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A STUDY OF THE LOCAL CONVERGENCE OF A FIFTH ORDER ITERATIVE METHOD[1]

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We present a local convergence study of a fifth order iterative method to approximate a locally unique root of nonlinear equations. The analysis is discussed under the assumption that first order Fréchet derivative satisfies the Lipschitz continuity condition. Moreover, we consider the derivative free method that obtained through approximating the derivative with divided difference along with the local convergence study. Finally, we provide computable radii and error bounds based on the Lipschitz constant for both cases. Some of the numerical examples are worked out and compared the results with existing methods.

Key words : Nonlinear equations; iterative methods; local convergence; divided differences.

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1. Introduction

In this paper, we concerned with the problem of approximating a locally unique solution α of the nonlinear equations

$$F(x) = 0. \tag{1}$$

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Such nonlinear problems arise from the major disciplines of mathematics, engineering and applied science. In the regards of applicability of nonlinear system, we have several examples in the available literature 1-6]. For instance, the problem of investigating coarse-grained dynamical properties of neuronal networks in kinetic theory discussed in [8]. In addition, Nejat and Ollivier [9] presented the effect of discretization order on preconditioning and convergence of high-order Newton-Krylov unstructured flow solver in computational fluid dynamics. In 2008, Grosan and Abraham [10], showed the applicability of the system of nonlinear equations in neurophysiology, kinematics syntheses problem, chemical equilibrium problem, combustion problem and economics modeling problem. Very recently, Awawdeh and Tsoulos *et al*. [11, 12] solved the reactor and steering problems by phrasing them like systems of nonlinear equations.

The solution of nonlinear equations play an important role in the field Mathematics and applied science. Analytic solution of such problems are not available often. Then, we have to focus on iterative methods that depends on the iteration process. These iterative methods can be divided into the categories of one point and multi-point methods, with and without memory. We have the well known quadratic convergent iterative method is the classical Newton's method. In the past and recent years, many authors have developed several robust and efficient iterative methods with higher convergence order in the available literature. But it is very important to discuss the local and semilocal convergence analysis for them.

The study of local convergence of higher order iterative methods can be analyzed under different continuity conditions in Banach spaces (see, [13, 14]). Argyros and George [15] developed the local convergence analysis of third order Halley-like methods under Lipchitz continuity conditions and it is given for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k) \\ u_k &= y_k + (1-a)F'(x_k)^{-1}F(x_k) \\ z_k &= y_k - \gamma A_{a,k}F'(x_k)^{-1}F(x_k) \\ x_{k+1} &= z_k - \eta B_{a,k}F'(x_k)^{-1}F(z_k), \end{aligned} \tag{2}$$

where, $\eta, \gamma, a \in (-\infty, \infty) - \{0\}$, $H_{a,k} = \frac{1}{a}F'(x_k)^{-1}\Big(F'(u_k) - F'(x_k)\Big)$, $A_{a,k} = I - \frac{1}{2}H_{a,k}\Big(I - \frac{1}{2}H_{a,k}\Big)$, $B_{a,k} = I - H_{1,k} + H_{a,k}^2$. The local convergence of Chebyshev-Halley-type method discussed in [17], which is given by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k) \\ z_k &= x_k - \Big(1 + (F(x_k) - 2\eta F(y_k))^{-1}F(y_k)\Big)F'(x_k)^{-1}F(x_k) \end{aligned}$$

$$x_{k+1} = z_k - \left(F^{'}(x_k) + \bar{F}''(x_k)(z_k - x_k)\right)^{-1} F(z_k), \quad k \geq 0, 1, 2, \ldots \tag{3}$$

where, $\bar{F}^{''}(x_k) = 2F(y_k)F^{'}(x_k)^2 F(x_k)^{-2}$ and η is a parameter. The order of this family is at least five for any value of η and for $\eta = 1$, it is six.

In this paper, we analyze the local convergence of fifth order iterative method which was proposed by Cordero *et al*. [16]. The local convergence study is based only the first order Fréchet derivative that satisfies the Lipschitz continuity condition. The existence and uniqueness region of the solution is established. Numerical examples worked out and convergence balls for each of them are obtained. We compare these results with the convergence balls of existing methods (2) and (3). In addition, we discuss the local convergence of derivative free iterative method that obtained through approximating the derivative by divided differences. Some numerical examples worked out and the convergence regions computed.

This paper is divided into four sections and organized as follows. In Section 1, we form the introduction. The local convergence study is performed in Section 2. In addition, the existence and uniqueness region of the solution is derived along with some numerical examples. In Section 3, the local convergence of the derivative free iterative method is discussed and also the computation of existence and uniqueness region of solution with numerical examples. Finally, the concluding remarks mentioned in the Section 4.

2. Local Convergence Analysis

In this section, we consider a fifth order iterative method proposed in [16] and its local convergence analysis under Lipschitz conditions on $F^{'}$. It is given for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1} F(x_k) \\ z_k &= x_k - 2(F'(x_k) + F'(y_k))^{-1} F(x_k) \\ x_{k+1} &= z_k - F'(y_k)^{-1} F(z_k), \end{aligned} \tag{4}$$

where x_0 is the starting point. In 2012, Cordero *et al*. [16], presented the fifth order of convergence using Taylor series on higher order Fréchet derivative without obtaining the convergence balls. Even though only first-order derivative appears in the expression (4). They also assumed that the starting point x_0 is sufficiently close to the solution without estimating this closeness. Now, we have addressed these problems using only first order Fréchet derivative.

Suppose that $B(v, \rho)$ and $\bar{B}(v, \rho)$ denote the open and closed balls, respectively with center v and radius ρ. Let $F : D \subseteq X \to Y$ be a Fréchet differentiable operator defined on open domain D such that for $\alpha \in D$, $L_0 > 0$, $L > 0$ and $\forall x, y \in D$, we have

$$F(\alpha) = 0, \quad F'(\alpha)^{-1} \in L(Y, X), F'(\alpha)^{-1} \neq 0 \tag{5}$$

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \leq L_0\|x - \alpha\|, \ \ \forall\, x \in D \tag{6}$$

$$\|F'(\alpha)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \ \ \forall\, x, y \in B\left(\alpha, \frac{1}{L_0}\right) \subseteq D. \tag{7}$$

Lemma 1 — If the nonlinear operator F satisfies the above assumptions, then $\forall x \in B(\alpha, \frac{1}{L_0})$, we have

$$\begin{aligned} \|F'(\alpha)^{-1}F'(x)\| &\leq 1 + L_0\|x - \alpha\|, \\ \|F'(\alpha)^{-1}F'(\alpha + t(x - \alpha))\| &\leq 1 + L_0\|x - \alpha\| \quad \forall\, t \in]0, 1[, \\ \|F'(\alpha)^{-1}F(x)\| &\leq (1 + L_0\|x - \alpha\|)\|x - \alpha\|. \end{aligned}$$

PROOF : The proof is trivial and can be seen in [13].

The following result describes the local convergence of iterative method (4) in Theorem 1.

***Theorem* 1** — *Let F be a nonlinear operator satisfying assumptions (5), (6) and (7). Then, the sequence $\{x_{k+1}\}$ generated by (4) is well defined for $x_0 \in B(\alpha, r_3)$ and converges to α, where, r_3 is the smallest positive root of s_3. Also, we obtain the following inequalities for $k \geq 0$:*

$$\|y_k - \alpha\| \leq g_1(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_3, \tag{8}$$

$$\|z_k - \alpha\| \leq g_2(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_3, \tag{9}$$

$$\|x_{k+1} - \alpha\| \leq g_3(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_3, \tag{10}$$

where, s_3, g_1, g_2 and g_3 are auxiliary functions defined in the proof. If there exists a $R \in [r_3, \frac{2}{L_0})$ such that $\overline{B}(\alpha, R) \subseteq D$, then α is the unique solution in $\overline{B}(\alpha, R)$.

PROOF : Since $x_0 \in D$ and using (6), we get

$$\|I - F^{'}(\alpha)^{-1}F^{'}(x_0)\| = \|F^{'}(\alpha)^{-1}(F^{'}(x_0) - F^{'}(\alpha))\| \leq L_0\|x_0 - \alpha\| < 1,$$

for $\|x_0 - \alpha\| < \frac{1}{L_0}$. Therefore, by Banach Lemma, $F^{'}(x_0)^{-1}$ exists and

$$\|F^{'}(x_0)^{-1}F^{'}(\alpha)\| \leq \frac{1}{1 - L_0\|x_0 - \alpha\|}. \tag{11}$$

Thus, y_0 is well defined. From the first equation of (4) for $k = 0$, we get

$$\begin{aligned} y_0 - \alpha &= x_0 - \alpha - F'(x_0)^{-1}F(x_0) \\ &= -F'(x_0)^{-1}\Big(F(x_0) - F'(x_0)(x_0 - \alpha)\Big) \\ &= -F'(x_0)^{-1}F'(\alpha)\int_0^1 F'(\alpha)^{-1}[F'(\alpha + t(x_0 - \alpha)) - F'(x_0)](x_0 - \alpha)dt. \end{aligned}$$

By using adequately Banach Lemma, the assumptions and denoting $e_0 = \|x_0 - \alpha\|$, we have

$$\|y_0 - \alpha\| \leq \frac{L\|x_0 - \alpha\|}{2\,(1 - L_0\|x_0 - \alpha\|)}\|x_0 - \alpha\| \leq g_1(e_0)e_0, \tag{12}$$

where,

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)}.$$

Obviously g_1 is an increasing function, and by taking $r_1 = \frac{2}{L+2L_0}$ it follows:

$$0 \leq g_1(t) < 1, \ \forall\, t \in [0, r_1). \tag{13}$$

Using (12) and (13), we get

$$\|y_0 - \alpha\| \leq g_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\|.$$

Again using (4) for $k = 0$, we get

$$\begin{aligned} z_0 - \alpha &= x_0 - \alpha - 2\big(F'(y_0) + F'(x_0)\big)^{-1}F(x_0) \\ &= (F'(y_0) + F'(x_0))^{-1}\Big[(F'(y_0) + F'(x_0))(x_0 - \alpha) - 2F(x_0)\Big] \\ &= -(F'(y_0) + F'(x_0))^{-1}\big[F(x_0) - F'(x_0)(x_0 - \alpha) + F(x_0) - F'(y_0)(x_0 - \alpha)\big] \\ &= -(F'(y_0) + F'(x_0))^{-1}F'(\alpha)\Big[2F'(\alpha)^{-1}\int_0^1[F'(\alpha + t(x_0 - \alpha)) - F'(x_0)](x_0 - \alpha)dt \\ &+ F'(\alpha)^{-1}(F'(x_0) - F'(y_0))(x_0 - \alpha)\Big]. \end{aligned}$$

To follow with the study of existence and bound for the product $(F'(y_0) + F'(x_0))^{-1}F'(\alpha)$ we observe that

$$-\frac{1}{2}F'(\alpha)^{-1}[F'(x_0) + F'(y_0) - 2F'(\alpha)] = I - \frac{1}{2}F'(\alpha)^{-1}[F'(x_0) + F'(y_0)] = I - A,$$

where $A = \frac{1}{2}F'(\alpha)^{-1}[F'(x_0) + F'(y_0)]$.

By adopting Banach Lemma, we have

$$\begin{aligned}\|I-A\| &= \left\|\frac{1}{2}F'(\alpha)^{-1}\left[F'(x_0)+F'(y_0)-2F'(\alpha)\right]\right\| \\ &= \left\|\frac{1}{2}\left[F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))+F'(\alpha)^{-1}(F'(y_0)-F'(\alpha))\right]\right\| \\ &\leq \frac{1}{2}\left(L_0\|x_0-\alpha\|+L_0\|y_0-\alpha\|\right) \\ &\leq \frac{1}{2}\left(L_0e_0+L_0g_1(e_0)e_0\right)=p_1(e_0)\end{aligned}$$

where $p_1(t) = \frac{1}{2}L_0(1+g_1(t))t$ is an increasing function such that $p_1(0) = 0$ and $p_1(r_1) = \frac{1}{2}L_0r_1\left(1+g_1(r_1)\right) = L_0r_1 < 1$ and so one has:

$$\|2(F'(x_0)+F'(y_0))^{-1}F'(\alpha)^{-1}\| \leq \frac{1}{1-p_1(e_0)}.$$

Then, turning to the expression of $z_0-\alpha$, we have

$$\begin{aligned}\|z_0-\alpha\| &\leq \frac{1}{2(1-p(\|x_0-\alpha\|))}\left[2L\int_0^1\|\alpha+t(x_0-\alpha)-x_0\|\|x_0-\alpha\|dt+L\|x_0-y_0\|\|x_0-\alpha\|\right] \\ &\leq \frac{1}{2(1-p(\|x_0-\alpha\|))}\left[2L\frac{\|x_0-\alpha\|^2}{2}+L(\|x_0-\alpha\|+\|y_0-\alpha\|)\|x_0-\alpha\|\right] \\ &\leq \frac{1}{2(1-p(e_0))}\left[Le_0+L(e_0+g_1(e_0)e_0)\right]e_0 \\ &\leq \frac{Le_0(2+g_1(e_0))}{2(1-p(e_0))}e_0 = g_2(e_0)e_0 \end{aligned} \tag{14}$$

where

$$g_2(t) = \frac{Lt(2+g_1(t))}{2(1-p_1(t))}.$$

So, we consider $s_2(t) = g_2(t)-1$ having that $s_2(0) = -1$ and $s_2(r_1) > 0$. Therefore, $s_2(t)$ has at least one zero in $(0, r_1)$ and let r_2 be the smallest one. Therefore, $0 < r_2 < r_1$ and

$$0 \leq g_2(t) \leq 1, \ \forall\, t \in [0, r_2). \tag{15}$$

Then, by using (14) and (15), we get

$$\|z_0-\alpha\| \quad \leq g_2(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\|.$$

Again using (4) for $k = 0$, we have

$$x_1-\alpha = z_0-\alpha-F'(y_0)^{-1}F'(\alpha)F'(\alpha)^{-1}F(z_0).$$

Since, $y_0 \in D$ and using (4), we obtain

$$\begin{aligned}\|I - F'(\alpha)^{-1}F'(y_0)\| &\le \|F'(\alpha)^{-1}(F'(\alpha) - F'(y_0))\| \\ &\le L_0\|y_0 - \alpha\| \le L_0 g_1(e_0)e_0 = p_2(e_0) < 1\end{aligned}$$

where

$$p_2(t) = L_0 g_1(t)t.$$

Then, $\exists\ (F'(\alpha)^{-1}F'(y_0))^{-1}$ and

$$\|F'(y_0)^{-1}F'(\alpha)\| \le \frac{1}{1 - p_2(e_0)}.$$

Therefore, by using Lemma 1, we get

$$\begin{aligned}\|x_1 - \alpha\| &\le \|z_0 - \alpha\| + \frac{1}{1 - p_2(e_0)}(1 + L_0\|z_0 - \alpha\|)\|z_0 - \alpha\| \\ &= \left(1 + \frac{1 + L_0 g_2(e_0)e_0}{1 - p_2(e_0)}\right) g_2(e_0)e_0 = g_3(e_0)e_0, \qquad (16)\end{aligned}$$

where

$$g_3(t) = \left(1 + \frac{1 + L_0 g_2(t)t}{1 - p_2(t)}\right) g_2(t).$$

Consider $s_3(t) = g_3(t) - 1$. Then, $s_3(0) = -1$ and $s_3(r_2) > 0$. Therefore, $s_3(t)$ has at least one zero in $(0, r_2)$ and let r_3 the smallest one. Therefore, $0 < r_3 < r_2$

$$0 \le g_3(t) \le 1\ \forall\ t \in [0, r_3), \qquad (17)$$

then by using (16) and (17), we have

$$\|x_1 - \alpha\| \le g_3(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\| < \eta.$$

Hence, Theorem 1 holds for $k = 0$. Changing x_0, y_0, z_0 and x_1 by x_k, y_k, z_k, x_{k+1}, we get the inequalities (8)-(10) $\forall\ k \ge 0$. Since, $\|x_{k+1} - \alpha\| \le \|x_k - \alpha\| < r_3$, this gives $x_{k+1} \in B(\alpha, r_3)$. Also $g_3(t)$ is an increasing function in $[0, r_3)$, since $g_3'(t) > 0\ \forall\ t \in [0, r_3)$. Thus, we get

$$\begin{aligned}\|x_{k+1} - \alpha\| &\le g_3(e_0)\|x_k - \alpha\| \le g_3(e_0)g_3(e_0)\|x_{k-1} - \alpha\| \\ &\le g_3(e_0)^2 g_3(e_0)\|x_{k-2} - \alpha\| \le \ldots \le g_3(e_0)^{k+1}\|x_0 - \alpha\|.\end{aligned}$$

Therefore, $\lim_{k\to\infty} x_k = \alpha$ as $g_3(t) < 1$.

For getting the uniqueness ball for root α, let $\beta \in B(\alpha, R)$ be such that $F(\beta) = 0$ and $\beta \neq \alpha$. Consider $P = \int_0^1 F^{'}(\beta + t(\alpha - \beta))dt$.

Using expression (1), we obtain

$$\|F^{'}(\alpha)^{-1}(P - F^{'}(\alpha))\| \leq \int_0^1 L_0\|\beta + t(\alpha - \beta) - \alpha\|dt \leq \frac{L_0}{2}\|\alpha - \beta\| = \frac{L_0}{2}R < 1,$$

therefore, by Banach Lemma, P^{-1} exists. Then,

$$0 = F(\alpha) - F(\beta) = P(\alpha - \beta),$$

we obtain $\alpha = \beta$. □

2.1 *Numerical examples*

In this subsection, we consider numerical examples to demonstrate the applicability of our work. Moreover, we compare our results with the local convergence of a modified Halley-like and Chebyshev-Halley-type methods (2) and (3), respectively.

Example 1 : Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0, 1)$. Define F on D for $v = (x, y, z)$ by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right).$$

Clearly, $\alpha = (0, 0, 0)$, $F'(\alpha) = F'(\alpha)^{-1} = diag\{1, 1, 1\}$, $L_0 = e - 1$, and $L = e$. Then, we have

$$r_3 = 0.13125 < r_2 = 0.21657 < r_1 = 0.32495.$$

Example 2 : Consider the system of nonlinear equations, $F(x) = 0$, associated to the nonlinear operator $F : R^2 \to R^2$ given by

$$F(x_1, x_2) = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix}$$

where $F_1(x_1, x_2) = 2x_1 - \frac{1}{9}x_1^2 - x_2$ and $F_2(x_1, x_2) = -x_1 + 2x_2 - \frac{1}{9}x_2^2$.

Clearly $\alpha = (9, 9)^T$ is a solution of above nonlinear system and $\forall\ (x, y) \in R^2$, we obtain

$$\|F'(\alpha)^{-1}(F'(x) - F'(y))\| = \frac{2}{9}\|x - y\|$$
$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| = \frac{2}{9}\|x - \alpha\|.$$

Table 1: Values of parameters

Examples	a	γ	η
1	1.0125	0.3	0.03
2	1	$1/9$	$2/9$
3	1	0.575	0.003

Taking $L_0 = \frac{2}{9}$ and $L = \frac{2}{9}$, we get

$$r_3 = 1.44284 < r_2 = 2.25000 < r_1 = 3.00000.$$

Example 3 : Consider the nonlinear Hammerstein type integral equation given by

$$F(x(s)) = x(s) - 5\int_0^1 s\,t\,x(t)^3 dt, \tag{18}$$

with $x(s)$ in $\mathcal{C}[\prime, \infty]$.

Clearly $\alpha = 0$. We work in the domain $B(0,1) \subseteq \mathcal{C}[\prime, \infty]$, then we have $L_0 = 7.5$ and $L = 15$, so we get

$$r_3 = 0.02481 < r_2 = 0.04185 < r_1 = 0.06667.$$

Now, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively. The value of parameters used by these methods are listed in Table 1. The radius of a convergence ball of a fifth order method (4) is compared with method (2) and method (3) in Table 2. We can observe that the larger radius of convergence ball is obtained by our approach.

Table 2: Comparison of radius of a ball

Examples	Method (4)	Method (2)	Method (3)
1	0.13125	0.02726	0.00892
2	1.44284	0.55264	0.06989
3	0.02481	0.00709	0.00755

3. The Derivative Free Method and Its Local Convergence Analysis

In this section our purpose is to complete the study of iterative method (4), when we use adequate approximation of the derivatives by divided differences. So, now the aim is to obtain the local convergence study in this case.

In order to obtain derivative free iterative methods we approximate derivatives by divided differences. That is an operator $[x, y; F]$ verifying

$$[x, y; F](x - y) = F(x) - F(y), \forall x, y \in D$$

and if F is Fréchet differentiable at $a \in D$ then $[\alpha, \alpha; F] = F'(\alpha)$. One can see different approximations of divided differences in [18, 19].

We consider the derivative free iterative method given for $k = 0, 1, 2, \dots$ by

$$\begin{aligned} y_k &= x_k - [x_k, x_k + F(x_k); F]^{-1} F(x_k) \\ z_k &= x_k - 2([x_k, x_k + F(x_k); F] + [y_k, y_k + F(y_k); F])^{-1} F(x_k) \\ x_{k+1} &= z_k - [y_k, y_k + F(y_k); F]^{-1} F(z_k), \end{aligned} \tag{19}$$

where x_0 is the starting point.

We use the following assumptions for setting the local convergence study in this case. Let $K_0 > 0$, $K > 0$ and $\forall x, y, u, v \in D$, we have $F(\alpha) = 0$, $F'(\alpha)^{-1} \neq 0$, in D, moreover

$$\|F'(\alpha)^{-1}([x, y; F] - [u, v; F])\| \leq K(\|x - u\| + \|y - v\|), \tag{20}$$

$$\|F'(\alpha)^{-1}([x, y; F] - [\alpha, \alpha; F])\| \leq K_0(\|x - \alpha\| + \|y - \alpha\|), \tag{21}$$

$$\|F(x) - F(\alpha)\| \leq L\|x - \alpha\|. \tag{22}$$

The next result describes the local convergence theorem for the derivative free iterative method (19).

***Theorem* 2** — *Let F the nonlinear operator satisfying assumptions (20), (21) and (22). Then, the sequence $\{x_{k+1}\}$ generated by (19) is well defined for any starting point $x_0 \in B(\alpha, \rho_3)$ and converges to α, where ρ_3 is the smallest positive root of function q_3. Also, we obtain the following inequalities for $k \geq 0$:*

$$\|y_k - \alpha\| \leq h_1(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < \rho_3, \tag{23}$$

$$\|z_k - \alpha\| \leq h_2(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < \rho_3, \tag{24}$$

$$\|x_{k+1} - \alpha\| \leq h_3(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < \rho_3, \tag{25}$$

where, h_1, h_2, h_3, q_3 are auxiliary functions defined in the proof and ρ_3 is the smallest root of $q_3(t)$. Moreover, if there exists a $R_1 \in [\rho_3, \frac{1}{K_0})$ such that $\overline{B}(\alpha, R_1) \subseteq D$, then α is the unique solution in $\overline{B}(\alpha, R_1)$.

PROOF : Since $x_0 \in D$ and using (21), we get

$$\begin{aligned}\|F'(\alpha)^{-1}([x_0, x_0 + F(x_0); F] - [\alpha, \alpha; F])\| &\leq K_0(\|x_0 - \alpha\| + \|x_0 + F(x_0) - \alpha\| \\ &\leq K_0(\|x_0 - \alpha\| + (1 + L)\|x_0 - \alpha\|) \\ &= K_0(2 + L)\|x_0 - \alpha\| < 1,\end{aligned}$$

for $\|x_0 - \alpha\| < \frac{1}{K_0(2+L)}$. Therefore, by Banach Lemma, $[x_0, x_0 + F(x_0); F]^{-1}$ exists and

$$\|[x_0, x_0 + F(x_0); F]^{-1} F'(\alpha)\| \leq \frac{1}{1 - K_0(2 + L)\|x_0 - \alpha\|}. \tag{26}$$

Thus, y_0 is well defined. Using (19) for $k = 0$, we get

$$\begin{aligned}y_0 - \alpha &= x_0 - \alpha - [x_0, x_0 + F(x_0); F]^{-1} F(x_0) \\ &= [x_0, x_0 + F(x_0); F]^{-1} F'(\alpha) F'(\alpha)^{-1} \left([x_0, x_0 + F(x_0); F] - [x_0, \alpha; F]\right)(x_0 - \alpha).\end{aligned}$$

Using expression (20), we have

$$\|F'(\alpha)^{-1}([x_0, x_0 + F(x_0); F] - [x_0, \alpha; F])\| \leq K(\|x_0 + F(x_0) - \alpha\|) \leq K(1 + L)\|x_0 - \alpha\|.$$

Therefore,

$$\begin{aligned}\|y_0 - \alpha\| &\leq \left(\frac{K(1 + L)\|x_0 - \alpha\|}{1 - K_0(2 + L)\|x_0 - \alpha\|}\right)\|x_0 - \alpha\| \\ &= h_1(e_0)e_0,\end{aligned} \tag{27}$$

where, $h_1(t) = \frac{K(1+L)t}{1-K_0(2+L)t}$ and $e_0 = \|x_0 - \alpha\|$.

Consider the function $q_1(t) = h_1(t) - 1$. Then, $q_1(0) = -1$ and $q_1(\frac{1}{K_0(2+L)}) \to +\infty$. Therefore, $q_1(t)$ has at least one zero in $\left(0, \frac{1}{K_0(2+L)}\right)$ and let ρ_1 be such a smallest zero. Therefore, $0 < \rho_1 < \frac{1}{K_0(2+L)}$, and

$$0 \leq h_1(t) < 1 \; \forall \, t \in [0, \rho_1). \tag{28}$$

Adopting the expressions (27) and (28), we get

$$\|y_0 - \alpha\| \leq h_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\|.$$

Again using (19) for $k = 0$, we get

$$\begin{aligned} z_0 - \alpha &= x_0 - \alpha - 2\big([x_0, x_0 + F(x_0); F] + [y_0, y_0 + F(y_0); F]\big)^{-1} F(x_0) \\ &= \big([x_0, x_0 + F(x_0); F] + [y_0, y_0 + F(y_0); F]\big)^{-1} F'(\alpha) F'(\alpha)^{-1} \Big([x_0, x_0 + F(x_0); F] \\ &+ [y_0, y_0 + F(y_0); F] - 2[x_0, \alpha; F]\Big)(x_0 - \alpha). \end{aligned}$$

Now, we study the existence and bound for the product $\big([x_0, x_0 + F(x_0); F] + [y_0, y_0 + F(y_0); F]\big)^{-1} F'(\alpha)$. Then, we observe that

$$\begin{aligned} &\frac{1}{2} F'(\alpha)^{-1} \Big(\big([x_0, x_0 + F(x_0); F] + [y_0, y_0 + F(y_0); F]\big) - 2F'(\alpha)\Big) \\ &= I - \frac{1}{2} F'(\alpha)^{-1} [\big([x_0, x_0 + F(x_0); F] + [y_0, y_0 + F(y_0); F]\big)] = I - C \end{aligned}$$

where $C = \frac{1}{2} F'(\alpha)^{-1} [\big([x_0, x_0 + F(x_0); F] + [y_0, y_0 + F(y_0); F]\big)]$.

By adopting Banach Lemma, we have

$$\begin{aligned} \|I - C\| &\leq \frac{1}{2}\Big(\|F^{'}(\alpha)^{-1}([x_0, x_0 + F(x_0); F] - F^{'}(\alpha))\| + \|F^{'}(\alpha)^{-1}([y_0, y_0 + F(y_0); F] \\ &\quad - F^{'}(\alpha))\|\Big) \\ &\leq \frac{K_0}{2}\Big((\|x_0 - \alpha\| + \|x_0 + F(x_0) - \alpha\|) + (\|y_0 - \alpha\| + \|y_0 + F(y_0) - \alpha\|) \Big) \\ &\leq \frac{K_0}{2}\Big((2 + L)\|x_0 - \alpha\| + (2 + L)\|y_0 - \alpha\|\Big) \\ &\leq \frac{K_0}{2}\Big((2 + L)\|x_0 - \alpha\| + (2 + L) h_1(\|x_0 - \alpha\|)\|x_0 - \alpha\|\Big) \\ &= \phi_1(e_0), \end{aligned}$$

where $\phi_1(t) = \frac{K_0}{2}(2 + L)(1 + h_1(t))t$ is an increasing function such that $\phi_1(0) = 0$ and $\phi_1(\rho_1) = \frac{K_0}{2}(2 + L)\rho_1\,(1 + h_1(\rho_1)) = \frac{K_0}{2}(2 + L)\rho_1 < 1$ and so we have

$$\|C^{-1}\| = \Big\|2\Big([x_0, x_0 + F(x_0); F] + [y_0, y_0 + F(y_0); F]\Big)^{-1} F'(\alpha)\Big\| \leq \frac{1}{1 - \phi_1(e_0)}.$$

Therefore,

$$\begin{aligned} \|z_0 - \alpha\| &\leq \tfrac{K}{2(1 - \phi_1(\|x_0 - \alpha\|))}\Big(\|x_0 + F(x_0) - \alpha\| + \|y_0 - x_0\| + \|y_0 + F(y_0) - \alpha\|\Big)\|x_0 - \alpha\| \\ &\leq \tfrac{K}{2(1 - \phi_1(e_0))}\Big((1 + L)e_0 + (1 + h_1(e_0))e_0 + (1 + L)h_1(e_0)e_0\Big)e_0 \\ &\leq \tfrac{K}{2(1 - \phi_1(e_0))}\Big((2 + L)(1 + h_1(e_0))e_0\Big)e_0 \\ &= h_2(e_0)e_0, \end{aligned} \tag{29}$$

where

$$h_2(t) = \frac{K}{2(1-\phi_1(t))}\Big((2+L)(1+h_1(t))t\Big).$$

Consider $q_2(t) = h_2(t) - 1$. Then, $q_2(0) = -1$ and $q_2(\rho_1) = \frac{K\rho_1(2+L)}{1-K_0(2+L)\rho_1} > 0$. Therefore, $q_2(t)$ has at least one zero in $(0, \rho_1)$ and let ρ_2 be such a smallest zero. Therefore, $0 < \rho_2 < \rho_1$ and

$$0 \le h_2(t) \le 1 \; \forall \, t \; \in \; [0, \rho_2). \tag{30}$$

Using expressions (29) and (30), we get

$$\|z_0 - \alpha\| \quad \le h_2(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\|.$$

Taking $k = 0$ in (19), we have

$$x_1 - \alpha = z_0 - \alpha - [y_0, y_0 + F(y_0); F]^{-1}F'(\alpha)F'(\alpha)^{-1}F(z_0) \quad .$$

Since $y_0 \in D$, we have

$$\begin{aligned}
\|F^{'}(\alpha)^{-1}([y_0, y_0 + F(y_0); F] - [\alpha, \alpha; F])\| &\le K_0(\|y_0 - \alpha\| + \|y_0 + F(y_0) - \alpha\| \\
&\le K_0(\|y_0 - \alpha\| + (1 + L)\|y_0 - \alpha\|) \\
&\le K_0(2 + L)h_1(e_0)e_0 = \phi_2(e_0) < 1,
\end{aligned}$$

where $\phi_2(t) = K_0(2 + L)h_1(t)t$.

Thus, by Banach Lemma, we have

$$\|[y_0, y_0 + F(y_0); F]^{-1}F'(\alpha)\| \le \frac{1}{1 - \phi_2(e_0)}.$$

Therefore,

$$\begin{aligned}
\|x_1 - \alpha\| &\le \|[y_0, y_0 + F(y_0); F]^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}([y_0, y_0 + F(y_0); F] - [z_0, \alpha; F])\|\|z_0 - \alpha\| \\
&\le \frac{K(\|y_0 - z_0\| + \|y_0 + F(y_0) - \alpha\|)}{1 - \phi_2(e_0)}\|z_0 - \alpha\|.
\end{aligned}$$

As

$$\|y_0 - z_0\| \le \|y_0 - \alpha\| + \|z_0 - \alpha\| \le (h_1(t) + h_2(t))\|x_0 - \alpha\|.$$

$$\begin{aligned}
\|x_1 - \alpha\| &= \frac{K\Big(\,(h_1(e_0) + h_2(e_0))\,e_0 + (1 + L)h_1(e_0)e_0\Big)}{1 - \phi_2(e_0)}h_2(e_0)\|x_0 - \alpha\| \\
&= h_3(e_0)e_0.
\end{aligned} \tag{31}$$

where

$$h_3(t) = \Big(\frac{K\big((2+L)h_1(t)t + h_2(t)t\big)}{1-\phi_2(t)}\Big)h_2(t).$$

Consider $q_3(t) = h_3(t) - 1$. Then, $q_3(0) = -1$ and $q_3(\rho_2) > 0$. Therefore, $q_3(t)$ has at least one zero in $(0, \rho_2)$ and let ρ_3 be such a smallest zero. Therefore, $0 < \rho_3 < \rho_2$, and

$$0 \le h_3(t) \le 1 \ \forall\ t \ \in \ [0, \rho_3). \tag{32}$$

Using (31) and (32), we get

$$\|x_1 - \alpha\| \quad \le h_3(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\|.$$

Thus, theorem holds for $k = 0$. Changing x_0, y_0, z_0 and x_1 by x_k, y_k, z_k, x_{k+1}, we get the inequalities (23)-(25) $\forall k \ge 0$. Since, $\|x_{k+1} - \alpha\| \le \|x_k - \alpha\| < r_3$, this gives $x_{k+1} \in B(\alpha, \rho_3)$. Also $h_3(t)$ is an increasing function in $[0, \rho_3)$, since $h_3^{'}(t) > 0\ \forall t \in [0, \rho_3)$. Thus, we obtain

$$\begin{aligned}\|x_{k+1} - \alpha\| \le h_3(e_0)\|x_k - \alpha\| &\le h_3(e_0)h_3(e_0)\|x_{k-1} - \alpha\|,\\ &\le h_3(e_0)^2 h_3(e_0)\|x_{k-2} - \alpha\| \le \cdots \le h_3(e_0)^{k+1}\|x_0 - \alpha\|.\end{aligned}$$

Therefore, $\lim\limits_{k\to\infty} x_k = \alpha$ as $h_3(t) < 1$.

For uniqueness part, let $P_1 = [\alpha, \beta; F]$ where $F(\beta) = 0$ and $\beta \in \overline{B}(\alpha, R_1)$. Thus, we have

$$\|F^{'}(\alpha)^{-1}(P_1 - F^{'}(\alpha))\| \le K_0\left(\|\alpha - \alpha\| + \|\beta - \alpha\|\right) \le K_0 R_1 < 1,$$

therefore, by Banach Lemma, P_1^{-1} exists. Then,

$$0 = F(\alpha) - F(\beta) = P_1(\alpha - \beta),$$

we obtain $\alpha = \beta$. □

3.1 *Numerical examples*

In this subsection, we consider numerical examples to demonstrate the applicability of our work. Moreover, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively by taking the examples from [12].

Example 4 : Let $X = Y = \mathbb{R}$, $D = (-1, 1)$. Define F on D by

$$F(x) = e^x - 1.$$

Clearly, $\alpha = 0$, $F'(\alpha) = F'(\alpha)^{-1} = 1$, $K_0 = \frac{e-1}{2}$, and $L = e$. Then, we have

$$r_3 = 0.100343 < r_2 = 0.101834 < r_1 = 0.109801.$$

Example 5 : Let $X = Y = \mathbb{R}$, $D = (-1, 1)$. Define F on D by

$$F(x) = x^2 - 1.$$

Clearly, $\alpha = 1$, $K_0 = K = \frac{1}{2}$, and $L = 2$. Then, we have

$$r_3 = 0.265055 < r_2 = 0.267949 < r_1 = 0.285714.$$

Example 6 : Consider the nonlinear Hammerstein type integral equation given by

$$F(x(s)) = x(s) - 5 \int_0^1 s\, t\, x(t)^3 dt, \tag{33}$$

with $x(s)$ in $\mathcal{C}[\prime, \infty]$.

Clearly $\alpha = 0$. Taking $K_0 = 3.75$, $K = 7.5$ and $L = 8.5$ we get

$$r_3 = 0.008636 < r_2 = 0.008708 < r_1 = 0.009039.$$

The value of parameters are listed in Table 1. The radius of a convergence ball of a derivative free fifth order method (3) is compared with the existing methods and depicted in Table 3. We can observe that except in example 5, all other examples larger radius of convergence ball is obtained by our approach. In example 5, we observe that the larger radius of convergence is obtained as compared to the method (3).

Table 3: Comparison of radius of a ball

Examples	Method (19)	Method (2)	Method (3)
4	0.100343	0.02726	0.00892
5	0.265055	0.55264	0.06989
6	0.008636	0.00709	0.00755

4. CONCLUSIONS

In this paper, we discussed the local convergence of two fifth order iterative methods for solving nonlinear equations in Banach spaces. We included the corresponding study when we consider the derivative free method obtained through approximating the derivatives by divided difference, getting a complete analysis of this iterative method. This analysis established under the assumption that the first order Fréchet derivative satisfies the Lipschitz continuity condition for first case and similar condition for the derivative free method involving only the first-order derivative at the exact solution. Finally, for each method, some numerical examples worked out and computed the radii of convergence. We also have compared these results with existing methods and observed that our results are more efficient.

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REFERENCES

1. A. Constantinides and N. Mostoufi, *Numerical Methods for Chemical Engineers with MATLAB Applications*, Prentice Hall PTR, New Jersey, (1999).
2. J. M. Douglas, *Process Dynamics and Control*, Prentice Hall, Englewood Cliffs, (1972).
3. M. Shacham, An improved memory method for the solution of a nonlinear equation, *Chem. Eng. Sci.*, **44** (1989), 1495-1501.
4. J. M. Ortega and W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, Academic Press, New-York, (1970).
5. J. R. Sharma and H. Arora, A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations, *Numer. Algorithms*, **67** (2014), 917-933.
6. I. K. Argyros, A. A. Magreńan, and L. Orcos, Local convergence and a chemical application of derivative free root finding methods with one parameter based on interpolation, *J. Math. Chem.*, **54** (2016), 1404-1416.
7. E. L. Allgower and K. Georg, *Lectures in Applied Mathematics*, American Mathematical Society (Providence, RI) **26**, 723-762.
8. A. V. Rangan, D. Cai, and L. Tao, Numerical methods for solving moment equations in kinetic theory of neuronal network dynamics, *J. Comput. Phys.*, **221** (2007), 781-798.
9. A. Nejat and C. Ollivier-Gooch, Effect of discretization order on preconditioning and convergence of a high-order unstructured Newton-GMRES solver for the Euler equations, *J. Comput. Phys.*, **227** (2008), 2366-2386.
10. C. Grosan and A. Abraham, A new approach for solving nonlinear equations systems, *IEEE Trans. Syst. Man Cybernet Part A: System Humans*, **38** (2008), 698-714.

11. F. Awawdeh, On new iterative method for solving systems of nonlinear equations, *Numer. Algorithms*, **54** (2010), 395-409.

12. I. G. Tsoulos and A. Stavrakoudis, On locating all roots of systems of nonlinear equations inside bounded domain using global optimization methods, *Nonlinear Anal. Real World Appl.,* **11** (2010), 2465-2471.

13. E. Martínez, S. Singh, J. L. Hueso, and D. K. Gupta, Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces, *Appl. Math. Comput.,* **281** (2016), 252-265.

14. S. Singh, D. K. Gupta, E. Martínez, and J. L. Hueso, Semi local and local convergence of a fifth order iteration with Fréchet derivative satisfying Hölder condition, *Appl. Math. Comput.,* **276** (2016), 266-277.

15. I. K. Argyros and S. George, Local convergence of modified Halley-like methods with less computation of inversion, *Novi. Sad.J. Math.*, **45** (2015), 47-58.

16. I. K. Argyros, R. Behl, and S. S. Motsa, Local Convergence of an Efficient High Convergence Order Method Using Hypothesis Only on the First Derivative *Algorithms* 2015, 8, 1076-1087; doi:10.3390/a8041076.

17. A. Cordero, J. L. Hueso, E. Martínez, and J. R. Torregrosa, Increasing the convergence order of an iterative method for nonlinear systems, *Appl. Math. Lett.,* **25** (2012), 2369-2374.

18. I. K. Argyros and A. A. Magreñán, A study on the local convergence and dynamics of Chebyshev-Halley-type methods free from second derivative, *Numer. Algorithms* **71** (2016), 1-23.

19. M. Grau-Sánchez, Á Grau, asnd M. Noguera, Frozen divided difference scheme for solving systems of nonlinear equations, *J. Comput. Appl. Math.,* **235** (2011), 1739-1743.

20. M. Grau-Sánchez, M. Noguera, and S. Amat, On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods, *J. Comput. Appl. Math.,* **237** (2013), 363-372.