step 7 : else if $|N_c(a)| = 3$ and $|N_c(b)| = 3$ then both the vertices a and b are critical. Hence Bob wins.

Note that Step 2 is always possible by Observation 4.2. Also, when a vertex a is fixed in the above algorithm, $|N_c(a)| = 2$ and we can find an uncolored vertex b such that $|N_{uc}(a) \cap N_{uc}(b)| = 1$ and $|N_c(a) \cap N_c(b)| = 1$ by using Observations 4.2 and 4.3. Observe that, from the third move of Alice, all her moves are forced moves. Hence Step 5 is possible.

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