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EXISTENCE, MULTIPLICITY AND NUMERICAL EXAMPLES FOR SCHRÖDINGER SYSTEMS WITH NONSTANDARD *P*(*X*)-GROWTH CONDITIONS

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In this paper, we deal with the Schrödinger's problems, in the first part we study the theoretical side, we show the existence of at least three weak solutions, our main tools are based on variational inequalities, more precisely, using the three critical points theorem due to Ricceri, existence and multiplicity results are established. In the second part, we are interested in the application side, more exactly, we examine some computational problems on the discretization of finite elements of the $p(x)$-Laplacian, we propose a quasi-Newton minimization approach for the solution, our numerical tests show that these algorithms are able to resolve the problems with $p(x)$-Laplacian, for different values of $p(x)$.

Key words : Nonlinear boundary value problems; variable exponent sobolev space; multiplicity results; quasi-Newton minimization.

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1. INTRODUCTION

Over the last decade, the variable exponent Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have been a subject of active research area (we refer to [15-19] for the fundamental properties of these spaces).

These investigations are stimulated mainly by the development of the studies of problems in Elasticity, Electrorheological fluids, Image Processing, Flow in Porous Media, Calculus of Variations, for the study of Differential Equations with $p(x)$-growth (see Acerbi and Mingione $[1]$, Ruzicka $[32]$, Zhikov [37]). Among these problems, the study of $p(x)$-Laplacian problems via variational methods is an interesting topic. A lot of researchers have devoted their work to this area (see Djellit, Youbi and Tas [14], Fan [16 – 19], Ogras, Mashiyev, Avci and Yucedag [29], Yin and Yang [34], Xu and An [35]).

We refer to the $p(x)$-Laplace operator $\Delta_{p(x)}u = div\left(|\nabla u|^{p(x)-2}\nabla u\right)$, where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearity than p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties, for example, we can not use the Lagrange Multiplier Theorem in many problems involving this operator.

In recent years, the problems of $p(x)$-Laplace operator have been studied by many authors using various methods.

Djellit, youbi and Tas in [14], show the existence of nontrivial solutions for the following $(p(x), q(x))$-Laplacian system, using Mountain Pass theorem

$$\begin{cases} -\Delta_{p(x)}u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta_{q(x)}v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N. \end{cases}$$

Xu and An in [35], obtains the the existence and multiplicity of solutions for elliptic systems, by the critical point theory

$$\begin{cases} -div\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N, \\ -div\left(|\nabla v|^{q(x)-2}\nabla v\right) + |v|^{q(x)-2}v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N, \\ (u,v) \in W^{1,p(x)}\left(\mathbb{R}^N\right) \times W^{1,q(x)}\left(\mathbb{R}^N\right). \end{cases}$$

In [27], Kristaly guaranteed the existence of an interval $\Lambda \subseteq [0,+\infty[$ such that for each $\lambda \in \Lambda$ for the quasilinear elliptic systems

$$\begin{cases} -\Delta_p u = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a strip-like domain and $\lambda > 0$ is a parameter, has at least two distinct nontrivial solutions, using an abstract critical point result of Ricceri and the Principle of Symmetric Criticality.

The minimax method has been used extensively in constructing critical points for functionals defined in Hilbert and Banach spaces as solutions of nonlinear partial differential equations. In particular, when the problems possess symmetry, one constructs multiple critical points by the minimax method; this is the general Ljusternik-Schnirelman type theory.

In the first part of the paper, we have generalized the articles $[2, 27, 30, 31]$ to the variable exponent, we obtain multiplicity results, we consider the schrödinger systems with Dirichlet boundary condition,

$$\begin{cases} -\Delta_{p_1(x)}u_1 + V(x)\left|u_1\right|^{p_1(x)-2}u_1 = \lambda\frac{\partial F}{\partial u_1}(x, u_1, ..., u_n) & \text{in } \Omega, \\ -\Delta_{p_2(x)}u_2 + V(x)\left|u_2\right|^{p_2(x)-2}u_2 = \lambda\frac{\partial F}{\partial u_2}(x, u_1, ..., u_n) & \text{in } \Omega, \\ . \\ . \\ . \\ -\Delta_{p_n(x)}u_n + V(x)\left|u_n\right|^{p_n(x)-2}u_n = \lambda\frac{\partial F}{\partial u_n}(x, u_1, ..., u_n) & \text{in } \Omega, \\ u_i = 0 \text{ for } 1 \leq i \leq n & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Delta_{p_i(x)}u_i = div\left(\left|\nabla u_i\right|^{p_i(x)-2}\nabla u_i\right)$ is the $p_i(x)$ $-$Laplacian operator, p_i are continuous functions such that $1 < p_i(x)$, for every $x \in \mathbb{R}^N$ $(N \geq 2)$ and for $1 \leq i \leq n$, $\Omega \subset \mathbb{R}^N$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $\lambda > 0$. However, $F : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $F(., t_1, ..., t_n)$ is continuous in $\bar{\Omega}$ for all $(t_1, ..., t_n)$, and $F(x, ..., .)$ is a C^1 function in $\mathbb{R}^N$ for almost every $x \in \bar{\Omega}$ We assume that the potential $V \in L^\infty(\Omega)$ satisfies the condition

$$ess \inf_{x\in\Omega} V > 0 \tag{V}$$

Our approach is based on a three critical points theorem due to Ricceri proved in $[31]$, recalled below for the reader's convenience (Theorem 2.6). Our main result, (Theorem 3.5) under a new assumptions ensures the existence of an open interval $\Lambda \subseteq [0; +\infty[$ and a positive real number ρ, such that, for each $\lambda \in \Lambda$, problem (3.1) admits at least three weak solutions whose norms in X_Ω are less than ρ.

In the second part, we propose a quasi-Newton minimization approach for the solution, we consider the schrödinger systems

$$\begin{cases} -\Delta_{p_i(x)}u_i + V(x)\left|u_i\right|^{p_i(x)-2}u_i = g_i(x) \text{ for } x \in \Omega, \\ u_i = 0 \text{ for } x \in \partial\Omega. \end{cases} \tag{4.1}$$

The systems (4.1) is viewed as one of the typical examples of a large class of nonlinear problems. It contains most of the essential diffculties in studies of finite element approximations. For this class of systems, many existing techniques in the finite element method, for example, the linearization method and deformation procedure, do not seem to work well. Finite element approximations of p-Laplacian have been extensively studied in the literature, for example, in [4-8, 10, 12, 21, 23, 25, 38].

In $[38]$, Zhou, Huang, and Feng proposed a hybrid conjugate gradient algorithm with weighted preconditioner of the problem

$$\begin{cases} -div\left(\left|\nabla u\right|^{p-2}\nabla u\right) = f, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

The algorithm that they use can efficiently solve the minimizing problem of general function deriving from finite element discretization of the p-Laplacian.

Huang, Li and Liu $[23]$ proposed a steepest descent algorithm with weighted preconditioner which is solved by an algbric multigrid method. The decent algorithm has excellent computing efficiency for both p large or relatively small, for example, p = 1000 and p = 1.5.

The paper is organized as follows. In Section 2, we recall some results involving the space $W^{1,p(x)}$ which can be found in [15-19], section 3 is devoted to the existence and multiplicity result for the system (3.1), in Section 4, we propose a quasi-Newton minimization algorithm , we present numerical results for the system (4.1) , for a differents values of $p(x)$, and finally end, in Section 5, with some conclusions and discussions.

2. Notations and Preliminaries

In this section, we recall some results on variable exponents Lebesgue and Sobolev spaces found in [15-19] and their references.

Let Ω be an open subset of $\mathbb{R}^N$,

$$C_+(\Omega) = \left\{h \in C(\Omega) : \inf_{x\in\Omega} h(x) > 1\right\}.$$

For every $h \in C_+(\Omega)$, denote

$$h^- = \inf_{x\in\Omega} h(x), \quad h^+ = \sup_{x\in\Omega} h(x).$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

Which is equipped with the norm, so-called Luxemburg norm

$$|u|_{p(x)} = |u|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

and $\left(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\Omega)} \right)$ becomes a Banach space.

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega); \ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and it can be equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$ and it is equipped with the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

If $p^- > 1$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

In the following discussions, we will use the product space

$$X_{\Omega} = W_{p_1(x),p_2(x),\dots,p_n(x)}(\Omega) = W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega) \times \dots \times W_0^{1,p_n(x)}(\Omega),$$

which is equipped with the norm $\|u\| = \|(u_1, u_2, \dots, u_n)\|_{p_1(x),p_2(x),\dots,p_n(x)} = \sum_{i=1}^{n} \|u_i\|_{p_i(x)}$,
$\forall (u_1, u_{2,\dots,} u_n) \in W_{p_1(x),p_2(x),\dots,p_n(x)}(\Omega)$.

The space $W^*_{p_1(x),p_2(x),\dots,p_n(x)}(\Omega)$ denotes the dual space of $W_{p_1(x),p_2(x),\dots,p_n(x)}(\Omega)$ and equipped with the norm $\|.\|_{*,p_1(x),p_2(x),\dots,p_n(x)}$.

Proposition 2.1 — (see [15-19]). The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}.$$

Proposition 2.2 — (see [15-19]). If we denote $\rho(u) = \int_{\Omega} |u|^{p(x)} dx$, $\forall u \in L^{p(x)}(\Omega)$, then

1. $|u|_{p(x)} < 1\,(=1;>1) \iff \rho(u) < 1\,(=1;>1)$;
2. $|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;
   $|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;
3. $|u|_{p(x)} \to 0\ (\infty) \iff \rho(u) \to 0\ (\infty)$;

Proposition 2.3 (see [15-19]). Let p and q be measurable functions such that $p(x) \in L^{\infty}(\Omega)$ and $1 \leq p(x)\,q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. then,

$$\begin{aligned} |u|_{p(x)q(x)} &\leq 1 \Longrightarrow |u|_{p(x)q(x)}^{p^+} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q'(x)}^{p^-}, \\ |u|_{p(x)q(x)} &\geq 1 \Longrightarrow |u|_{p(x)q(x)}^{p^-} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q'(x)}^{p^+}. \end{aligned}$$

Definition 2.4 — Define $1 < p(x) < N$ and for all $x \in \Omega$, let define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

where $p^*(x)$ is the so-called critical Sobolev exponent of $p(x)$.

Proposition 2.5 — (see [15-19]). Let $p(x) \in C_+^{0,1}(\Omega)$, that is, Lipschitz-continuous function defined on Ω, then there exists a positive constant c such that

$$\|u\|_{p^*(x)} \leq c\,\|u\|_{p(x)'}$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

We recall for reader's convenience the three critical points Theorem of $[31]$ and a Proposition of $[30]$.

***Theorem* 2.6** — *Let X_Ω be a reflexive real Banach space, $\Phi : X_\Omega \to \mathbb{R}$ a continuously Gateaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X_Ω, whose Gateaux derivative admits a continuous inverse on X_Ω^*; $J : X_\Omega \to \mathbb{R}$ a C^1 functional with compact Gateaux derivative. Assume that*

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda J(u)) = +\infty,$$

for all $\lambda \in [0, +\infty[$, and that there exists a continuous concave function $h : [0, +\infty[\to \mathbb{R}$ such that,

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda J(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda J(u) + h(\lambda)),$$

then, there exist an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* ρ *such that, for each* $\lambda \in \Lambda$, *the equation*

$$\Phi'(u) + \lambda J'(u) = 0,$$

has at least three solutions in X_Ω *whose norms are less than* ρ.

Proposition 2.7 — Let X_Ω be a non-empty set and Φ, J two real functions on X_Ω. Assume that there are $r > 0$ and $u_1, u_2 \in X_\Omega$ such that

$$\begin{aligned} \Phi(u_1) &= J(u_1) = 0,\ \Phi(u_2) > r, \\ \sup_{x\in\Phi^{-1}(]-\infty,r])} J(u) &< r\frac{J(u_2)}{\Phi(u_2)}. \end{aligned}$$

Then, for each ρ satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} J(u) < \rho < r\frac{J(u_2)}{\Phi(u_2)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{u\in X_\Omega}(\Phi(u) + \lambda(\rho - J(u))) < \inf_{u\in X_\Omega}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - J(u))).$$

3. Multiplicity Result

First, we give the assumptions that we use in this section

(F_1) $F(x,0,...,0) = 0$ for almost every $x \in \bar{\Omega}$.

(F_2) There exist functions $a_i \in L^{\alpha_i}$ and $a_i \geq 1$ for $1 \leq i \leq n$, such that

$$\frac{\partial F(x,u_1,...,u_n)}{\partial u_i} \leq a_i\left(\sum_{j=1}^{n}|u_i|^{q_j^- - 1}\right),$$

for almost every $x \in \bar{\Omega}$, where $\sum\limits_{i=1}^{n}\frac{1}{\alpha_i} = 1$, and $1 < q_j(x) < \inf(p_j(x))$.

(F_3) There exist $\theta > 0$; $r > 0$ and $\omega \in X_\Omega$ such that

$\sum\limits_{i=1}^{n}\frac{1}{p_i^+}\int\limits_\Omega\left(|\nabla\omega_i|^{p_i(x)} + V(x)|\omega_i|^{p_i(x)}\right)dx > r$ and

$meas(\Omega)\max\limits_{(x,t_1,...,t_n)\in\bar{\Omega}\times K}F(x,t_1,...,t_n) < \frac{\theta}{c}\frac{\int\limits_\Omega F(x,\omega_1,...,\omega_n)dx}{\sum\limits_{i=1}^{n}\prod\limits_{j=1,j\neq i}^{n}p_j^-\int\limits_\Omega\left(|\nabla\omega_i|^{p_i(x)}+V(x)|\omega_i|^{p_i(x)}\right)dx)}$; where

$$K = \left\{(t_1,...,t_n),\ \sum_{i=1}^{n}\frac{|t_i|^{p_i(x)}}{p_i^+} \leq \frac{\theta}{\prod\limits_{i=1}^{n}p_i^+},\ \theta > 0\right\}.$$

We define Φ, Ψ :$X_\Omega \to \mathbb{R}$ by

$$\begin{aligned}\Phi(u_1, u_2, ..., u_n) &= \sum_{i=1}^{n} \int_{\Omega} \frac{1}{p_i(x)} \left(|\nabla u_i(x)|^{p_i(x)} + V(x) |u_i|^{p_i(x)} \right) dx. \\ \Psi(u_1, u_2, ..., u_n) &= -\int_{\Omega} F(x, u_1, ..., u_n)\, dx.\end{aligned}$$

It is easy to verify that Φ, $\Psi \in C^1(X_\Omega, \mathbb{R})$ and that for any $v \in X_\Omega$,

$$\begin{aligned}\left\langle \Phi'(u), (v) \right\rangle &= \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i(x)|^{p_i(x)-2} \nabla u_i(x) \nabla v_i(x) + V(x) |u_i|^{p_i(x)-2} u_i v_i dx. \\ \left\langle \Psi'(u), (v) \right\rangle &= -\sum_{i=1}^{n} \int_{\Omega} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)\, v_i dx.\end{aligned}$$

Definition 3.1 — $(u_1, u_2, ..., u_n)$ is called a weak solution of the system (3.1) if

$$\int_{\Omega} \sum_{i=1}^{n} \left(|\nabla u_i(x)|^{p_i(x)-2} \nabla u_i(x) \nabla v_i(x) + V(x) |u_i|^{p_i(x)-2} u_i v_i \right) dx$$

$$-\lambda \int_{\Omega} \sum_{i=1}^{n} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)\, v_i(x)\, dx = 0, \tag{3.2}$$

for every $(v_1, v_2, ..., v_n) \in X_\Omega$.

It follows that we can look for weak solution of (3.2) by applying Theorem 2.6.

We will use the following lemmas to get our main results.

Lemma 3.2 — Under the assumptions (F_1) and (F_2), the functional Ψ is well defined, and it is of class C^1 on $W_{p_1(x), p_2(x), ..., p_N(x)}(\Omega)$. Moreover its derivative is

$$\langle \Psi'(u_1, u_2, ..., u_n), (v_1, ..., v_n) \rangle = -\sum_{i=1}^{n} \int_{\Omega} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)\, v_i dx.$$

PROOF : For all pair of real functions $(u, v) \in X_\Omega^2$, under the assumptions (F_1) and (F_2) we can write

$$
\begin{aligned}
F(x,u_1,...,u_n) &= \int_0^{u_1} \frac{\partial F}{\partial s}(x,s,...,u_n)\,ds + F(x,0,...,u_n)\,,\\
F(x,u_1,...,u_n) &= \int_0^{u_1} \frac{\partial F}{\partial s}(x,s,...,u_n)\,ds + \int_0^{u_2} \frac{\partial F}{\partial s}(x,0,s,...,u_n)\,ds + F(x,0,0,...,u_n)\,,\\
F(x,u_1,...,u_n) &= \int_0^{u_1} \frac{\partial F}{\partial s}(x,s,...,u_n)\,ds + \int_0^{u_2} \frac{\partial F}{\partial s}(x,0,s,...,u_n)\,ds\\
&\quad +...+ \int_0^{u_n} \frac{\partial F}{\partial s}(x,0,0,...,0,s)\,ds + F(x,0,0,...,0)\,,\\
F(x,u_1,...,u_n) &\le \int_0^{u_1} \left|\frac{\partial F}{\partial s}(x,s,...,u_n)\right| ds + \int_0^{u_2} \left|\frac{\partial F}{\partial s}(x,0,s,...,u_n)\right| ds\\
&\quad +...+ \int_0^{u_n} \left|\frac{\partial F}{\partial s}(x,0,0,...,0,s)\right| ds,\\
\int_\Omega F(x,u_1,...,u_n)\,dx &\le c(|a_1|_{L^{\alpha_1}}\left[\|u_1\|_{q_1(x)}^{q_1^-} + ... + \|u_n\|_{q_n(x)}^{q_n^- -1}\|u_1\|_{q_1(x)}\right]\\
&\quad + |a_2|_{L^{\alpha_2}}\left[\|u_2\|_{q_2(x)}^{q_2^-} + ... + \|u_n\|_{q_n(x)}^{q_n^- -1}\|u_2\|_{q_2(x)}\right] + ... + |a_n|_{L^{\alpha_n}}\|u_N\|_{q_n(x)}^{q_n^-}),\\
\int_\Omega F(x,u_1,...,u_n)\,dx &< c\sum_{i=1}^{n}\left(|a_i|_{L^{\alpha_i}}\left[\|u_1\|_{q_1(x)}^{q_1^- -1}\|u_i\|_{q_i(x)} + ... + \|u_n\|_{q_n(x)}^{q_n^- -1}\|u_i\|_{q_i(x)}\right]\right),\\
&< \infty.
\end{aligned}
$$

Hence, Ψ is well defined.

Moreover,one can see easily that Ψ' is also well defined on X_Ω.

Indeed,using (F_2) for all $v \in X_\Omega$, we have

$$
\begin{aligned}
\Psi'(u)(v) &= -\sum_{i=1}^{n}\int_\Omega \frac{\partial F}{\partial u_i}(x,u_1,...,u_n)\,v_i dx\\
&\le -\sum_{i=1}^{n}\int_\Omega a_i\left(|u_1|^{q_1^- -1} + .. + |u_n|^{q_n^- -1}\right)|v_i|\,dx
\end{aligned}
$$

$$\Psi'(u)\,(v) \leq -c(\int\limits_{\Omega} a_1 \left[|u_1|_{q_1(x)}^{q_1^- -1} + ... + |u_n|_{q_n(x)}^{q_n^- -1} \right] |v_1|\, dx$$
$$+ ... + \int\limits_{\Omega} a_n \left[|u_1|_{q_1(x)}^{q_1^- -1} + ... + |u_n|_{q_n(x)}^{q_n^- -1} \right] |v_n|\, dx),$$

and applying Propositions $2.1 - 2.3$ and 2.5, it follows that

$$\begin{aligned} \int\limits_{\Omega} \frac{\partial F}{\partial u_1}(x, u_1, ..., u_n)\, v_1 dx &\leq c(|a_1|_{L^{\alpha_1}} \|u_1\|_{q_1(x)}^{q_1^- -1} \|v_1\|_{q_1(x)} \\ &\quad + ... + |a_1|_{L^{\alpha_1}} \|u_n\|_{q_n(x)}^{q_n^- -1} \|v_1\|_{q_1(x)}), \\ &< \infty, \end{aligned}$$

and similarly, for $1 < i \leq n$, we have

$$\begin{aligned} \int\limits_{\Omega} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)\, v_i dx &\leq c(|a_i|_{L^{\alpha_i}} \|u_1\|_{q_1(x)}^{q_1^- -1} \|v_i\|_{q_i(x)} \\ &\quad + ... + |a_i|_{L^{\alpha_i}} \|u_n\|_{q_n(x)}^{q_n^- -1} \|v_i\|_{q_i(x)}), \\ &< \infty.\square \end{aligned}$$

Lemma 3.3 — $\Phi' : X_\Omega \to X_\Omega^*$ is strictly monotone operator and it is a homeomorphism.

PROOF : For any $\xi, \eta \in \mathbb{R}^N$, we have the following inequalities, from which we can get the strictly monotonicity of Φ'.

$\left[\left(|\xi_i|^{p(x)-2} \xi_i - |\eta_i|^{p(x)-2} \eta_i \right) (\xi_i - \eta_i) \right] . \left(|\xi_i|^{p(x)} + |\eta_i|^{p(x)} \right)^{\frac{p(x)-1}{p(x)}} \geq (p\,(x) - 1)\, |\xi_i - \eta_i|^{p(x)}$, $1 < p\,(x) < 2$

$\left(|\xi_i|^{p(x)-2} \xi_i - |\eta_i|^{p(x)-2} \eta_i \right) (\xi_i - \eta_i) \geq \left(\frac{1}{2}\right)^{p(x)} |\xi_i - \eta_i|^{p(x)}, p\,(x) \geq 2.$

Since $\Phi'(u) = \sum\limits_{i=1}^{n} \Phi'(u_i)$, then Φ' is strictly monotone.

By the strictly monotonicity, Φ' is an injection.

Let us show that Φ' is coercive.

$$\lim_{\|u\| \to \infty} \frac{(\Phi'(u), u)}{\|u\|} = \lim_{\|u\| \to \infty} \frac{\sum\limits_{i=1}^{n} \int\limits_{\Omega} \left(|\nabla u_i(x)|^{p_i(x)} + V(x)\, |u_i|^{p_i(x)} \right) dx}{\|u\|} = \infty;$$

Thus Φ' is a surjection in view of Minty-Browder Theorem (see [36]). Hence Φ' has an inverse mapping $[\Phi']^{-1} : X_\Omega^* \to X_\Omega$. Therefore, the continuity of $[\Phi']^{-1}$ is sufficient to ensure Φ' to be a homeomorphism.

If $f_k, f \in X_\Omega^*$, $f_k \to f$, let $u_k = [\Phi']^{-1}(f_k)$, $u = [\Phi']^{-1}(f)$ then $\Phi'(u_k) = f_k$, $\Phi'(u) = f$, since $f_k \to f$ we can assume that $u_k \rightharpoonup u$, Φ' is of type (S_+), then $u_k \to u$, so $[\Phi']^{-1}$ is continuous. □

Lemma 3.4 — Under assumptions $(F_1) - (F_2)$, Φ is lower weakly semicontinuous in $W_{p_1(x),p_2(x),\ldots,p_N(x)}(\Omega)$.

PROOF : Let $(u_k)_{k\in\mathbb{N}} \subset X_\Omega$ and $u \subset X_\Omega$ such that $u_k \rightharpoonup u$.

The functional $J : X_\Omega \to \mathbb{R}$ defined as $J(u_i) = \int\limits_\Omega \frac{1}{p_i(x)} \left(|\nabla u_i|^{p_i(x)}\right) dx$, $(1 \le i \le n)$ is weakly lower semi-continuous in X_Ω, we have

$$J(u_i) \le \liminf J(u_{k_i}).$$

Let $H(u_i) = \int\limits_\Omega \frac{1}{p_i(x)} \left(V(x)|u_i|^{p_i(x)}\right) dx$, then

$$|H(u_{k_i}) - H(u_i)| \le \frac{c}{p^-} \int\limits_\Omega \left| |u_{k_i}|^{p_i(x)} - |u_i|^{p_i(x)} \right| dx.$$

By using the inequality

$$\left| |a|^{p_i(x)} - |b|^{p_i(x)} \right| \le \beta |a-b| (|a| + |b|)^{p_i(x)-1} \ \forall a, b \in \mathbb{R},$$

where $\beta > 0$ is the constant of Hölder inequality, we obtain

$$|H(u_{k_i}) - H(u_i)| \le \beta \frac{c}{p^-} \int\limits_\Omega |u_{k_i} - u_i| (|u_{k_i}| + |u_i|)^{p_i(x)-1} dx.$$

With continuous and compact injection in the Sobolev space, it follows that

$$\lim_{k\to\infty} H(u_{k_i}) = H(u_i).$$

Furthermore, we have

$$\begin{aligned} |\Phi(u)| &= \left| \sum_{i=1}^n (J(u_i) + H(u_i)) \right|, \\ &\le \sum_{i=1}^n |J(u_i) + H(u_i)|. \end{aligned}$$

By the properties of the lower limit of sequences of real numbers, we obtain

$$\Phi(u) \le \liminf \Phi(u_k). \square$$

***Theorem* 3.5** — *Assume* (F_1), (F_2), (F_3) *and* (V) *hold, then there exist an open interval* $\Lambda \subseteq [0, +\infty[$ *and a positive real number* ρ *such that, for each* $\lambda \in \Lambda$, *problem* (P) *has at least three weak solutions in* X_Ω *whose norms are less than* ρ.

PROOF : For each $u \in X_\Omega$,

$\Phi : X_\Omega \to \mathbb{R}$ is a sequentially weakly lower semicontinuous, C^1 functional, whose derivative admits a continuous inverse on X^*; $\Psi : X_\Omega \to \mathbb{R}$ a C^1 functional with compact derivative.

Hence, the weak solutions of (3.1) are exactly the solutions of the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0.$$

By (F_2), for each $\lambda > 0$ one has that

$$\begin{aligned}
\Phi(u) + \lambda\Psi(u) &> \frac{1}{p^+}\sum_{i=1}^{n}\int_\Omega |\nabla u_i|^{p_i(x)} + c|\,|u_i|^{p_i(x)}\; dx - \lambda\int_\Omega F(x, u_1, ..., u_n)\,dx, \\
\int_\Omega F(x, u_1, ..., u_n)\,dx &\leq C\sum_{i=1}^{n}[\|u_i\|_{q_i(x)}^{q_i^-} + \|u_1\|_{q_1(x)}^{q_1^- -1}\|u_i\|_{q_i(x)} \\
&\quad + ... + \|u_n\|_{q_n(x)}^{q_n^- -1}\|u_i\|_{q_i((x))}], \\
\Phi(u) + \lambda\Psi(u) &> 2\min\left(\frac{1}{p^+}, \frac{c}{p^+}\right)\sum_{i=1}^{n}\|u_i\|_{p_i(x)}^{p_i^-} \\
&\quad -\lambda C\sum_{i=1}^{n}\left(\|u_1\|_{q_1(x)}^{q_1^- -1}\|u_i\|_{q_i(x)} + ... + \|u_n\|_{q_n(x)}^{q_n^- -1}\|u_i\|_{q_i(x)}\right).
\end{aligned}$$

Clearly, $\Phi(u) + \lambda\Psi(u)$ tends to infinity as $\|(u_1, u_2, ..., u_n)\|_{p_1(x), p_2(x), ..., p_n(x)} \to \infty$, since $q_i(x) < \inf(p_i(x)), (1 \leq i \leq n)$.

So one of the assumption of Theorem 3.5 holds.

For each $u \in X_\Omega$,

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty, r])}(-\Psi(u)) &= \sup_{\sum_{i=1}^{n}\int_\Omega \frac{|\nabla u_i|^{p_i(x)}}{p_i(x)} + V(x)\frac{|u_i|^{p_i(x)}}{p_i(x)}dx \leq r} \int_\Omega F(x, u_1(x), ..., u_n(x))\,dx, \\
&\leq meas(\Omega) \max_{(x, t_1, ..., t_n)\in\bar{\Omega}\times K} F(x, t_1, ..., t_n).
\end{aligned}$$

Now by (F_3), there exist $r>0$ and $\omega\in X_\Omega$ such that

$$\begin{aligned} meas\,(\Omega)\max_{(x,t_1,...,t_n)\in\bar{\Omega}\times K}F(x,t_1,...,t_n) &< \frac{\theta}{c}\frac{\int\limits_\Omega F(x,\omega_1,...,\omega_n)\,dx}{\sum\limits_{i=1}^n\prod\limits_{j=1,j\neq i}^n p_j^-\left(\int\limits_\Omega\left(|\nabla\omega_i|^{p_i(x)}+V(x)\,|\omega_i|^{p_i(x)}\right)dx\right)}, \\ &< \frac{\theta}{c\prod\limits_{i=1}^n p_i^-}\frac{\int\limits_\Omega F(x,\omega_1,...,\omega_n)\,dx}{\sum\limits_{i=1}^n\int\limits_\Omega\frac{|\nabla\omega_i|^{p_i(x)}+V(x)|\omega_i|^{p_i(x)}}{p_i^-}dx}, \\ &< \frac{\theta}{c\prod\limits_{i=1}^n p_i^-}\frac{\int\limits_\Omega F(x,\omega_1,...,\omega_n)\,dx}{\sum\limits_{i=1}^n\int\limits_\Omega\frac{|\nabla\omega_i|^{p_i(x)}+V(x)|\omega_i|^{p_i(x)}}{p_i(x)}dx}, \\ &= r\frac{-\Psi(\omega)}{\Phi(\omega)}. \end{aligned}$$

Then $\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u))<r\frac{-\Psi(\omega)}{\Phi(\omega)}$.

By the Proposition 2.7 we fix ρ such that

$$\sup_{x\in\Phi^{-1}(]-\infty,r])}(-\Psi(u))<\rho<r\frac{-\Psi(u)}{\Phi(u_2)}.$$

One has

$$\sup_{\lambda\geq0}\inf_{u\in X}(\Phi(u)+\lambda(\rho+\Psi(u)))<\inf_{u\in X}\sup_{\lambda\geq0}(\Phi(u)+\lambda(\rho+\Psi(u))).$$

At this point, the conclusion follows directly from Theorem 2.6, if we choose $h(\lambda)=\lambda\rho$, $u_1=0$, $u_2=\omega$, and $J=-\Psi$.

Now we want to point out a simple consequence of Theorem 3.5.

Fix $x_0\in\Omega$ and pick r_1,r_2 with $0<r_1<r_2$, such that $S(x_0,r_1)\subset S(x_0,r_2)\subseteq\Omega$.

Put

$$\kappa(N,p,r_1,r_2)=\frac{1}{r_2-r_1}\left(\frac{c\pi^{\frac{N}{2}}\left(r_2^N-r_1^N\right)}{\Gamma(1+\frac{N}{2})}\right)^{\frac{1}{p(x)}},$$

where Γ denotes the Gamma function and $m(\Omega)$ is the Lebesgue measure of the set Ω.

Corollary 3.6 — Let $h:\mathbb{R}\to\mathbb{R}$ is a continuous function. Put $H(t)=\int\limits_0^t h(\xi)\,d\xi$ for each $t\in\mathbb{R}$ and assume that there exist four positive constants θ,τ,η and s with $(\tau\kappa)^{p(x)}>\theta$, $s<p(x)$ such that

1. $H(t) \geq 0$, for each $t \in [0, \tau]$.
2. $m(\Omega)(\tau\kappa)^{p(x)} \max\limits_{t \in [-\sqrt[p]{\theta}, \sqrt[p]{\theta}]} H(t) < \theta r_1^N \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} H(\tau)$.
3. $H(t) \leq \eta(1 + |t|^s)$ for each $t \in \mathbb{R}$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and positive real number q such that, for each $\lambda \in \Lambda$, problem

$$\begin{cases} \Delta_{p(x)} u + \lambda h(u) = 0 \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

admits at least three solutions in X_Ω whose norms are less than q.

Example 3.7 : Consider the problem

$$\begin{cases} div\left(|\nabla u|^{p(x)-2} \nabla u\right) + \lambda\left(e^{-u} u^{13}(14 - u)\right) = 0, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases} \tag{3.3}$$

where $\Omega = \left\{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 9\right\}$, $p(x) = 1 + \left(x^2 + y^2\right)$. Taking into account $c = \frac{10\sqrt{2}}{2\sqrt[4]{\pi}}$, choosing $x_0 = (0, 0)$, $r_1 = 1$, $r_2 = 2$ and $h(u) = e^{-u} u^{13}(14 - u)$ for each $u \in \mathbb{R}$, all the assumptions of Corollary 3.6, are satisfied by choosing, for instance $\theta = 1$, $\tau = 10$, $s = 1$ and η sufficiently large. So there exist an open interval $\Lambda \subseteq [0, +\infty[$ and positive real number q such that, for each $\lambda \in \Lambda$, problem (3.3) admits at least three solutions in $W_0^{1,p(x)}(\Omega)$ whose norms are less than q.

4. Quasi-Newton Minimization Approach

In this section, we propose a quasi-Newton minimization approach for the solution. We consider the System

$$\begin{cases} -\Delta_{p_i(x)} u_i + V(x) |u_i|^{p_i(x)-2} u_i = g_i(x) \text{ for } x \in \Omega, \\ u_i = 0 \text{ for } x \in \partial\Omega. \end{cases} \tag{4.1}$$

Where Ω is an open bounded subset of $\mathbb{R}^N$ with $\partial\Omega$ Lipschitz continuous, $p_i \in P^{\log}$, that is p_i is a measurable function, and $\frac{1}{p_i}$ is globally log-Hölder continuous. Moreover, we assume $g_i \in L^{p_i'}(\Omega)$ (where p_i' denotes the dual variable exponent of $p_i(x)$).

The Schrödinger System (4.1), admits a weak solution $\underline{u}$ satisfying

$$\underline{u} = \arg \inf_{v \in X} J(v),$$

where

$$J(u) = \sum_{i=1}^{n} \int \frac{|\nabla u_i|^{p_i(x)}}{p_i(x)} + V(x) \frac{|u_i|^{p_i(x)}}{p_i(x)} - \int g_i u. \tag{4.2}$$

Or, equivalently,

$$J'(\underline{u})\, v = 0, \; \forall v \in X. \tag{4.3}$$

where,

$$J'(\underline{u})\, v = \sum_{i=1}^{n} \int |\nabla u_i|^{p_i(x)-2} \nabla u_i \nabla v_i + V(x) |u_i|^{p_i(x)-2} u_i v_i - \int g_i v. \tag{4.4}$$

A common way [10, 11, 23, 38] to tackle the problem is the direct minimization, in a suitable finite dimensional subspace of X, of the functional J in Eq (4.2) , rather than solving the nonlinear equation (4.3). However, to our knowledge, ad hoc minimization algorithms were developed only for the p-constant case [5,23,38], whereas only general purpose methods such as the quasi Newton method BFGS (Broyden–Fletcher–Goldfarb–Shanno) have been used for the $p(x)$-variable case [11].

In this section, we minimize $J(u)$ employing a new quadratic model which makes use of the exact second differential $J''(u)$, only slightly regularized in order to handle possible analytic or numerical degeneracy when $|\nabla u|$ is small and $p_i(x)$ is close to the extreme values p_i^- or p_i^+.

The result is an efficient and robust algorithm converging faster than those available in literature, both for the p-constant case and the $p(x)$-variable one.

4.1 *Minimization problem*

We minimize $J(u)$ in a suitable finite element subspace of X and we call $\underline{u}^h$ the solution,

$$\underline{u}^h = \arg \min_{v^h \in X_0^h} J\left(v^h\right) \Leftrightarrow J'\left(\underline{u}^h\right) v^h = 0, \; \forall v^h \in X_0^h.$$

Given a regular triangulation of a polygonal approximation Ω_h of the domain, we select the subspace $X_0^h \subset X$ of continuous piecewise linear functions which are zero at the boundaries of Ω_h. Since for $p_i \neq 2$ problem (4.1) is degenerate quasi-linear elliptic, its solution has a limited regularity (see, for instance, [24]) and therefore higher-order finite element approximations do not worth, (see [4]).

For the variable exponent case, $p(x)$ is approximated by continuous piecewise linear functions as well, even if a local approximation by constant functions is possible (see [7, 8]). Given the approximation $u^k = \left(u_1^k, u_2^k, ..., u_n^k\right) \in X_0^h$ of the solution $\underline{u}^h$ at iteration k, we look for a direction $d^k \in V_0^h$ such that

$$J\left(u^k + \alpha_k d^k\right) < J\left(u^k\right).$$

The descent direction d^k is called steepest descent direction if

$$J'\left(u^k\right)d^k = -\left\|J'\left(u^k\right)\right\|_* \cdot \left\|d^k\right\|,$$

where $\|.\|$ is a suitable norm in X_0^h and $\|.\|_*$ its dual norm. The idea (see [23,38]) is to find d^k as the solution of

$$d^k : b_k\left(d^k, v\right) = -J'\left(u^k\right)v,\ \forall v \in X_0^h.$$

Where $b_k(.,.)$ is a suitable bilinear form depending on iteration k. The choice of b_k characterizes the minimization method.

The extension to non-homogeneous Dirichlet boundary conditions is straightforward. The solution $\underline{u}$ belongs to the variable exponent Sobolev space $X_f = W_f^{1,p_1(x)} \times ... \times W_f^{1,p_n(x)} = \{v \in W^{1,p_1(x)} \times ... \times W^{1,p_n(x)}, v = f \text{ sur } \partial\Omega\}$, and its piecewise approximation must be in $X_{f_h}^h$, that is the space of continuous piecewise linear functions whose value of $\partial\Omega_h$ is f_h, where f_h is chosen to approximate the Dirichlet boundary data. The search directions are still in the space X_0^h.

4.1.1 *Gradient-based directions*

The choice in [23], for the p_i-constant case, is $d^k = w^k$, where

$$b_k\left(w^k, v\right) = \begin{cases} \int\left(\epsilon + \left|\nabla u^k\right|^{p_i-2}\right)\nabla w^k.\nabla v,\ p_i > 2. \\ \int\left(\epsilon + \left|\nabla u^k\right|\right)^{p_i-2}\nabla w^k.\nabla v,\ p_i < 2. \end{cases} \tag{4.5}$$

The bilinear form $b_k(.,.)$ corresponds to a simple linearization of $J'\left(u^k\right)v$. The parameter ϵ is introduced in order to handle possible analytic or numerical degeneracy where $\left|\nabla u^k\right|$ is small. In fact, for $p_i >> 2$ the term $\left|\nabla u^k\right|^{p_i-2}$ may underflow even if $\left|\nabla u^k\right| > 0$. On the other hand, for $p_i < 2$ the same term may overflow. We notice that the parameter ϵ is introduced only for finding the descent direction and not for regularizing the original $p(x)$-Laplacian functional J. With the above choice, the authors in [23] proved a convergence result $\left(J\left(u^k\right) \to J(u)\right)$ only for the case $p_i > 2$. Their complicated proof is hardly extendible to the case $p_i < 2$ or to the general case with variable $p(x)$. The direction w^k is called in [23] preconditioned steepest descent. The scalar value α_k is chosen by exact line search

$$\alpha_k = \arg\min_{\alpha} J\left(u^k + \alpha d^k\right). \tag{4.6}$$

In [38], w^k is computed for all $1 < p < \infty$, using the first definition in (4.5). The descent direction is then computed by

$$d^k = w^k + \beta_k d^{k-1},$$

where,

$$\beta_k = \max\left\{0, \min\left\{\frac{w^{k^T} w^k}{w^{k-1^T} w^{k-1}}, \frac{\left(w^k - w^{k-1}\right)^T w^k}{w^{k-1^T} w^{k-1}}\right\}\right\}.$$

The definition of β_k corresponds to an hybridization of the popular Fletcher–Reeves and Polak–Ribière-Polyak parameters for the nonlinear conjugate gradient method (see also [3]). Direction d^k is called in [38] hybrid conjugate gradient. The scalar α_k is chosen as in Eq.(4.6).

4.1.2 *Quasi-Newton direction*

Our proposal for the direction d^k in the general case $p(x)$ is the following. We start with the second differential of J, which is well defined for $p_i(x) \geq 2$,

$$\begin{aligned} J''(u)(v,w) &= \sum_{i=1}^{n} \int (p_i(x) - 2)\left(|\nabla u_i|^{p_i(x)-4} (\nabla u_i.\nabla w_i)(\nabla u_i.\nabla v_i)\right. \\ &\quad \left. + V(x)|u_i|^{p_i(x)-4}(u_i.w_i)(u_i.v_i)\right) \\ &\quad + \sum_{i=1}^{n} \int \left(|\nabla u_i|^{p_i(x)-2}(\nabla v_i.\nabla w_i) + V(x)|u_i|^{p_i(x)-2}(v_i.w_i)\right). \end{aligned} \tag{4.7.a}$$

$$\begin{aligned} &= \sum_{i=1}^{n} \int |\nabla u_i|^{p_i(x)-2}\left((p_i(x)-2)\left(\frac{\nabla u_i}{|\nabla u_i|}.\nabla w_i\right)\left(\frac{\nabla u_i}{|\nabla u_i|}.\nabla v_i\right) + (\nabla v_i.\nabla w_i)\right) \\ &+ \sum_{i=1}^{n} \int V(x)|u_i|^{p_i(x)-2}\left((p_i(x)-2)\left(\frac{u_i}{|u_i|}.w_i\right)\left(\frac{u_i}{|u_i|}.v_i\right) + (v_i.w_i)\right). \end{aligned} \tag{4.7.b}$$

$$\begin{aligned} &= \sum_{i=1}^{n} \int |\nabla u_i|^{p_i(x)-2}\left((p_i(x)-2)(sign(\nabla u_i).\nabla w_i)(sign(\nabla u_i).\nabla v_i) + (\nabla v_i.\nabla w_i)\right) \\ &+ \sum_{i=1}^{n} \int V(x)|u_i|^{p_i(x)-2}\left((p_i(x)-2)(sign(u_i).w_i)(sign(u_i).v_i) + (v_i.w_i)\right), \end{aligned} \tag{4.7.c}$$

where we denoted

$$sign(\nabla u_i) = \frac{\nabla u_i}{\sqrt{|\nabla u_i|^2 + (1 - sign\left(|\nabla u_i|^2\right))}} = \begin{cases} \frac{\nabla u_i}{|\nabla u_i|} \text{ if } \nabla u_i \neq 0, \\ 0 \text{ if } \nabla u_i = 0. \end{cases}$$

$$sign(u_i) = \frac{u_i}{\sqrt{|u_i|^2 + (1 - sign\left(|u_i|^2\right))}} = \begin{cases} \frac{u_i}{|u_i|} \text{ if } u_i \neq 0, \\ 0 \text{ if } u_i = 0. \end{cases}$$

Formula $(4.7.c)$ is well defined and numerically computable for $p(x) \geq 2$ even if ∇u_i is zero somewhere. On the other hand it is in general still not positive definite and not defined if $p(x) < 2$ and $\nabla u_i = 0$ somewhere. Therefore we modify $|\nabla u_i|^{p_i(x)-2}$ and $|u_i|^{p_i(x)-2}$in $(4.7.c)$ into

$$|\nabla u_i|_\epsilon^{p_i(x)-2} = \epsilon + \left(\epsilon^2 . \left(1 - sign\left(|\nabla u_i|^2\right)\right) + |\nabla u_i|^2\right)^{\frac{p(x)}{p(x)-2}}.$$

$$|u_i|_\epsilon^{p_i(x)-2} = \epsilon + \left(\epsilon^2 . \left(1 - sign\left(|u_i|^2\right)\right) + |u_i|^2\right)^{\frac{p(x)}{p(x)-2}}. \tag{4.8}$$

In this way, we accomplish both the regularizations in Eq.(4.5), since $p(x)$ can be simultaneously very large in some regions and small in some other regions. Hence, the regularized second differential is

$$J_\epsilon''(u)(v,w) = \sum_{i=1}^{n} \int |\nabla u_i|_\epsilon^{p_i(x)-2} \left((p_i(x) - 2)(sign(\nabla u_i).\nabla w_i)(sign(\nabla u_i).\nabla v_i) + (\nabla v_i.\nabla w_i)\right)$$

$$+ \sum_{i=1}^{n} \int V(x) |u_i|_\epsilon^{p_i(x)-2} \left((p_i(x) - 2)(sign(u_i).w_i)(sign(u_i).v_i) + (v_i.w_i)\right). \tag{4.9}$$

Therefore our descent direction is $d^k = w^k$ defined by

$$w^k : b_k\left(v, w^k\right) = -J'\left(u^k\right) v,\ \forall v \in X_0^h. \tag{4.10}$$

Where $b_k\left(v, w^k\right) = J_\epsilon''\left(u^k\right)\left(v, w^k\right)$. In this way, we are in practice approximating $J(u)$ by a quadratic positive definite model

$$J(u) \approx J\left(u^k\right) + J'\left(u^k\right)\left(u - u^k\right) + \frac{1}{2} J_\epsilon''\left(u^k\right)\left(\left(u - u^k\right), \left(u - u^k\right)\right),$$

from which the name quasi-Newton. Other regularizations would be possible, by replacing $|\nabla u_i|^{p_i(x)-2}$ with $\epsilon + (\epsilon + |\nabla u_i|)^{p_i(x)-2}$ (see, for instance, [23]) and $|u_i|^{p_i(x)-2}$ with $\epsilon + (\epsilon + |u_i|)^{p_i(x)-2}$. Or with $\epsilon + \left(\epsilon^2 + |\nabla u_i|^2\right)^{\frac{p_i(x)}{p_i(x)-2}}$, (see, for instance, [6]), which is similar to the idea used in [22], where the problem itself, and not only the second differential, is regularized in the same way. Our choice (4.8) turned out to be the most effective in the numerical experiments. We notice that the choice of the gradient-based directions [23,38] can be generalized for the $p(x)$-variable case as

$$w^k : P_\epsilon\left(u^k\right)\left(v, w^k\right) = -J'\left(u^k\right) v,\ \forall v \in X_0^h. \tag{4.11}$$

and

$$d^k = w^k,\ \text{preconditioned steepest descent } [23], \tag{4.12.a}$$

$$d^k = w^k + \beta_k d^{k-1}, \text{ hybrid conjugate gradient } [38]. \tag{4.12.b}$$

The scaling length α_k in $u^k + \alpha_k d^k$ is found by a backtracking line search method based on sufficient decrease condition (Armijo's rule). Together with reasonable assumptions on $J''_\epsilon\left(u^k\right)$, this is enough to guarantee convergence to a stationary point of $J(u)$ (see, for instance, [25, Th. 3.2.4]).

4.2 *Numerical examples*

We implemented the quasi-Newton in FreeFem++(see $[21]$), for the solution of two-dimensional problems, being the extension to three dimensions straightforward. The numerical solution at iteration k is denoted by

$$u^k(x,y) = \sum_j u_j^k \phi_j(x,y),$$

where $\phi_j(x,y)$ is the jth nodal finite element basis function. In the following numerical examples, the initial guess $u^0(x,y)$ is always the solution of Poisson's problem corresponding to $p = 2$ and $V = 0$. The descent directions w^k in (4.10) and (4.11) are approximated by the linear conjugate gradient method.

The exit criterion (see [12, p. 160]) is

$$\max_j \left| \frac{J'\left(u^k\right)\phi_j \circ u_j^k}{J\left(u^k\right)} \right| \leq 10^{-6},$$

where $J'\left(u^k\right)\phi_j$ is defined in (4.10) and $\circ$ denotes Hadamard's product. For comparison, in [23,38] the initial guess is $u^0(x,y) = 0$, w^k is computed by a multigrid solver, the bisection method and the golden section method are used in the line search, respectively and the exit criterion is

$$\frac{\sqrt{b_k\left(d^k, d^k\right)}}{\sqrt{b_0\left(d^0, d^0\right)}} \leq 10^{-6}.$$

The solution of Eq.(4.10) is obtained by the default linear conjugate gradient method provided by FreeFem++, which employs the diagonal preconditioner.

4.2.1 *p-constant case*

Example 4.1 : In this example, we present another method of discretization, by introducing an iteration index, we plot the discretized solution, according to the values of p.

Given the following boundary value problem involving the p_i-laplacian operator $\Delta_{p_i} = div$

$(|\nabla u|^{p_i-2}\nabla u)$, with $|\nabla u|^{p_i-2} = \left(\sum_{j=1}^{N}\left(\frac{\partial u}{\partial x_j}\right)^2\right)^{\frac{p_i-2}{2}}$ and $1 < p_i < \infty$ on a domain Ω :

$$\begin{cases} -\Delta_{p_i} u_i + V(x)\,|u_i|^{p_i-2}\,u_i = g_i(x)\,,\ \text{for } x \in \Omega, \\ u_i = 0 \text{ for } x \in \partial\Omega. \end{cases}$$

Weak solutions of the last problem satisfy the following equation:

$$\sum_{i=1}^{n}\int_{\Omega}(|\nabla u_i|^{p_i-2}\langle\nabla u_i, \nabla v_i\rangle + V(x)\,|u_i|^{p_i-2}\langle u_i, v_i\rangle - g_i v_i)dx = 0,$$

We solved the equation iteratively by introducing the dependant variable

$$S^k = \begin{cases} 1,\ \text{if } k = 1, \\ \gamma S^{k-1} + (1-\gamma)\left|\nabla u_i^k\right|^{p_i-2},\ \text{if } k > 1. \end{cases}$$

$k \in \mathbb{N}$, is the iteration index. The value of γ affects the convergence properties of the solution process. In the example, γ was chosen to be 0.1, which seems to work for $p \geq 1.2$ but becomes less stable for $1 < p \leq 1.2$. For $p < 1.2$, you seem to need a γ that converges (slowly enough) to one. $\gamma = 0$, seems to be unstable or at least oscillatory for all p.

Using the technique of S^k, with one equation for $g = 1$ and $V = 0$, in Free Fem ++ 3.26, according to the values of p and γ, we obtain our maximum and minimum, see the Figure 1.

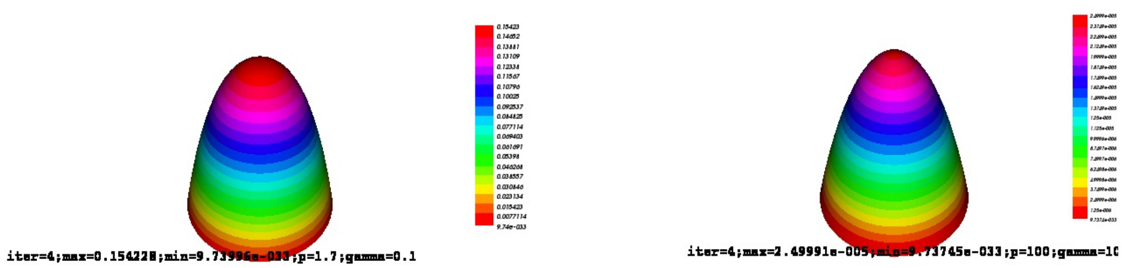

Figure 1:

Example 4.2 : This case is taken from [23, 38], with, $\Omega = (0,1)$ and $g = 1$, $V = 0$. The exact solution is

$$\underline{u}(x,y) = \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{1}{p-1}}\left(1 - \left(x^2+y^2\right)^{\frac{p}{2p-2}}\right).$$

The corresponding value of $J(\underline{u})$ is

$$J(\underline{u}) = \pi \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{(p-1)^2}{p(2-3p)}.$$

The disk $B(0,1)$ is discretized with different meshes. We notice that for the relatively small value $p = 4$, the shape is very close to the cone $1 - \sqrt{x^2 + y^2}$ corresponding to the limit $p \to \infty$. On the other hand, in the limit $p \to 1^+$ the solution tends to zero with a cake like shape.

We shows Now, the numerical solutions on mesh $D1$ for the different cases of p:

	$p = 1.02$	$p = 1.1$	$p = 4$	$p = 1000$
iters	117	75	8	145
J err	0.450996	0.0232178	0.000981594	0.00716519
$W^{1,p}$ err	0.566756	0.0977252	0.0262471	0.278893

We selected a range of constant p values from 1.02 to 1000. The value 1.02 was chosen because in [23] the smallest successfully tried value was 1.06, whereas the value 1000 was the maximum tested in [23, 38]. We also considered the value 1.1 because used in [23] and claimed to overflow in [38]. We notice that the iteration number weakly depends on the mesh size, especially for not too large values of p. This property was already observed in [23, 38] for the gradient-based methods and therein named “mesh independence”.

4.2.2 *p(x)-variable case*

Example 4.3 : This case is the two-dimensional extension of the one-dimensional example reported in [11], with $\Omega = (-1,1)^2$, $f = 0$ and

$$p(x,y) = \begin{cases} \frac{1-\epsilon}{\epsilon}|x| + 1 + \epsilon \text{ if } |x| \leq \epsilon, \\ 2, \text{ if } \epsilon < |x| \leq 1. \end{cases}$$

where ϵ is a small parameter and $p(0,y) \to 1^+$ when $\epsilon \to 0^+$. The non-homogeneous Dirichlet boundary conditions are such that the exact solution is

$$\underline{u}(x,y) = \begin{cases} (U(|x|) - U(0))sign(x), \text{ if } |x| \leq \epsilon, \\ (C(|x|-1) + B)sign(x), \text{ if } \epsilon < |x| \leq 1. \end{cases}$$

where C is set to 1.3, and for $0 < |x| \leq \epsilon$,

$$U(x) = \frac{\left(\frac{1-\epsilon}{\epsilon}x + \epsilon\right)\exp\left(\frac{\ln C}{\frac{1-\epsilon}{\epsilon}x+\epsilon}\right) - \ln C.Ei\left(\frac{\ln C}{\frac{1-\epsilon}{\epsilon}x+\epsilon}\right)}{\frac{1-\epsilon}{\epsilon}},$$

and $B = U(\epsilon) - U(0) + C(1-\epsilon)$. The function $Ei(x)$ is the exponential integral defined as

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt.$$

For small values of ϵ the solution has a steep gradient along $x = 0$. For instance, for $\epsilon = 0.02, \partial_x \underline{u}(0, y) = C^{\frac{1}{\epsilon}} = 1.3^{50}$. As correctly observed in [11], a more efficient and accurate finite element approximation would require a discontinuous Galerkin approach. For this reason, we report the Luxemburg norm in $L^{p(x)}$ space of the relative error. In fact, even if the solution is in $W^{1,p(x)}$ space, due to the steep gradient along $x = 0$, we had no reliable numerical approximation of $\|\nabla \underline{u}\|_{L^{p(x)}}$, on the uniform grid we used (N = 101 points in each direction).

	$\epsilon = 0.02$	$\epsilon = 0.04$	$\epsilon = 0.08$	$\epsilon = 0.1$
iters	3	3	3	3
$L^{p(x)}$ rel err	0.391598	0.0965927	0.0153809	0.00794818

We see that, for relatively small values of ϵ the quasi-Newton method takes only three iterations. On the other hand, if we use the BFGS method implemented in FreeFem++ (the same method was chosen by the authors in Ref. [11] for the one-dimensional example), then the maximum number of allowed iterations is reached. The hybrid preconditioned Conjugate Gradient method, never applied before to the $p(x)$-Laplacian, is better than BFGS but in any case worse than our quasi-Newton method.

5. Conclusions

In the first, we study the theoretical side for the Schrodinger's problems, under a new assumptions we ensures the existence of an open interval $\Lambda \subseteq [0; +\infty[$ and a positive real number ρ, such that, for each $\lambda \in \Lambda$, problem (3.1) admits at least three weak solutions whose norms in X_Ω are less than ρ. In the second, we developed a minimization approach for the Schrodinger's problem based on a quadratic model of the objective functional with a regularized second differential (quasi-Newton minimization). We have carried out several numerical examples in two space dimensions with constant p or variable $p(x)$, verified the results against existing analytic solutions, and found that our method outperforms those available in literature, both in number of iterations. In particular, the quasi-Newton approach proved to be robust and efficient for values of p very small (up to 1.05) or very large (up to 1000) and for examples of $p(x)$ varying on the domain in a range between p_1 and p_2 with $1.02 \le p_1 \le 2$ and $2 \le p_2 \le 4$.

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