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FORMULÆ FOR MULTI-PARAMETER GAUSSIAN q-BINOMIAL SUMS WITH APPLICATIONS

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Some Gaussian q-binomial sum identities from [3] are further generalized, introducing two additional parameters. We prove the claimed results by q-calculus. Finally we present applications to the generalized Fibonomial sums as corollaries.

Key words : Fibonomial coefficient; Fibonacci number; Lucas number; sum.

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1. Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2,\ V_1 = p.$$

For $n \geq k \geq 1$, define the generalized Fibonomial coefficient by

$$\left\{ {n \atop k} \right\}_U := \frac{U_1U_2\dots U_n}{(U_1U_2\dots U_k)(U_1U_2\dots U_{n-k})}$$

with $\left\{ {n \atop 0} \right\}_U = \left\{ {n \atop n} \right\}_U = 1$.

When $p = 1$, we obtain the usual Fibonomial coefficient, denoted by $\left\{{n \atop k}\right\}_F$. The generalized Fibonomial coefficient is a generalization of the usual binomial coefficients. For more details about the usual Fibonomial and generalized Fibonomial coefficients, see [2, 6, 7].

Our approach will be as follows. We will use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n(1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ with $(x;q)_0 = 1$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

As stated, this defines the q-Pochhammer symbol only for nonnegative integers. It is extended as follows: One forms the infinite product

$$(x;q)_\infty := \prod_{j\ge 0}(1-xq^j),$$

and notices that

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty},$$

thanks to cancellations. The right-hand side, however, makes sense for any $n \in \mathbb{C}$; in particular for negative integers we have

$$(x;q)_{-n} = \frac{(x;q)_\infty}{(xq^{-n};q)_\infty} = \frac{(x;q)_\infty}{(xq^{-n};q)_n (x;q)_\infty} = \frac{1}{(xq^{-n};q)_n}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n}(1+xq^k),$$

and *Rothe's* formula [1] is

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k).$$

Quite recently, the authors [5] computed some Gaussian q-binomial sums. In order to prove these sums, their approach is to use q-analysis, in particular a formula of Rothe, and computer algebra. For example, they showed that for any $n \geq 1$,

$$\sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} \left(1 - q^k\right) = (1 - q^n) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q$$

and its Fibonomial corollary:

$$\sum_{k=1}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_U (-1)^{\binom{k}{2}} U_k = U_n \left\{ \begin{matrix} 2n-1 \\ n \end{matrix} \right\}_U.$$

For all n such that $2n - 1 \geq r$, they showed that

$$\sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}\left(k^2 - k(2r+1)\right)} \left(1 + q^k\right)^{2r+1} = -2^{2r} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

and its generalized Fibonomial-Lucas corollary:

$$\sum_{k=1}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_U (-1)^{\frac{k(k+(-1)^r)}{2}} V_k^{2r+1} = -4^r \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}_U.$$

Recently the authors [3, 4] proved various Fibonomial-Fibonacci and Lucas sum identities: they converted them into q-notation and proved them by q-calculus. In particular, in [4], the authors proved eight Gaussian q-binomial sum identities. We recall these results here for the readers convenience: Let n and m be both nonnegative integers, then:

$$\sum_{k=0}^{2n} [1 \pm q^{(2m-1)k}] (-q)^{-(m+n)k + \binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q = P_{n,m} \sum_{k=1}^{m} [1 \pm q^{(4k-2)n}] (-q)^{-(2k-1)(m+n-k)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1} [1 \pm q^{2mk}] (-q)^{-(m+n)k + \binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q = P_{n,m} \sum_{k=0}^{m} [1 \pm q^{2k(2n+1)}] (-q)^{k(2k-2m-2n-1)} \begin{bmatrix} 2m \\ 2k \end{bmatrix}_q,$$

$$\sum_{k=0}^{2n} (-1)^k [1 \pm q^{(2m-1)k}] (-q)^{-(m+n)k + \binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q = P_{n,m} \sum_{k=0}^{m-1} [1 \pm q^{4kn}] (-q)^{k(2k-2m-2n+1)} \begin{bmatrix} 2m-1 \\ 2k \end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}(-1)^k[1\pm q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q$$
$$=-P_{n,m}\sum_{k=1}^{m}[1\pm q^{(2k-1)(2n+1)}](-q)^{(2k-1)(k-m-n-1)}\begin{bmatrix}2m\\2k-1\end{bmatrix}_q,$$

where

$$P_{n,m}=\begin{cases}2(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m} & \text{if } n\geq m,\\ (-q)^{\binom{m-n}{2}}(-q^2;q^2)^{-1}_{m-n-1} & \text{if } n<m.\end{cases}$$

All the above identities hold for general q, and results about generalized Fibonacci and Lucas numbers came out as corollaries for the special choice of q. We recall some of the mentioned Fibonomial-Fibonacci and Lucas sums identities: if n and m are both *nonnegative* integers, then

1.
$$\sum_{k=0}^{2n}\left\{\begin{matrix}2n\\k\end{matrix}\right\}_U U_{(2m-1)k}=T_{n,m}\sum_{k=1}^{m}\left\{\begin{matrix}2m-1\\2k-1\end{matrix}\right\}_U U_{(4k-2)n},$$

2.
$$\sum_{k=0}^{2n+1}\left\{\begin{matrix}2n+1\\k\end{matrix}\right\}_U U_{2mk}=T_{n,m}\sum_{k=0}^{m}\left\{\begin{matrix}2m\\2k\end{matrix}\right\}_U U_{(2n+1)2k},$$

3.
$$\sum_{k=0}^{2n}\left\{\begin{matrix}2n\\k\end{matrix}\right\}_U (-1)^k V_{(2m-1)k}=T_{n,m}\sum_{k=0}^{m-1}\left\{\begin{matrix}2m-1\\2k\end{matrix}\right\}_U V_{4kn},$$

4.
$$\sum_{k=0}^{2n+1}\left\{\begin{matrix}2n+1\\k\end{matrix}\right\}_U (-1)^k V_{2mk}=-T_{n,m}\sum_{k=1}^{m}\left\{\begin{matrix}2m\\2k-1\end{matrix}\right\}_U V_{(2n+1)(2k-1)},$$

where

$$T_{n,m}=\begin{cases}\prod_{k=0}^{n-m}V_{2k} & \text{if } n\geq m,\\ \prod_{k=1}^{m-n-1}V_{2k}^{-1} & \text{if } n<m.\end{cases}$$

In this paper, we consider generalizations of the above eight Gaussian q-binomial sums with two additional integer parameters. Then we prove the claimed results by q-calculus, in particular, by Rothe's formula.

As applications of these sums, corollaries are given, which were the main motivation of this study.

2. The Main Result

As generalizations of the eight sum formulæ mentioned in the introductory section, we present the following results which include two additional integer parameters t and r.

***Theorem* 1** — *(1) For all integers n, m, r and t such that $2n+t \geq 0$ and $2m+t-1 \geq 0$,*

$$\sum_{k=0}^{2n+t} [1 \pm q^{r+(2m+t-1)k}](-q)^{-(m+n)k+\binom{k-t+1}{2}} \begin{bmatrix} 2n+t \\ k \end{bmatrix}_q = P_{n,m} \times \sum_{k=-\lceil t/2 \rceil+1}^{m} [1 \pm q^{r+(2k+t-1)(2n+t)}](-q)^{k(2k-1)-(2k+t-1)(m+n)} \begin{bmatrix} 2m+t-1 \\ 2k+t-1 \end{bmatrix}_q.$$

(2) For all integers n, m, r and t such that $2n+t \geq 0$ and $2m+t-1 \geq 0$,

$$\sum_{k=0}^{2n+t} (-1)^k [1 \pm q^{r+(2m+t-1)k}](-q)^{-(m+n)k+\binom{k-t+1}{2}} \begin{bmatrix} 2n+t \\ k \end{bmatrix}_q = (-1)^t P_{n,m} \times \sum_{k=\lceil t/2 \rceil}^{m+t-1} [1 \pm q^{r+(2k-t)(2n+t)}](-q)^{(k-t)(2k-2t+1)-(2k-t)(m+n)} \begin{bmatrix} 2m+t-1 \\ 2k-t \end{bmatrix}_q$$

where $P_{n,m}$ is defined as before.

Clearly when $r = 0$ and $t \in \{0, 1\}$ in Theorem 1, we obtain the results given in [4].

3. Proofs

We will only prove the first formula of Theorem 1 since all the other verifications are very similar. Before the proof, for later use, we will present a lemma:

Lemma 1 — For all t,

$$(1+\mathbf{i})(-\mathbf{i}^{2n+2t+1}q^{-n}; q)_{2n+1} = 2\mathbf{i}^{n(n+1)} q^{-\binom{n+1}{2}} (-q^2; q^2)_n.$$

Proof : For the sake of brevity, we only consider the case t is odd. The other case is similarly done. Then we must show that

$$(1+\mathbf{i})(\mathbf{i}^{2n+1}q^{-n}; q)_{2n+1} = 2\mathbf{i}^{n(n+1)} q^{-\binom{n+1}{2}} (-q^2; q^2)_n.$$

We consider two subcases: If n is odd, then by the definition of the q-Pochhammer notation the

LHS of the claim is

$$\begin{aligned}(1+\mathbf{i})\prod_{k=0}^{2n}(1-\mathbf{i}^{2n+1}q^{k-n}) &= (1+\mathbf{i})^2\prod_{k=1}^{n}(1+\mathbf{i}q^{-k})\prod_{k=1}^{n}(1+\mathbf{i}q^{k})\\ &= 2\mathbf{i}\cdot\mathbf{i}^n q^{-\binom{n+1}{2}}\prod_{k=1}^{n}(1-\mathbf{i}q^{k})\prod_{k=1}^{n}(1+\mathbf{i}q^{k})\\ &= 2\cdot\mathbf{i}^{n+1}q^{-\binom{n+1}{2}}\prod_{k=1}^{n}(1+q^{2k}),\end{aligned}$$

which is the formula for odd n.

Similarly for even n, we write the LHS of the claim by the definition of q-Pochhammer notation as

$$\begin{aligned}(1+\mathbf{i})\prod_{k=0}^{2n}(1-\mathbf{i}^{2n+1}q^{k-n}) &= (1-\mathbf{i})(1+\mathbf{i})\prod_{k=0}^{n-1}(1-\mathbf{i}q^{k-n})\prod_{k=1}^{n}(1-\mathbf{i}q^{k})\\ &= 2(-\mathbf{i})^n q^{-\binom{n+1}{2}}\prod_{k=1}^{n}(1+\mathbf{i}q^{k})\prod_{k=1}^{n}(1-\mathbf{i}q^{k})\\ &= 2(-\mathbf{i})^n q^{-\binom{n+1}{2}}\prod_{k=1}^{n}(1+q^{2k}),\end{aligned}$$

which is the desired formula. □

PROOF OF THEOREM 1 :

$$\begin{aligned}&\sum_{k=0}^{2n+t}[1\pm q^{r+(2m+t-1)k}](-q)^{-(m+n)k+\binom{k-t+1}{2}}\begin{bmatrix}2n+t\\k\end{bmatrix}_q\\ &=\sum_{k=0}^{2n+t}(-q)^{-(m+n)k+\binom{k-t+1}{2}}\begin{bmatrix}2n+t\\k\end{bmatrix}_q\\ &\pm q^r\sum_{k=0}^{2n+t}q^{(2m+t-1)k}(-q)^{-(m+n)k+\binom{k-t+1}{2}}\begin{bmatrix}2n+t\\k\end{bmatrix}_q\\ &=\sum_{k=0}^{2n+t}(-q)^{\binom{k}{2}+\binom{t}{2}-k(m+n+t-1)}\begin{bmatrix}2n+t\\k\end{bmatrix}_q\\ &\pm q^r\sum_{k=0}^{2n+t}q^{(2m+t-1)k}(-q)^{\binom{k}{2}+\binom{t}{2}-k(m+n+t-1)}\begin{bmatrix}2n+t\\k\end{bmatrix}_q\\ &=(-q)^{\binom{t}{2}}I_1\pm q^r(-q)^{\binom{t}{2}}I_2.\end{aligned}$$

Notice that

$$(-1)^{\binom{k}{2}}=\frac{1+\mathbf{i}}{2}(-\mathbf{i})^k+\frac{1-\mathbf{i}}{2}\mathbf{i}^k.$$

Consequently, I_1 and I_2 can both be evaluated by two applications of Rothe's formula each.

$$\begin{aligned}
I_1 &= \frac{1+\mathbf{i}}{2}\sum_{k=0}^{2n+t}(-1)^{k(m+n+t)}q^{\binom{k}{2}-k(m+n+t-1)}\begin{bmatrix}2n+t\\k\end{bmatrix}_q \mathbf{i}^k\\
&+\frac{1-\mathbf{i}}{2}\sum_{k=0}^{2n+t}(-1)^{k(m+n+t-1)}q^{\binom{k}{2}-k(m+n+t-1)}\begin{bmatrix}2n+t\\k\end{bmatrix}_q \mathbf{i}^k\\
&=\frac{1+\mathbf{i}}{2}(-(-1)^{m+n+t}q^{-m-n-t+1}\mathbf{i};q)_{2n+t}\\
&+\frac{1-\mathbf{i}}{2}(-(-1)^{m+n+t-1}q^{-m-n-t+1}\mathbf{i};q)_{2n+t}.
\end{aligned}$$

The evaluation of I_2 is similar.

The final simplification is done using the above lemma. Several cases must be distinguished, according to the parity of m, n, t. We skip the simple but boring details that are very similar for all the instances that appear in the theorem.

4. Applications

Finally we give some applications of our results by taking $q=(1-\sqrt{5})/(1+\sqrt{5})$:

(1) For all integers n, m, r and t such that $2n+t\geq 0$ and $2m+t-1\geq 0$,

$$\sum_{k=0}^{2n+t}\begin{Bmatrix}2n+t\\k\end{Bmatrix}_U U_{r+(2m+t-1)k}=T_{n,m}\sum_{k=-\lceil t/2\rceil+1}^{m}\begin{Bmatrix}2m+t-1\\2k+t-1\end{Bmatrix}_U U_{r+(2k+t-1)(2n+t)}.$$

(2) For all integers n, m, r and t such that $2n+t\geq 0$ and $2m+t-1\geq 0$

$$\sum_{k=0}^{2n+t}(-1)^k\begin{Bmatrix}2n+t\\k\end{Bmatrix}_U U_{r+(2m+t-1)k}=(-1)^t T_{n,m}\sum_{k=\lceil t/2\rceil}^{m+t-1}\begin{Bmatrix}2m+t-1\\2k-t\end{Bmatrix}_U U_{r+(2k-t)(2n+t)}.$$

(3) For all integers n, m, r and t such that $2n+t\geq 0$ and $2m+t-1\geq 0$

$$\sum_{k=0}^{2n+t}\begin{Bmatrix}2n+t\\k\end{Bmatrix}_U U_{r+(2m+t-1)k}=T_{n,m}\sum_{k=-\lceil t/2\rceil+1}^{m}\begin{Bmatrix}2m+t-1\\2k+t-1\end{Bmatrix}_U U_{r+(2k+t-1)(2n+t)}.$$

(4) For all integers n, m, r and t such that $2n+t\geq 0$ and $2m+t-1\geq 0$

$$\sum_{k=0}^{2n+t}(-1)^k\begin{Bmatrix}2n+t\\k\end{Bmatrix}_U U_{r+(2m+t-1)k}=(-1)^t T_{n,m}\sum_{k=\lceil t/2\rceil}^{m+t-1}\begin{Bmatrix}2m+t-1\\2k-t\end{Bmatrix}_U U_{r+(2k-t)(2n+t)},$$

where $T_{n,m}$ is defined as before.

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