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ASYMPTOTIC NORMALITY OF CONDITIONAL MODE ESTIMATION FOR FUNCTIONAL DEPENDENT DATA

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Based on the local polynomial smoother idea, we construct a local linear estimator of the conditional mode for dependent functional covariables. Precisely, observations are assumed to be a sequence of stationary α-mixing random variables. Then, we establish the asymptotic normality of the constructed estimator.

Key words : Functional data analysis; local linear method; kernel method; conditional mode; asymptotic normality.

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1. Introduction and Motivations

Nonparametric estimation in models containing functional data has been the subject of many studies over the last decade. This area of statistical research, called non-parametric functional data analysis (NFDA), is concerned with non-parametric modeling of data in the form of curves, images or objects. For a general overview on this subject, one can refer to the monographs by Ferraty and Romain [20] and Ferraty and Vieu [19] as well as to the references therein.

It is very common in NFDA that the statistical prediction of a scalar response from a functional explanation variable is performed by estimating the conditional expectation of Y given X. However, the regression function is not efficient enough in some situations. For example, when the conditional density is asymmetric or it is multi-modal. In this situation, the conditional mode would be more efficient than the regression function. Motivated by this importance, the study of the conditional mode has attracted the attention of many researchers (see Ferraty *et al*. [14], Khardani *et al*. [17] and Ould-Saïd and Tatachak [24], and more recently Dabo-Niang *et al*. [7], for more discussion and motivation). Let's point out that in the above works the classical Nadaraya-Watson estimator of the conditional mode has been considered, but it is well known that the local linear estimation method has more advantages than the latter (see Fan and Gijbels [13] in the finite dimensional framework). In fact, this type of estimator makes it possible, among other things, to improve the term of the bias of the Nadaraya-Watson estimator and to avoid its boundary effects. Notice that the local linear estimation of the conditional mode was introduced in NFDA by Demongeot *et al*. [8], who proved the almost-complete convergence of the local linear estimator of the conditional mode for independent and identically distributed (i.i.d.) data. We also return to Laksaci *et al*. [18] for the case of spatial functional data. Notice also that some authors have been interested in the linear local estimation of the regression operator in NFDA (see, for instance, Baìllo and Grané [1] who deals with the case where the regressor is in a Hilbert space, and Boj *et al*. [3] for alternative versions of the linear local estimator).

On the other hand, Barrientos *et al*. [2] studied a so-called fast functional version of the linear local estimator when the regressor belongs to a semi-metric space. In this paper, we will use this last approach to construct a conditional mode estimator. Our main goal would be to study its asymptotic properties when data are dependent. In addition, in order to use the conditional model as a predictor in time series analysis, we will focus on the strongly mixing data framework. This case has attracted the attention of several researchers in the finite dimensional statistics (see, for instance, Collomb *et al*. [5] for the strong convergence of the kernel estimator of the conditional mode, Ould-Saïd [26] for the conditional mode prediction for ergodic processes, as well as Bouzebda *et al*. [4], for the latest advances and references).

In fact, in this article, we prove, under certain standard conditions, the asymptotic normality of the local linear estimator when we approach the problem of the prediction of a functional time series by estimating the conditional mode. For this purpose we will construct, in Section 2, the linear local estimator of the conditional model. In Section 3, we will introduce and discuss necessary conditions for establishing the asymptotic normality of the estimator. Finally, details of technical lemmas of the proof of the main result are given in the appendix (see Section 4).

2. Local Linear Estimator Construction

Let $(X_i, Y_i)_{1\leq i\leq n}$ be a stationary α-mixing process taking values in $\mathfrak{F}\times\mathbb{R}$ where $\mathfrak{F}$ is a semi-metric space equipped with the semi-metric $d(\cdot,\cdot)$. We assume that there exists a regular version of the conditional probability of Y given X and that, for a given $x\in\mathfrak{F}$, there is some compact subset $S=[\theta-\xi,\theta+\xi]$, $\xi>0$, such that the conditional density of Y given $X=x$ has a unique mode $\theta(x)$ in S, which is defined by

$$f^x(\theta(x)) = \sup_{y\in S} f^x(y).$$

A local linear estimator of $\theta(x)$ is defined by

$$\widehat{f}^x(\widehat{\theta}(x)) = \sup_{y\in S} \widehat{f}^x(y), \tag{1}$$

where the conditional density function estimator is "a", the minimizer of the following criterion

$$\min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} \Big(h_H^{-1} H^{(1)}(h_H^{-1}(y-Y_i)) - a - b\beta(X_i,x)\Big)^2 K(h_K^{-1}\delta(x,X_i)), \tag{2}$$

where $\beta(\cdot,\cdot)$ and $\delta(\cdot,\cdot)$ are a known bi-functional operators which are defined from $\mathfrak{F}^2$ into $\mathbb{R}$ such that $|\delta(x,z)|=d(x,z)$ and $\delta=\varphi(\beta)$ where φ is a measurable function, K is a kernel function, $H^{(1)}$ is the first derivative of a given distribution function H and $h_K=h_{K,n}$ (resp. $h_H=h_{H,n}$) is a sequence of positive real numbers.

More precisely, the local linear estimator $\widehat{f}^x(y)$, of $f^x(y)$, is then $\widehat{a}$ which is the first component of the pair (a,b) solution of the minimization problem (2). However, if for all $z\in\mathfrak{F}$, $\beta(z,z)=0$, then $\widehat{f}^x(y)$ is explicitly defined by

$$\widehat{f}^x(y) = \frac{\sum_{i=1}^{n}\sum_{k=1}^{n} W_{ik}(x) H^{(1)}(h_H^{-1}(y-Y_k))}{h_H \sum_{i=1}^{n}\sum_{k=1}^{n} W_{ik}(x)}, \tag{3}$$

where $W_{ik}(x) = \beta(X_i,x)\left(\beta(X_i,x)-\beta(X_k,x)\right) K(h_K^{-1}\delta(x,X_i))K(h_K^{-1}\delta(x,X_k))$, with the convention $0/0=0$ (see Barrientos *et al.* [2], for some details).

By reducing the double sum in (3) into a single one, and set

$$\Delta_k := K_k^{-1} \sum_{i=1}^{n} W_{ik} = \sum_{i=1}^{n}\beta_i^2 K_i - \sum_{i=1}^{n}\beta_i K_i\,\beta_k,$$

with $\beta_i = \beta(X_i, x)$, $K_i = K(h_K^{-1}\delta(x, X_i))$ and $H_k^{(1)} = H^{(1)}(h_H^{-1}(y - Y_k))$, we get the following simple formula of $\widehat{f}^x(y)$

$$\widehat{f}^x(y) = \frac{\sum_{k=1}^{n} \Delta_k K_k H_k^{(1)}}{h_H \sum_{k=1}^{n} \Delta_k K_k}.$$

Furthermore, by defining

$$\widehat{f}_N^x(y) = \frac{1}{n h_H \mathbb{E}[\Delta_1 K_1]} \sum_{k=1}^{n} \Delta_k K_k H_k^{(1)} \text{ and } \widehat{f}_D^x = \frac{1}{n\mathbb{E}[\Delta_1 K_1]} \sum_{k=1}^{n} \Delta_k K_k,$$

we can write

$$\widehat{f}^x(y) = \frac{\widehat{f}_N^x(y)}{\widehat{f}_D^x}.$$

3. Main Result

3.1 *Assumptions and notations*

In what follows, x (resp. y) is a fixed functional element in the space $\mathfrak{F}$ (resp. $\mathbb{R}$), $\mathcal{N}_x$ (resp. $\mathcal{N}_y$) will denote a fixed neighborhood of x (resp. of y) and

$$\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(X, x) \leq r_1).$$

Moreover, we will set, for any $l \in \{0, 2\}$

$$\Lambda_l(s) = \mathbb{E}\left[\lambda_l(X, y) - \lambda_l(x, y) | \beta(x, X) = s\right] \text{ where } \lambda_l(x, y) = \frac{\partial^l f^x(y)}{\partial y^l}.$$

In order to establish our asymptotic results we need the following hypotheses.

(H1)

(i) For any $r > 0$, $\phi_x(r) := \phi_x(-r, r) > 0$ where $\lim_{r \to 0} \phi_x(r) = 0$.

(ii) There exist a function $\Psi_x(\cdot)$ such that

$$\lim_{h_K \to 0} \frac{\phi_x(t h_K, h_K)}{\phi_x(h_K)} = \Psi_x(t) \text{ for all } t \in [-1, 1].$$

(H2) For any $l \in \{0, 2\}$, the quantities $\Lambda_l^{(2)}(0)$ exist.

(H3)

$$\sup_{i \neq j} \mathbb{P}((X_i, X_j) \in B(x, h_K) \times B(x, h_K)) \leq \psi_x(h_K),$$

where $\psi_x(h_K)$ is such that there exists $\epsilon \in \left]0, \frac{a+1}{a-1}\right]$ for which

$$0 < \psi_x(h_K) = O\left(\phi_x^{1+\epsilon}(h_K)\right).$$

(H4) The stochastic process $(X_i, Y_i)_{i \in \mathbb{N}}$ is α-mixing whose coefficient verifies that there exist $a > 1$, $C > 0$ such that for all $n \in \mathbb{N}$

$$\alpha(n) \leq C\, n^{-a}.$$

(H5) There exist some positive constants C_1 and C_2 such that the bandwidths satisfy

$$C_2 n^{1-a} \leq \phi_x(h_K) \leq C_1 n^{\frac{1}{1-a}} \text{ and } \lim_{n \to +\infty} \frac{\log(n)}{n h_H^3 \phi_x(h_K)} = 0,$$

with $a > 1$.

(H6) The bi-functions δ and β satisfy that, there exist some positive constants C_1 and C_2 such that, for all $z \in \mathfrak{F}$

$$C_1|\delta(x,z)| \leq |\beta(x,z)| \leq C_2|\delta(x,z)|.$$

(H7)

(i) The kernel K is a positive and differentiable function, for which the support is within $(-1, 1)$.

(ii) The jth order derivatives $H^{(j)}$ for $j = 1, 2$, are bounded and Lipschitzian functions.

Before announcing the main result, we introduce the following quantities in order to provide bias and variance dominant terms of $\widehat{f}^x(y)$.

$$M_j = K^j(1) - \int_{-1}^{1} (K^j(u))' \Psi_x(u) du \text{ where } j = 1, 2,$$

$$N(a,b) = K^a(1) - \int_{-1}^{1} (u^b K^a(u))' \Psi_x(u) du \text{ for all } a > 0 \text{ and } b = 2, 4.$$

3.2 *Comments on the assumptions*

Observe that assumptions (H1)-(H2) and (H6) are not unduly restrictive, and are common in the setting of functional local linear fitting (see, for instance, Barrientos *et al*. [2] and Demongeot *et* *al*. [11] among others). Concerning the first part of the assumption (H1), the reader will find in the book by Ferraty and Vieu [15] a deeper discussion on the links between this assumption, the semi-metric d and the small ball concentration properties, whereas the second part, of this assumption, will play a key role in our methodology, in particular when we will have to compute the exact constant terms involved in our asymptotic result. In order to quantify the expression of the covariance term we need the assumption (H3). Notice also that the conditions on the smoothing parameters h_K and h_H are standard. On the other hand, the boundedness of the kernel K in the assumption (H7)(i) is also standard; and assumptions (H5)-(H7)(ii) are some technical conditions which make the theorem proof fast and brief.

We are now ready for enouncing the asymptotic normality of the estimator $\widehat{\theta}(x)$. Notice that despite there are already various asymptotic results on the asymptotic behaviour of the classical kernel method (see, for instance, Dobo-Niang and Laksaci [6], Ezzahrioui and Ould-Saïd [11] or Ferraty *et al*. [14]), to the best of our knowledge this constitutes the first result on the asymptotic normality of the local linear estimator of the conditional mode for functional strongly mixing processes.

***Theorem* 1** — *Under assumptions (H1)-(H7), we have*

$$\left(\frac{n\,h_H^3\phi_x(h_K)(f^{x(2)}(\theta(x)))^2}{V_{HK}(x,\theta(x))}\right)^{1/2}\left(\widehat{\theta}(x)-\theta(x)\right)\xrightarrow{D}\mathcal{N}(0,1),$$

where

$$V_{HK}(x,\theta(x))=\frac{M_2}{M_1^2}f^x(\theta(x))\int(H^{(2)}(t))^2dt,$$

and $\xrightarrow{D}$ *denoting the convergence in distribution.*

3.3 *Proof of Theorem 1.*

Based on the Taylor expansion of $\widehat{f}^{x(1)}(.)$ in the neighborhood of $\theta(x)$ and according to the assumptions (H2) and (H3), we have

$$\widehat{\theta}(x)-\theta(x)=-\frac{\widehat{f}^{x(1)}(\theta(x))}{\widehat{f}^{x(2)}(\bar{\theta}(x))}=-\frac{\widehat{f}_N^{x(1)}(\theta(x))}{\widehat{f}_N^{x(2)}(\bar{\theta}(x))}, \tag{4}$$

where $\bar{\theta}(x)$ is between $\widehat{\theta}(x)$ and $\theta(x)$.

By using (4), we obtain the following decomposition

$$\begin{aligned}&\sqrt{nh_H^3\phi_x(h_K)\left(\widehat{f}_N^{x(2)}(\bar{\theta}(x))\right)^2}(\theta(x)-\widehat{\theta}(x))\\ &= \sqrt{nh_H^3\phi_x(h_K)}\left[\widehat{f}_N^{x(1)}(\theta(x))-\mathbb{E}[\widehat{f}_N^{x(1)}(\theta(x))]\right]+\sqrt{nh_H^3\phi_x(h_K)}\mathbb{E}\left[\widehat{f}_N^{x(1)}(\theta(x))\right]\\ &=: J+\acute{J}. \end{aligned}\tag{5}$$

Then, the rest of the proof of this theorem is based on the following lemmas for which proofs are given in the appendix.

Lemma 2 — (See Rachdi *et al*. [27]). Under the assumptions (H2), (H3) and (H5)(ii)-(H7)(ii), we have

$$\acute{J} \to 0 \text{ as } n \to +\infty.$$

Lemma 3 — Under the assumptions of Theorem 1, we have

$$J \xrightarrow{D} \mathcal{N}(0, V_{HK}(x, \theta(x)).$$

Lemma 4 — Under assumptions (H5), (H7)(ii) and (H3)(i), we have

$$\widehat{f}_N^{x(2)}(\theta(x)) \xrightarrow{P} f^{x(2)}(\theta(x)),$$

where $\xrightarrow{P}$ denotes the convergence in probability.

4. Appendix

PROOF OF LEMMA 3 : Denote

$$\sqrt{nh_H^3\phi_x(h_K)}\left[\widehat{f}_N^{x(1)}(\theta(x)) - \mathbb{E}[\widehat{f}_N^{x(1)}(\theta(x))]\right] = \frac{(nh_H^3\phi_x(h_K))^{\frac{1}{2}}}{nh_H^2\mathbb{E}(\Delta_1 K_1)}\sum_{i=1}^{n} L_i(x,\theta(x)) = S_n,$$

where

$$L_i(x,\theta(x)) = \Delta_i K_i H_i^{(2)} - \mathbb{E}[\Delta_i K_i H_i^{(2)}].$$

Then, we consider the following decomposition

$$S_n = T_{1,k}T_{1,i} - \mathbb{E}[T_{1,k}T_{1,i}] - \left(T_{2,k}T_{2,i} - \mathbb{E}[T_{2,k}T_{2,i}]\right),$$

where

$$T_{1,k} = \frac{1}{n\mathbb{E}[\beta_1^2 K_1]}\sum_{k=1}^{n}\beta_k^2 K_k \quad , \quad T_{1,i} = \frac{\sqrt{nh_H^3\phi_x(h_K)}\mathbb{E}[\beta_1^2 K_1]}{h_H^2\mathbb{E}[\Delta_1 K_1]}\sum_{i=1}^{n} K_i H_i^{(2)},$$

$$T_{2,k} = \frac{1}{n\mathbb{E}[\beta_1 K_1]}\sum_{k=1}^{n}\beta_k K_k \quad \text{and} \quad T_{2,i} = \frac{\sqrt{nh_H^3\phi_x(h_K)}\mathbb{E}[\beta_1 K_1]}{h_H^2\mathbb{E}[\Delta_1 K_1]}\sum_{i=1}^{n}\beta_i K_i H_i^{(2)}.$$

Obviously, the result in Lemma 3 can be deduced directly from the following statements.

Claim 1 :

$$T_{1,k}T_{1,i} - \mathbb{E}[T_{1,k}T_{1,i}] \xrightarrow{D} \mathcal{N}(0, V_{HK}(x,\theta(x))).$$

Claim 2 :

$$T_{2,k}T_{2,i} - \mathbb{E}[T_{2,k}T_{2,i}] \xrightarrow{P} 0.$$

The rest of this proof will be devoted to the demonstration of these two claims.

PROOF OF CLAIM 1 : We can write

$$\begin{aligned} & T_{1,k}T_{1,i} - \mathbb{E}[T_{1,k}T_{1,i}] \\ & \qquad = T_{1,i} - \mathbb{E}[T_{1,i}] + ((T_{1,k}-1)T_{1,i} - \mathbb{E}[(T_{1,k}-1)T_{1,i}]). \end{aligned}$$

So, it suffices to prove the following two results

$$((T_{1,k}-1)T_{1,i} - \mathbb{E}[(T_{1,k}-1)T_{1,i}]) \xrightarrow{P} 0. \tag{6}$$

$$T_{1,i} - \mathbb{E}[T_{1,i}] \xrightarrow{D} \mathcal{N}(0, V_{HK}(x,\theta(x))). \tag{7}$$

PROOF OF (6) : Firstly, by using the Cauchy-Schwartz inequality we obtain that

$$\begin{aligned} \mathbb{E}\,|(T_{1,k}-1)T_{1,i} - \mathbb{E}[(T_{1,k}-1)T_{1,i}]| &\leqslant 2\mathbb{E}\mid (T_{1,k}-1)T_{1,i} \mid \\ &\leq \sqrt{E[(T_{1,k}-1)^2]}\sqrt{\mathbb{E}[T_{1,i}^2]}. \end{aligned}$$

Then, all it remains to prove is

$$\mathbb{E}[(T_{1,k}-1)^2] \to 0, \text{ as } n \to +\infty. \tag{8}$$

$$\mathbb{E}[T_{1,i}^2] \to V_{HK}(x,\theta(x)), \text{ as } n \to +\infty. \tag{9}$$

PROOF OF (8) : Firstly, we can write

$$\mathbb{E}[(T_{1,k}-1)^2] = n\,\mathit{var}[T_{1,1}] + 2\,\mathit{cov}(T_{1,k}, T_{1,l}). \tag{10}$$

For the first term on the right hand side of (10), we have

$$\begin{aligned} n\,\mathit{var}[T_{1,1}] &= \frac{1}{n^2\mathbb{E}^2[\beta_1^2 K_1]}\, n\,\mathit{var}[\beta_1^2 K_1] \\ &= O\left(\frac{1}{n\phi_x(h_K)}\right). \end{aligned}$$

Next, before computing the second term of (10) we treat the general case of $cov(T_{j,k}, T_{j,l})$ where $j \in \{1, 2\}$. So, by denoting for $c = 2$, $j = 1$ and for $c = 1$, $j = 2$, we have

$$\frac{1}{n\mathbb{E}[\beta_1^c K_1]} \sum_{k=1}^{n} \beta_k^c K_k = T_{j,k}.$$

Then

$$cov(T_{j,k}, T_{j,l}) \quad = \quad \frac{1}{n^2\mathbb{E}^2[\beta_1^c K_1]} \sum_{k \neq l} cov(\beta_k^c K_k, \beta_l^c K_l). \tag{11}$$

In order to adopt the same technique as in Masry [22], we define the sets E_1 and E_2 as follows

$$E_1 = \{(k,l) \in \{1, 2, \ldots, n\} \text{ such that } 1 \leq \mid k - l \mid \leq m_n\},$$

and

$$E_2 = \{(k,l) \in \{1, 2, \ldots, n\} \text{ such that } m_n + 1 \leq \mid k - l \mid \leq n - 1\},$$

where the sequence m_n is chosen such that $m_n \to +\infty$ as $n \to +\infty$.

Let $A_{1,n}$ and $A_{2,n}$ be the sum of covariances over the sets E_1 and E_2 respectively

$$A_{1,n} = \frac{1}{n^2\mathbb{E}^2[\beta_1^c K_1]} \sum_{E_1} cov(\beta_k^c K_k, \beta_l^c K_l),$$

and

$$A_{2,n} = \frac{1}{n^2\mathbb{E}^2[\beta_1^c K_1]} \sum_{E_2} cov(\beta_k^c K_k, \beta_l^c K_l).$$

For the sum $A_{1,n}$, from the stationarity property, we have

$$|cov(\beta_k^c K_k, \beta_l^c K_l)| \leq |\mathbb{E}\left[\beta_k^c \, \beta_l^c \, K_l \, K_k\right]| + \mathbb{E}^2[\beta_1^c K_1].$$

Moreover, assumption (H6) implies that

$$\text{for all } (k,c) \in (\mathbb{N}^*, \mathbb{N}) : \; K_i^k \, |\beta_i|^c \, h_K^{-c} \leq C K_1^k |\delta(x, X_i)|^c h_K^{-c},$$

and from the Jensen's inequality and under assumption (H3), it follows that

$$\begin{aligned} |\mathbb{E}[\beta_k^c.\beta_l^c \, K_l \, K_k]| \quad &\leq \quad \mathbb{E}[|\beta_k^c \, \beta_c^c \, K_l \, K_k|], \\ &\leq \quad C h_K^{2c} \psi_x(h_K). \end{aligned}$$

Then

$$|cov(\beta_k^c K_k, \beta_l^c K_l)| \leq C\, h_K^{2c}\, \psi_x(h_K) + \mathbb{E}^2[\beta_1^c K_1].$$

This last result together with the technical lemma A.1 in [2] leads directly to

$$\begin{aligned} |A_{1,n}| &\leq C\, n\, m_n \left(h_K^{2c}\psi_x(h_K) + \mathbb{E}^2[\beta_1^c K_1]\right) \frac{1}{n^2 \mathbb{E}^2[\beta_1^c K_1]} \\ &\leq \frac{C m_n}{n\phi_x(h_K)^{1-\epsilon}}. \end{aligned}$$

For the sum $A_{2,n}$, we use the inequality for the bounded mixing processes (see Proposition A.10(i) in [15]). This leads, for all $l \neq k$, to

$$|cov(\beta_k^c K_k, \beta_l^c K_l)| \leq C h_K^{2c}\, \alpha(|k-l|).$$

On the other hand, by using the inequality $\sum_{j \geq x+1} j^{-a} \leq \int_{u \geq x} u^{-a}$, we get

$$\sum_{k=1}^{n} \sum_{E_2} \alpha(|k-l|) \leq \frac{C\, n(m_n)^{1-a}}{a-1}. \tag{12}$$

Finally, we arrive at

$$|A_{2,n}| \leq \frac{C\,(m_n)^{1-a}}{n\,\phi_x^2(h_K)}.$$

Then, by choosing $m_n = (\phi_x^{1+\epsilon}(h_K))^{-1/a}$, we get

$$A_{1,n} \to 0 \text{ and } A_{2,n} \to 0 \text{ as } n \to \infty.$$

Consequently, in the case where $c = 2$ and $j = 1$, we have

$$\lim_{n \to +\infty} cov(T_{1,k}, T_{1,l}) = 0.$$

PROOF OF (9) : We start by writing

$$\mathbb{E}[T_{1,i}^2] = var[T_{1,i}] + \mathbb{E}^2[T_{1,i}]. \tag{13}$$

The first term on the right hand side of (13) tends to $V_{HK}(x,\theta(x))$ as $n\to\infty$ and details will be given in the proof of result (7). For the second term on the right hand side of (13), we use similar ideas as those used by Ezzahrioui and Ould-Saïd [12] (see Lemma 2) to deduce that

$$\sqrt{nh_H^3\phi_x(h_K)}\mathbb{E}\left[\frac{K_1H_1^{(2)}}{h_H^2\phi_x(h_K)}\right]\to 0 \text{ as } n\to+\infty. \tag{14}$$

Thus, it remains to check that

$$\lim_{n\to+\infty}\mathbb{E}[T_{1,i}^2]=V_{HK}(x,\theta(x)).$$

PROOF OF (7) : First, we calculate $\mathit{var}(T_{1,i})$

$$\mathit{var}[T_{1,i}]=n\mathit{var}[T_{1,1}]+2\frac{n\phi_x(h_K)\mathbb{E}^2[\beta_1^2K_1]}{h_H\mathbb{E}^2[\Delta_1K_1]}\sum_{i\neq j}\mathit{cov}(K_iH_i^{(2)},K_jH_j^{(2)}).$$

So, to prove $\lim_{n\to+\infty}\mathit{var}[T_{1,i}]=V_{HK}(x,\theta(x))$, it is necessary to establish the following results

$$\lim_{n\to+\infty}\mathit{var}[T_{1,i}]=\frac{M_2}{M_1^2}f^x(\theta(x))\int(H^{(2)}(t))^2dt, \tag{15}$$

$$\lim_{n\to+\infty}\frac{n\phi_x(h_K)\mathbb{E}^2[\beta_1^2K_1]}{h_H\mathbb{E}^2[\Delta_1K_1]}\sum_{i\neq j}\mathit{cov}(K_iH_i^{(2)},K_jH_j^{(2)})=0. \tag{16}$$

Notice that the proof of (16) is very close to the proof of (8). So, we focus only on (15). To do that we use the technical Lemma A.1 in Zhou and Lin [28], to have

$$\begin{aligned}\mathit{var}[T_{1,i}] &= \frac{n^2\phi_x(h_K)\mathbb{E}^2[\beta_1^2K_1]}{h_H\mathbb{E}^2[\Delta_1K_1]}\mathbb{E}\left[\left(K_1H_1^{(2)}\right)^2\right]\\ &\quad-\frac{\mathbb{E}^2[\beta_1^2K_1]\,n\,\phi_x^2(h_K)}{\mathbb{E}^2[\Delta_1K_1]}\mathbb{E}^2\left[\sqrt{nh_H^3\phi_x(h_K)}\frac{K_1H_1^{(2)}}{h_H^2\phi_x(h_K)}\right].\end{aligned} \tag{17}$$

Then, it follows from (14), that the second term on the right hand side of (17) tends to 0. Hence

$$\mathit{var}[T_{1,i}]=\frac{n^2\phi_x(h_K)\mathbb{E}^2[\beta_1^2K_1]}{h_H\mathbb{E}^2[\Delta_1K_1]}\mathbb{E}\left[K_1^2\mathbb{E}[(H_1^{(2)})^2\mid X]\right].$$

On one hand we have

$$\begin{aligned}&\frac{1}{h_H}\mathbb{E}\left[\left(H^{(2)}\left(\frac{y-Y_1}{h_H}\right)\right)^2\mid X_1\right]\\ &\qquad=\int_{\mathbb{R}}(H^{(2)}(t))^2[f^{(X_1)}(y-th_H)-f^x(y)]dt+\int_{\mathbb{R}}(H^{(2)}(t))^2f^x(y)dt,\end{aligned}$$

and by the continuity of f^x we deduce that

$$\mathbb{E}\left[K_1^2\left(H^{(2)}\left(\frac{y-Y_1}{h_H}\right)\right)^2 \mid X_1\right] \to \mathbb{E}[K_1^2]f^x(y)\int (H^{(2)}(t))^2dt \text{ as } n\to\infty.$$

On the other hand, by using again the technical Lemma A.1 in Zhou and Lin [28] we obtain

$$\frac{n^2\phi_x(h_K)\mathbb{E}[\beta_1^2K_1]}{\mathbb{E}^2[\Delta_1K_1]} \quad \to \frac{1}{\phi_x(h_K)M_1^2} \text{ as } n\to+\infty.$$

Finally, we arrive at

$$\lim_{n\to+\infty} \mathit{var}[T_{1,i}] = \frac{M_2}{M_1^2}f^x(\theta(x))\int (H^{(2)}(t))^2dt.$$

Now to establish the asymptotic normality of the conditional mode estimator dealing with dependent random variables, we start by writing

$$\begin{aligned} T_{1,i}-\mathbb{E}(T_{1,i}) &= \frac{\sqrt{nh_H^3\phi_x(h_K)}\mathbb{E}[\beta_1^2K_1]}{h_H^2\mathbb{E}[\Delta_1K_1]}\sum_{i=1}^n\left(K_iH_i^{(2)}-\mathbb{E}[K_iH_i^{(2)}]\right) \\ &= \sum_{i=1}^n \tilde{L}_i(x,\theta(x)), \end{aligned}$$

where

$$\tilde{L}_i(x,\theta(x)) = \left(\frac{n}{(n-1)^2h_H\phi_x(h_K)}\right)^{\frac{1}{2}}\frac{1}{M_1}\sum_{i=1}^n\left(K_iH_i^{(2)}-\mathbb{E}[K_iH_i^{(2)}]\right).$$

We base ourselves on the CLT by Liebscher [20] (see Corollary 2.2, Page 196), which rests on the asymptotic behavior of the quantity

$$\lim_{n\to\infty}\sum_{i=1}^n\mathbb{E}[\tilde{L}_i^2(x,\theta(x))], \tag{18}$$

in addition to the assumptions

$$\left\{\begin{array}{c} \text{There exists a sequence } \tau_n = o(\sqrt{n}) \text{ such that} \\ \tau_n \le \left(\max_{i=1,\dots,n} C_i\right)^{-1} \text{ where } C_i = ess\sup_{\omega\in\Omega}|\mathbb{E}\tilde{L}_i(x,\theta(x))| \\ \text{and } \dfrac{n}{\tau_n}\,\alpha(\epsilon\tau_n)\to 0 \text{ for all } \epsilon>0, \end{array}\right. \tag{19}$$

and

$$\begin{cases} \text{There exists a sequence } (m_n) \text{ of positive integers tending to} \infty \text{ such that} \\ nm_n\gamma_n = o(1) \text{ where } \gamma_n := \max_{1\leq i\neq j\leq n} \left(\mathbb{E}[|\mathbb{E}\tilde{L}_i(x,\theta(x))\mathbb{E}\tilde{L}_j(x,\theta(x))|]\right) \\ \text{and } \left(\sum_{j=m_n+1}^{\infty} \alpha(j)\right) \sum_{i=1}^{n} C_i = o(1). \end{cases} \tag{20}$$

It follows that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[\tilde{L}_i^2(x,\theta(x))] = V_{HK}(x,\theta(x)). \tag{21}$$

Concerning (19), the boundedness of H and K allows to obtain have $C_i = O\left(\frac{1}{\sqrt{nh_H\phi_x(h_K)}}\right)$.

Therefore, we can take $\tau_n = \sqrt{\frac{nh_H\phi_x(h_K)}{\log n}}$.

Furthermore, this choice gives, for all $\epsilon > 0$

$$\begin{aligned} \frac{n}{\tau_n}\,\alpha(\epsilon\tau_n) &\leq C\left(n^{1-(a+1)/2}(h_H\phi_x(h_K))^{-(a+1)/2}(\log n)^{(a+1)/2}\right) \\ &\leq Cn^{1-(a+1)/2+(a+1)/2(a-1)}(\log n)^{(a+1)/2} \\ &\leq Cn^{(3a-a^2)/2(a-1)}(\log n)^{(a+1)/2} \to 0 \text{ since } a > 3. \end{aligned}$$

Let us derive (20). On one hand, by using assumption (H5) and since $a > 1$, we obtain

$$\gamma_n = \max_{1\leq i\neq j\leq n} \left(\mathbb{E}[|\mathbb{E}\tilde{L}_i(x,\theta(x))\mathbb{E}\tilde{L}_j(x,\theta(x))|]\right) = O\left(\frac{h_H\phi_x(h_K)}{n}\right).$$

Next, by using the fact that

$$\sum_{j\geq x+1} j^{-a} \leq \int_{u\geq x} u^{-a} = \left[(a-1)x^{a-1}\right]^{-1} \tag{22}$$

we get

$$\sum_{j=m_n+1}^{\infty} \alpha(j) \leq \sum_{j=m_n}^{\infty} \alpha(j) \leq \int_{t\geq m_n} t^{-a}dt = \frac{m_n^{1-a}}{a-1},$$

thus,

$$\left(\sum_{j=m_n+1}^{\infty} \alpha(j)\right) \sum_{i=1}^{n} C_i = O\left(\frac{m_n^{1-a}}{a-1}\sqrt{\frac{n}{h_H\phi_x(h_K)}}\right).$$

We choose $m_n = \left\lfloor \left(\frac{h_H\phi_x(h_K)}{\log n}\right)^{1/(2(1-a))} \right\rfloor$, where $\lfloor . \rfloor$ denotes the function integer part. It is clear that under assumption (H5), $m_n \to \infty$. In addition, if we replace m_n by its expression, we obtain

$$\sum_{j=m_n+1}^{\infty} \alpha(j) \sum_{i=1}^{n} C_i = O(\log n)^{-1/2} = o(1),$$

and again, under assumption (H5), we have

$$\begin{aligned} m_n\gamma_n &\leq Cn^{-1-1/(2(1-a)}(h_H\phi_x(h_K))^{1+1/(2(1-a)}(\log n)^{-1/(2(1-a))} \\ &\leq n^{(-3+2a)/(2(1-a)}(h_H\phi_x(h_K))^{(3-2a)/(2(1-a)}(\log n)^{-1/(2(1-a))} \\ &\leq n^{-1-\eta(3-2a)/(2(1-a)}(\log n)^{-1/(2(1-a))} = o\left(n^{-1}\right). \end{aligned}$$

Finally, Claim 1 can be easily deduced from (18)-(20) and Corollary 2.2 by Liebscher [24].

PROOF OF CLAIM 2 : Following the same approach as the one used to show Claim 1, we show that

$$\mathbb{E}\left|T_{2,i} - \mathbb{E}[T_{2,i}]\right| \xrightarrow{L^1} 0 \text{ as } n \to +\infty, \tag{23}$$

and

$$\mathbb{E}\left|(T_{2,k}-1)T_{2,i} - \mathbb{E}[(T_{2,k}-1)T_{2,i}]\right| \to 0 \text{ as } n \to +\infty. \tag{24}$$

PROOF OF (23) : In order to show (23), it suffices to prove the L^2-consistency of $T_{2,i}$. For this, we have

$$\begin{aligned} &\mathbb{E}\left[(T_{2,i} - \mathbb{E}[T_{2,i}])^2\right] \\ &\quad = n\mathit{var}[T_{2,1}] + 2\frac{n\phi_x(h_K)\mathbb{E}^2[\beta_1^2K_1]}{h_H\mathbb{E}^2[\Delta_1K_1]}\sum_{i\neq j}\mathit{cov}(\beta_iK_iH_i^{(2)}, \beta_jK_jH_j^{(2)}). \end{aligned} \tag{25}$$

On one hand, we have

$$n\mathit{var}[T_{2,1}] = \frac{n^2\phi_x(h_K)\mathbb{E}^2[\beta_1K_1]}{h_H\mathbb{E}^2[\Delta_1K_1]}\left(\mathbb{E}[\beta_1^2K_1^2(H_1^{(2)})^2] - \mathbb{E}^2[\beta_1K_1H_1^{(2)}]\right).$$

Also, by using Lemma A.1 by Zhou and Lin [28] we prove that

$$\lim_{n\to+\infty} n\mathit{var}[T_{2,1}] = o(1).$$

On the other hand, using exactly the same arguments as for showing (16), we show that the second term on the right hand side of (25) tends to 0 as $n \to \infty$. With this the result (23) is thus proved.

PROOF OF (24) : Following the same steps as those used to get (11) and (6), we get

$$\mathbb{E}[(T_{2,k}-1)^2] = \frac{1}{n\mathbb{E}^2[\beta_1 K_1]} var[\beta_1 K_1] + 2cov(T_{2,k}, T_{2,l}).$$

The result (24) may then be directly obtained since, firstly

$$\frac{1}{n\mathbb{E}^2[\beta_1 K_1]} var[\beta_1 K_1] \leqslant \frac{1}{n\mathbb{E}^2[\beta_1 K_1]} \mathbb{E}[\beta_1^2 K_1^2] = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

and secondly, by (11) and for $c = 1$, $j = 2$, we get

$$cov(T_{2,k}, T_{2,l}) \to 0 \text{ as } n \to +\infty. \Box$$

PROOF OF LEMMA 4 : Observe that for n large enough

$$\left| \widehat{f}_N^{(2)(x)}(\bar{\theta}(x)) - \widehat{f}_N^{(2)(x)}(\theta(x)) \right| \leq 2 \sup_{y \in S} \left| \widehat{f}_N^{(2)(x)}(y) - f^{(2)(x)}(y) \right|.$$

By following the same ideas as those used by Laksaci *et al.* [18] in Lemma 7, we get

$$\sup_{y \in S} \left| \widehat{f}_N^{(2)(x)}(y) - f^{(2)(x)}(y) \right| \xrightarrow{P} 0 \text{ as } n \to +\infty. \tag{26}$$

So, the proof of this lemma is a consequence of (26). □

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