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NONLINEAR FRACTAL INTERPOLATION CURVES WITH FUNCTION VERTICAL SCALING FACTORS

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In this paper we present a method to generate new fractal interpolation curves. We ensure that attractors of nonlinear iterated function systems (IFSs) constructed by Geraghty contractions are graphs of some continuous functions which interpolate the given data. In particular, we give an explicit illustrative example to demonstrate the effectiveness of obtained results. Concerning IFSs, our methods and results extend known results from the literature.

Key words : Iterated function system; fractal interpolation function; fractal interpolation curve; Geraghty contraction; Rakotch contraction.

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1. INTRODUCTION

Fractal interpolation curves (see [1, 2, 9, 11]) have become a powerful tool for modeling many natural objects and have wide applications in mathematics and several other areas of applied sciences. How to construct fractal curves (rough curves) and analyse their complexity has become one of the most important topics in fractals (see [9]). The graph of a fractal interpolation function is an attractor of some iterated function system (see [1]). The concept of iterated function systems was introduced as a natural generalization of the well-known Banach contraction principle (see [2]). Iterated function systems have become powerful tools for construction and analysis of new fractal interpolation functions. In particular, the connectivity of attractors of iterated function systems is very important in the construction of fractal interpolation curves. The graphs of linear one variable fractal interpolation functions are always continuous curves.

In usual approaches, the existence of linear fractal interpolation functions follows from Banach's fixed point theorem (see [1, 2]). In order to construct new iterated function systems and fractal interpolation functions, one can use the well-known fixed point results obtained in the fixed point theory (see [5, 8-10]). As far as we know, the first significant generalization of Banach's principle was obtained by Rakotch in 1962. In [9], one presented a method to generate nonlinear fractal interpolation functions by using the Rakotch's fixed point theorem [6] instead of the Banach's fixed point theorem. The usual methods to obtain fractal interpolation curves are effective in the case of linear fractal interpolation curves (see [1, 2, 11]), however, they may not be applicable in the case of nonlinear fractal interpolation curves (cf. [9]). In 1973, Geraghty introduced a natural generalization of the Rakotch's fixed point theorem (see [3], p.607, Corollary 3.1, Corollary 3.2 and Corollary 3.3).

In this paper, in order to obtain new nonlinear fractal interpolation functions, we use special function vertical scaling factors, and we apply Geraghty's fixed point theorem [3] instead of Banach's fixed point theorem (or Rakotch's fixed point theorem). Our main results give a generalization of the corresponding results in [1, 2, 11], in particular [9]. And we give an explicit illustrative example to demonstrate the effectiveness of obtained results.

This paper is organized as follows. In Section 2 we recall some results needed in constructing new fractal interpolation functions. In Section 3 we propose a method to construct new type of nonlinear fractal interpolation curves by using Geraghty's fixed point theorem and function vertical scaling factors. Finally, in Section 4 we draw our conclusions.

2. Preparatory Facts

We describe some basic notions and theorems on fixed point theory. The following results will be the key in the proof of our main results.

Definition 2.1 — (see [10], p.100, cf. [3], p.607, Corollary 3.1, Corollary 3.2, Corollary 3.3).

(1) If for some function $\varphi : (0, +\infty) \to (0, +\infty)$ and a self-map f of a metric space (X, d), we have

$$\forall_{x,y \in X} \;\; d(f(x), f(y)) \leq \varphi(d(x, y)),$$

then we say that f is a φ-contraction.

(2) If f is a φ-contraction for some function $\varphi : (0, +\infty) \to (0, +\infty)$ such that for any $t > 0$, $\alpha(t) := \frac{\varphi(t)}{t} < 1$ and the function $(0, +\infty) \ni t \to \frac{\varphi(t)}{t}$ is non-increasing(so-called Geraghty I) (or non-decreasing(Geraghty II), or continuous(Geraghty III)), then we call such a function a Geraghty contraction.

Remark 2.2 : (see [6], see [3]). (1) Each Rakotch contraction is a Geraghty contraction, since a map $f : X \to X$ is a Rakotch contraction iff it is a φ-contraction for some function $\varphi : (0, +\infty) \to (0, +\infty)$ such that for any $t > 0$, $\alpha(t) := \frac{\varphi(t)}{t} < 1$ and the function $(0, +\infty) \ni t \to \frac{\varphi(t)}{t}$ is non-increasing.

(2) Each Banach contraction is a Rakotch contraction, since a map $f : X \to X$ is a Banach contraction iff it is a φ-contraction for a function $\varphi(t) = \alpha t$, for some $0 \leq \alpha < 1$.

***Theorem* 2.3** — *(see [3], p.607, Corollary 3.1, Corollary 3.2 and Corollary 3.3, cf. [10], cf. [5])*

(1) Let X be a complete metric space and $f : X \to X$ be a Geraghty contraction. Then there is a unique fixed point $k \in X$ of f, and for each $x \in X$,

$$\lim_{n \to +\infty} f^n(x) = k.$$

(2) Let X be a complete metric space and $\{X; f_1, \cdots, f_N\}$ be an iterated function system consisting of Geraghty contractions. Then there is a unique non-empty compact set $K \subset X$ such that

$$K = \bigcup_{i=1}^{N} f_i(K).$$

Now we describe some basic results on fractal interpolation theory.

Let N be a positive integer greater than one and $I := [x_0, x_N] \subset \mathbb{R}$. Let a set of data points $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, 2, \cdots, N\}$ be given, where $\{x_0, x_1, \cdots, x_N\}$ is a partition of I (i.e., $x_0 < x_1 < x_2 < \cdots < x_N$) and $y_0, y_1, \cdots, y_N$ are given real numbers.

Set $I_i := [x_{i-1}, x_i] \subset I$ and let $l_i : I \to I_i$ for $i = 1, 2, \cdots, N$ be contractive homeomorphisms such that

$$l_i(x_0) = x_{i-1}, l_i(x_N) = x_i,$$
$$|l_i(x') - l_i(x'')| \leq \lambda |x' - x''| \ \ whenever \ \ x', x'' \in I$$

for some $0 \leq \lambda < 1$. Let $K := I \times [a, b]$ for some $-\infty < a < b < +\infty$. Furthermore, let mappings $F_i : K \to [a, b]$ be continuous with, for some $k \geq 0$ and $0 \leq \alpha < 1$,

$$F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i.$$
$$|F_i(x', y') - F_i(x'', y'')| \leq k|x' - x''| + \alpha |y' - y''|$$

for all $x', x'' \in I$, $y', y'' \in [a, b]$, and $i = 1, 2, \cdots, N$.

Now define functions $w_i : K \to K$ for $i = 1, 2, \cdots, N$ by

$$w_i := (l_i(x), F_i(x, y)).$$

Barnsley presented the following famous result.

***Theorem* 2.4** — *(cf. [1], p.306, Theorem 1, cf. [2], p.217, Theorem 1, cf. [2], p.218, Theorem 2) The $IFS\{K, w_i : i = 1, 2, \cdots, N\}$ defined above has a unique nonempty compact set $G \subset \mathbb{R}^2$ such that*

$$G = \bigcup_{i=1}^{N} w_i(G).$$

Then G is the graph of a continuous function $f : I \to [a, b]$ which obeys

$$f(x_i) = y_i \quad for \quad i = 0, 1, \cdots, N.$$

The function $f(x)$ whose graph is the attractor of an IFS is called a fractal interpolation function corresponding to the data $\{(x_i, y_i) : i = 0, 1, \cdots, N\}$ (cf. [1], p.306, cf. [2], p.220).

Remark 2.5 : In accordance with the idea of Barnsley, researchers proposed many types of FIFs. In [1], [2], [9] and [11],

$$l_i(x) := \frac{x_i - x_{i-1}}{x_N - x_0} x + \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}.$$

(1) In [1], [2] and [11], the maps $w_i(x, y)$ are chosen so that functions $F_i(x, y)$ are Banach contraction with respect to the second variable. In the affine fractal interpolation function (cf. [1], p.308, cf. [2], p.214),

$$F_i(x, y) := c_i x + d_i y + f_i,$$

where $|d_i| < 1$, and in the fractal interpolation function with variable parameters (cf. [11], p.3-4),

$$F_i(x, y) := d_i(x) y + q_i(x),$$

where $\sup_{x \in I} |d_i(x)| < 1$.

(2) In [9], one type of fractal interpolation functions is considered, where the maps $w_i(x, y)$ are chosen so that functions $F_i(x, y)$ are Rakotch contraction with respect to the second variable. In the nonlinear fractal interpolation function (see [9]),

$$F_i(x, y) := c_i x + s_i(y) + f_i,$$

where s_i is some Rakotch contraction.

3. MAIN RESULTS

In this section, we give new type of nonlinear fractal interpolation curves.

Barnsley's functional condition for existence of a fractal interpolation function can be replaced by another functional conditions (see [9]). In order to obtain new nonlinear fractal interpolation curves, we use special function vertical scaling factors (cf. [11]) and Geraghty's fixed point theorem in the construction of curves. In particular, we improve upon a result proved by [9], and we give an explicit illustrative example to demonstrate the effectiveness of obtained results.

Let N be a positive integer greater than one and $I := [x_0, x_N] \subset \mathbb{R}$.

We will work in the compact metric space $K := I \times [a, b]$ for some $-\infty < a < b < +\infty$, with the Euclidean metric d_0.

Let a set of data points $\{(x_i, y_i) \in K : i = 0, 1, 2, \cdots, N\}$ be given, where $x_0 < x_1 < x_2 < \cdots < x_N$ and $y_0, y_1, y_2, \cdots, y_N \in [a, b]$.

Set $I_i := [x_{i-1}, x_i] \subset I$ and define contractive homeomorphisms $l_i : I \to I_i$ by

$$l_i(x) := a_i x + e_i,$$

where for all $i = 1, 2, \cdots, N$, the real numbers a_i, e_i are chosen to ensure that $l_i(I) = I_i$.

Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a non-decreasing function such that for any $t > 0$, $\alpha(t) := \frac{\varphi(t)}{t} < 1$ and the function $(0, +\infty) \ni t \to \frac{\varphi(t)}{t}$ is non-increasing (or non-decreasing, or continuous).

Let $d_i : I \to \mathbb{R}$ be a continuously differentiable function such that

$$\max_{x \in I} |d_i(x)| \leq 1.$$

Then by the differential mean value theorem, we can see that for some $L_{d_i} > 0$,

$$|d_i(x') - d_i(x'')| \leq L_{d_i} |x' - x''|,$$

where $x', x'' \in I$. Hence d_i is Lipschitz function defined on I satisfying $\max_{x \in I} |d_i(x)| \leq 1$.

Consider an IFS of the form $\{K; w_i, i = 1, 2, \cdots, N\}$ in which the maps are nonlinear transformations of the special structure

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l_i(x) \\ F_i(x, y) \end{pmatrix} = \begin{pmatrix} a_i x + e_i \\ c_i x + d_i(x) s_i(y) + f_i \end{pmatrix},$$

where the transformations are constrained by the data according to

$$w_i\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, w_i\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

for $i = 1, 2, \cdots, N$, and s_i are some Geraghty contractions (with the same function φ). Then for all $(x, y'), (x, y'') \in K \subset \mathbb{R}^2$,

$$\begin{aligned} |F_i(x, y') - F_i(x, y'')| &= |d_i(x)||s_i(y') - s_i(y'')| \\ &\le |s_i(y') - s_i(y'')| \le \varphi(|y' - y''|). \end{aligned}$$

That is, each $w_i(x, y)$ is chosen so that function $F_i(x, y)$ is Geraghty contraction with respect to the second variable.

Also, analytically, we obtain (compare with a_i, e_i, c_i, f_i of [9]).

$$\begin{aligned} a_i &= \frac{x_i - x_{i-1}}{x_N - x_0}, \\ e_i &= \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0} \\ c_i &= \frac{y_i - y_{i-1}}{x_N - x_0} - \frac{d_i(x_N)s_i(y_N) - d_i(x_0)s_i(y_0)}{x_N - x_0}, \\ f_i &= \frac{x_N y_{i-1} - x_0 y_i}{x_N - x_0} - \frac{x_N d_i(x_0)s_i(y_0) - x_0 d_i(x_N)s_i(y_N)}{x_N - x_0}. \end{aligned}$$

Denote by $C(I)$ the set of continuous functions $f : I = [x_0, x_N] \to [a, b]$.

Let $C^*(I) \subset C(I)$ denote the set of continuous functions $f : I \to [a, b]$ such that $f(x_0) = y_0$ and $f(x_N) = y_N$, that is,

$$C^*(I) := \{f \in C(I) : f(x_0) = y_0, f(x_N) = y_N\}.$$

Let $C^{**}(I) \subset C^*(I) \subset C(I)$ be the set of continuous functions that pass through the given data points $\{(x_i, y_i) \in K = [x_0, x_N] \times [a, b] : i = 0, 1, 2, ..., N\}$, that is,

$$C^{**}(I) := \{f \in C^*(I) : f(x_i) = y_i, i = 0, 1, \cdots, N\}.$$

Define a metric $d_{C(I)}$ on $C(I)$ by

$$d_{C(I)}(g, h) := \max_{x \in [x_0, x_N]} |g(x) - h(x)|$$

for all $g, h \in C(I)$. Then $(C(I), d_{C(I)})$, $(C^*(I), d_{C(I)})$ and $(C^{**}(I), d_{C(I)})$ are complete metric spaces.

For all $f \in C^*(I)$, define a mapping $T : C^*(I) \to C(I)$ by

$$Tf(x) := F_i(l_i^{-1}(x), f(l_i^{-1}(x)))$$

for $x \in [x_{i-1}, x_i]$ and $i = 1, 2, \cdots, N$.

Lemma 3.1 — $Tf \in C^{**}(I)$ for all $f \in C^*(I)$. That is, $T : C^*(I) \to C^{**}(I)$ and $T^n : C^{**}(I) \to C^{**}(I)$ for all $n \geq 2$.

PROOF : Since

$$w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, w_i \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

for $i = 1, 2, \cdots, N$, we obtain $l_i(x_0) = x_{i-1}$, $l_i(x_N) = x_i$, $l_i^{-1}(x_{i-1}) = x_0$, $l_i^{-1}(x_i) = x_N$, $F_i(x_0, y_0) = y_{i-1}$ and $F_i(x_N, y_N) = y_i$ for $i = 1, 2, \cdots, N$. Hence if $x_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, N$, then since $f \in C^*(I)$, we obtain

$$\begin{aligned} Tf(x_i) &= F_i(l_i^{-1}(x_i), f(l_i^{-1}(x_i))) \\ &= F_i(x_N, f(x_N)) = F_i(x_N, y_N) = y_i \end{aligned}$$

and if $x_i \in [x_i, x_{i+1}]$ for $i = 0, 1, 2, \cdots, N-1$, then since $f \in C^*(I)$, we obtain

$$\begin{aligned} Tf(x_i) &= F_{i+1}(l_{i+1}^{-1}(x_i), f(l_{i+1}^{-1}(x_i))) \\ &= F_{i+1}(x_0, f(x_0)) = F_{i+1}(x_0, y_0) = y_i. \end{aligned}$$

So $f(x_i) = y_i$ for all $i = 0, 1, 2, \cdots, N$ and $Tf(x)$ is continuous at each of the points $x_1, x_2, \cdots, x_{N-1}$.

By definition of the mapping T, $Tf(x)$ is continuous on the interval $[x_{i-1}, x_i]$ for all $i = 1, 2, \cdots, N$. Hence $Tf \in C^{**}(I)$ and $T^n : C^{**}(I) \to C^{**}(I)$ for all $n \geq 2$.

Using the technique introduced in [9], we can prove following Theorem.

***Theorem* 3.2** — *Let N be a positive integer greater than one. Let $\{K; w_i, i = 1, 2, \cdots, N\}$ denote the IFS defined above, associated with the set of data*

$$\{(x_i, y_i) : i = 0, 1, \cdots, N\}.$$

Then the operator T is a Geraghty contraction (considered as a map $T : C^*(I) \to C^*(I)$). Hence there is a unique continuous function $f : I \to [a, b]$ which is a fixed point of T. In particular, $f(x_i) = y_i$ for $i = 0, 1, \cdots, N$. Moreover, the graph G of f is invariant with respect to $\{K; w_1, \cdots, w_N\}$, i.e.,

$$G = \bigcup_{i=1}^{N} w_i(G).$$

PROOF : Since $F_i(x,y) = c_i x + d_i(x)s_i(y) + f_i$ and $\max_{x\in I}|d_i(x)| \le 1$, we obtain that for all $x \in I$, $y', y'' \in [a,b]$ and $i = 1, 2, \cdots, N$,

$$\begin{aligned}|F_i(x,y') - F_i(x,y'')| &\le |c_i x + d_i(x)s_i(y') + f_i - c_i x - d_i(x)s_i(y'') - f_i| \\ &\le |d_i(x)s_i(y') - d_i(x)s_i(y'')| \\ &\le \varphi(|y' - y''|),\end{aligned}$$

where $\varphi : (0,+\infty) \to (0,+\infty)$ is some non-decreasing function such that $\varphi(t) < t$ for $t > 0$ and $t \to \frac{\varphi(t)}{t}$ is non-increasing (or non-decreasing, or continuous). That is, each F_i is a Geraghty contraction (with the same function φ) with respect to the second variable. Hence for all $g, h \in C^*(I) \subset C(I)$, we have

$$\begin{aligned}d_{C(I)}(Tg,Th) &= \max_{x\in[x_0,x_N]}|Tg(x) - Th(x)| \\ &= \max_{i=1,2,\cdots,N}\max_{x\in[x_{i-1},x_i]}|Tg(x) - Th(x)| \\ &= \max_{i=1,2,\cdots,N}\max_{x\in[x_{i-1},x_i]}|F_i(l_i^{-1}(x), g(l_i^{-1}(x))) - F_i(l_i^{-1}(x), h(l_i^{-1}(x)))| \\ &\le \max_{i=1,2,\cdots,N}\sup_{x\in[x_{i-1},x_i]}\varphi(|g(l_i^{-1}(x)) - h(l_i^{-1}(x))|).\end{aligned}$$

Since $\varphi : (0,+\infty) \to (0,+\infty)$ is non-decreasing function and $l_i^{-1} : [x_{i-1}, x_i] \to [x_0, x_N]$ for all $i = 1, 2, \cdots, N$, we obtain that for $i_0 \in \{1, 2, \cdots, N\}$ and $x_0 \in [x_{i_0-1}, x_{i_0}]$,

$$\begin{aligned}\varphi(|g(l_{i_0}^{-1}(x_0)) - h(l_{i_0}^{-1}(x_0))|) \le&\varphi(\max_{x\in[x_{i_0-1},x_{i_0}]}|g(l_{i_0}^{-1}(x)) - h(l_{i_0}^{-1}(x))|) \\ \le&\varphi(\max_{x\in[x_0,x_N]}|g(x) - h(x)|) \\ =&\varphi(d_{C(I)}(g,h)).\end{aligned}$$

Since x_0 was arbitrary,

$$\sup_{x\in[x_{i_0-1},x_{i_0}]}\varphi(|g(l_{i_0}^{-1}(x)) - h(l_{i_0}^{-1}(x))|) \le\varphi(d_{C(I)}(g,h))$$

and since i_0 was arbitrary,

$$\max_{i=1,2,\cdots,N}\sup_{x\in[x_{i-1},x_i]}\varphi(|g(l_i^{-1}(x) - h(l_i^{-1}(x))|) \le\varphi(d_{C(I)}(g,h)).$$

Hence we obtain

$$\begin{aligned}d_{C(I)}(Tg,Th) \le& \max_{i=1,2,\cdots,N}\sup_{x\in[x_{i-1},x_i]}\varphi(|g(l_i^{-1}(x) - h(l_i^{-1}(x))|) \\ \le&\varphi(d_{C(I)}(g,h)).\end{aligned}$$

So we conclude that $T : C^*(I) \to C^{**}(I) \subset C^*(I)$ is a Geraghty contraction (with the same function φ) on the complete metric space $(C^*(I), d_{C(I)})$.

Theorem 2.3 (1) implies that T possesses a unique fixed point in $C^*(I)$. That is, there exists a continuous function $f \in C^*(I)$ such that for all $x \in [x_0, x_N]$,

$$Tf(x) = f(x).$$

Since $T : C^*(I) \to C^{**}(I)$ (by Lemma 3.1), we have $f = Tf \in C^{**}(I)$. That is, there is a continuous function f that passes through the given data points $\{(x_i, y_i) \in [x_0, x_N] \times [a, b] : i = 0, 1, 2, ..., N\}$.

Let G denote the graph of $f \in C^{**}(I)$, that is, $G := \{(x, f(x)) : x \in [x_0, x_N]\}$. Since f is a fixed point of the operator T and if $x \in [x_{i-1}, x_i]$, then

$$Tf(x) = F_i(l_i^{-1}(x), f(l_i^{-1}(x))),$$

we obtain that for all $x \in [x_0, x_N]$,

$$\begin{aligned} f(l_i(x)) =& Tf(l_i(x)) \\ =& F_i(l_i^{-1}(l_i(x)), f(l_i^{-1}(l_i(x)))) \\ =& F_i(x, f(x)). \end{aligned}$$

Since $w_i(x, y) = (l_i(x), F_i(x, y))$ for all for $i = 1, 2, \cdots, N$, we obtain that

$$\begin{aligned} w_i(G) =& w_i(\{(x, f(x)) : x \in [x_0, x_N]\}) \\ =& \{w_i(x, f(x)) : x \in [x_0, x_N]\} \\ =& \{(l_i(x), F_i(x, f(x))) : x \in [x_0, x_N]\} \\ =& \{(l_i(x), f(l_i(x))) : x \in [x_0, x_N]\} \\ =& \{(x, f(x)) : x \in [x_{i-1}, x_i]\}. \end{aligned}$$

Hence

$$\begin{aligned} G =& \{(x, f(x)) : x \in [x_0, x_N]\} \\ =& \bigcup_{i=1}^{N} \{(x, f(x)) : x \in [x_{i-1}, x_i]\} \\ =& \bigcup_{i=1}^{N} w_i(G). \end{aligned}$$

This completes the proof. □

Notice that Theorem 3.2 does not ensure that the IFS$\{K; w_i, i = 1, 2, \cdots, N\}$ has a unique invariant set. The uniqueness of invariant set is determined explicitly in Theorem 3.3.

Our main theorem (Theorem 3.3) in this section improves upon a result proved by [9].

***Theorem* 3.3** — *Let N be a positive integer greater than one. Let each s_i be a bounded function. Let $\{K; w_i, i = 1, 2, \cdots, N\}$ denote the IFS defined above, associated with the set of data*

$$\{(x_i, y_i) : i = 0, 1, \cdots, N\}.$$

Then there is a metric d_θ on $K = I \times [a, b]$, equivalent to the Euclidean metric d_0, such that for all $i = 1, \cdots, N$, w_i are Geraghty contraction maps with respect to d_θ. In particular, there exists a unique nonempty compact set $G \subset K = I \times [a, b]$ such that

$$G = \bigcup_{i=1}^{N} w_i(G).$$

PROOF : We define a metric d_θ on K by

$$d_\theta((x', y'), (x'', y'')) := |x' - x''| + \theta|y' - y''|,$$

where θ is a positive real number which is specified below.

Since $|d_i(x') - d_i(x'')| \leq L_{d_i}|x' - x''|$ and $F_i(x, y) := c_i x + d_i(x)s_i(y) + f_i$,

$$\begin{aligned}
|F_i(x', y') - F_i(x'', y'')| = \\
&= |c_i x' + d_i(x')s_i(y') + f_i - (c_i x'' + d_i(x'')s_i(y'') + f_i)| \\
&\leq |c_i||x' - x''| + |d_i(x')s_i(y') - d_i(x'')s_i(y'')| \\
&\leq |c_i||x' - x''| + |d_i(x')||s_i(y') - s_i(y'')| + |s_i(y'')||d_i(x') - d_i(x'')| \\
&\leq |c_i||x' - x''| + |s_i(y') - s_i(y'')| + \sup_{y'' \in D(s_i)} |s_i(y'')||d_i(x') - d_i(x'')| \\
&\leq (|c_i| + \sup_{y'' \in D(s_i)} |s_i(y'')|L_{d_i})|x' - x''| + |s_i(y') - s_i(y'')|,
\end{aligned}$$

where $D(s_i)$ denotes the domain of definition of s_i.

Let

$$k := \max_{i=1,2,\cdots,N} (|c_i| + \sup_{y'' \in D(s_i)} |s_i(y'')|L_{d_i}),$$

Then for all $(x', y'), (x'', y'') \in K$,

$$|F_i(x', y') - F_i(x'', y'')| \le k|x' - x''| + \varphi(|y' - y''|),$$

where $\varphi : (0, +\infty) \to (0, +\infty)$ is some non-decreasing function such that $\varphi(t) < t$ for $t > 0$ and $t \to \frac{\varphi(t)}{t}$ is non-increasing (or non-decreasing, or continuous). That is, each F_i is a Geraghty contraction (with the same function φ) with respect to the second variable, and Lipschitz with respect to the first variable. Hence we obtain for all $(x', y'), (x'', y'') \in K$,

$$\begin{aligned} d_\theta(w_i(x', y'), w_i(x'', y'')) =& d_\theta((l_i(x''), F_i(x', y')), (l_i(x''), F_i(x'', y''))) \\ =& |l_i(x') - l_i(x'')| + \theta|F_i(x', y') - F_i(x'', y'')| \\ \le& |a_i||x' - x''| + \theta(k|x' - x''| + \varphi(|y' - y''|)) \\ =& |a_i||x' - x''| + \theta k|x' - x''| + \theta\varphi(|y' - y''|) \\ \le& (|a_i| + \theta k)|x' - x''| + \theta\varphi(|y' - y''|). \end{aligned}$$

Let $(x', y'), (x'', y'') \in K$ and $(x', y') \neq (x'', y'')$. Since $\varphi : (0, +\infty) \to (0, +\infty)$ is non-decreasing function and $\varphi(t) < t$ for all $t > 0$, we obtain that

$$\begin{aligned} d_\theta(w_i(x', y'), w_i(x'', y'')) \le& (|a_i| + \theta k)|x' - x''| + \theta\varphi(|y' - y''|) \\ =& (|a_i| + \theta k)|x' - x''| + \theta\frac{\varphi(|y' - y''|)}{|x' - x''| + |y' - y''|}(|x' - x''| + |y' - y''|) \\ =& (|a_i| + \theta k + \theta\frac{\varphi(|y' - y''|)}{|x' - x''| + |y' - y''|})|x' - x''| \\ &+ \theta\frac{\varphi(|y' - y''|)}{|x' - x''| + |y' - y''|}|y' - y''| \\ \le& (|a_i| + \theta k + \theta\frac{\varphi(|x' - x''| + |y' - y''|)}{|x' - x''| + |y' - y''|})|x' - x''| \\ &+ \theta\frac{\varphi(|x' - x''| + |y' - y''|)}{|x' - x''| + |y' - y''|}|y' - y''| \\ \le& (|a_i| + \theta k + \theta)|x' - x''| + \theta\frac{\varphi(|x' - x''| + |y' - y''|)}{|x' - x''| + |y' - y''|}|y' - y''| \\ \le& \max\{|a_i| + \theta k + \theta, \frac{\varphi(|x' - x''| + |y' - y''|)}{|x' - x''| + |y' - y''|}\}(|x' - x''| + \theta|y' - y''|) \\ =& \max\{|a_i| + \theta k + \theta, \frac{\varphi(|x' - x''| + |y' - y''|)}{|x' - x''| + |y' - y''|}\}d_\theta((x', y'), (x'', y'')) \\ \le& \max\{\max_{i=1,2,\cdots,N}|a_i| + \theta k + \theta, \frac{\varphi(|x' - x''| + |y' - y''|)}{|x' - x''| + |y' - y''|}\}d_\theta((x', y'), (x'', y'')). \end{aligned}$$

Since $N > 1$, we obtain $0 < a_i := \frac{x_i - x_{i-1}}{x_N - x_0} < 1$ for all $i = 1, 2, \cdots, N$.

Let

$$\theta := \frac{1 - \max_{i=1,2,\cdots,N} |a_i|}{2(k+1)}.$$

Then $0 < \max_{i=1,2,\cdots,N} |a_i| + \theta k + \theta < 1$ and since $k \geq 0$, we obtain $0 < \theta < 1$.

Let for all $t > 0$,

$$\beta(t) := \max\{\max_{i=1,2,\cdots,N} |a_i| + \theta k + \theta, \frac{\varphi(t)}{t}\}.$$

Then because $\alpha(t) := \frac{\varphi(t)}{t}$ and $\alpha : (0, +\infty) \to [0, 1)$ is a non-increasing (or non-decreasing, or continuous), we can see that $\beta : (0, +\infty) \to [0, 1)$ is a non-increasing (or non-decreasing, or continuous) and for each $(x', y'), (x'', y'') \in K, (x', y') \neq (x'', y'')$,

$$d_\theta(w_i(x', y'), w_i(x'', y'')) \leq \beta(d((x', y'), (x'', y'')))d_\theta((x', y'), (x'', y'')),$$

where $d((x', y'), (x'', y'')) := |x' - x''| + |y' - y''|$. Since $0 < \theta < 1$, for all $(x', y'), (x'', y'') \in K, (x', y') \neq (x'', y'')$,

$$\theta|x' - x''| + \theta|y' - y''| \leq |x' - x''| + \theta|y' - y''| \leq |x' - x''| + |y' - y''|.$$

That is,

$$d_\theta((x', y'), (x'', y'')) \leq d((x', y'), (x'', y'')) \leq \theta^{-1} d_\theta((x', y'), (x'', y'')).$$

If $\beta : (0, +\infty) \to [0, 1)$ is a non-increasing, then

$$\begin{aligned} d_\theta(w_i(x', y'), w_i(x'', y'')) &\leq \beta(d((x', y'), (x'', y'')))d_\theta((x', y'), (x'', y'')) \\ &\leq \beta(d_\theta((x', y'), (x'', y'')))d_\theta((x', y'), (x'', y'')). \end{aligned}$$

If $\beta : (0, +\infty) \to [0, 1)$ is a non-decreasing, then

$$\begin{aligned} d_\theta(w_i(x', y'), w_i(x'', y'')) &\leq \beta(d((x', y'), (x'', y'')))d_\theta((x', y'), (x'', y'')) \\ &\leq \beta(\theta^{-1} d_\theta((x', y'), (x'', y'')))d_\theta((x', y'), (x'', y'')) \\ &\leq \beta'(d_\theta((x', y'), (x'', y'')))d_\theta((x', y'), (x'', y'')), \end{aligned}$$

where $\beta' : (0, +\infty) \to [0, 1)$ is a non-decreasing function $\beta' := \beta \circ \theta^{-1}$.

If $\beta : (0,+\infty) \to [0,1)$ is a continuous, then

$$d_\theta(w_i(x',y'),w_i(x'',y'')) \leq \beta(d((x',y'),(x'',y'')))d_\theta((x',y'),(x'',y'')) \leq \beta''(d_\theta((x',y'),(x'',y'')))d_\theta((x',y'),(x'',y'')),$$

where $\beta'' : (0,+\infty) \to [0,1)$ is a continuous function $\beta''(t) := \max_{x\in[t,\theta^{-1}t]} \beta(t)$. Hence w_i are Geraghty contractions in (K, d_θ).

On the other hand, metric d_θ is equivalent to the Euclidean metric d_0 on K (see [9]). So (K, d_θ) is a complete metric space. Hence $w_i : K \to K$ is a Geraghty contraction in (K, d_θ) and by Theorem 2.3 (1), there exists an unique fixed point in K.

By Theorem 2.3 (2), for the complete metric space (K, d_θ), there is a unique nonempty compact set $G \subset K$ such that

$$G = \bigcup_{i=1}^{N} w_i(G).$$

By the definition of Hausdorff metric, equivalence of two metrics implies the equivalence of Hausdorff metrics generated by them (see [8], p.91, Lemma 3.6). Hence for (K, d_0), there is a unique nonempty compact set $G \subset K$ such that

$$G = \bigcup_{i=1}^{N} w_i(G).$$

This completes the proof. □

Remark 3.4 : Our result is a substantial generalization of [1, 2, 9, 11].

The function whose graph is the attractor of an IFS as described in Theorem 3.2 and Theorem 3.3 generalizes the affine fractal interpolation function (see [1, 2]), the fractal interpolation function with variable parameters (see [11]) and the nonlinear fractal interpolation function (see [9]).

(1) In the affine fractal interpolation function (cf. [1], p. 308, Example 1, see [2], p.214), $d_i(x) \equiv d_i$ $(|d_i| < 1)$ and for all $t > 0$,

$$\varphi(t) := \max_{i=1,2,\cdots,N} |d_i|t.$$

(2) In the fractal interpolation function with variable parameters (cf. [11], p.3), $d_i(x)$ are Lipschitz functions defined on I satisfying $\sup_{x\in I} |d_i(x)| < 1$, and for all $t > 0$,

$$\varphi(t) := \max_{i=1,2,\cdots,N} \max_{x\in I} |d_i(x)|t.$$

(3) In the nonlinear fractal interpolation function (cf. [9]), $d_i(x) \equiv 1$ and $\varphi : (0,+\infty) \to (0,+\infty)$ is a non-decreasing function such that for any $t > 0$, $\alpha(t) := \frac{\varphi(t)}{t} < 1$ and the function $(0,+\infty) \ni t \to \frac{\varphi(t)}{t}$ is non-increasing.

In order to demonstrate the effectiveness of results in this section, we now present an explicit illustrative example.

Example 3.5 : Let $\varphi_1(t) := \frac{t}{1+t}$, $\varphi_2(t) := \max(2|\sin(t/2)|,\ t/2)$ and $\varphi(t) := \max(\varphi_1(t), \varphi_2(t))$ for $t \in (0,+\infty)$. Then $\varphi : (0,+\infty) \to (0,+\infty)$ is a non-decreasing function and $t \to \frac{\varphi(t)}{t}$ is non-increasing function.

Let a set of data $\{(x_i, y_i) : i = 0,1,2,3,4\}$ be given, where $x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.8$, $x_4 = 1$ and $y_0 = 0.2$, $y_1 = 0.5$, $y_2 = 0.3$, $y_3 = 0.8$, $y_4 = 0.6$.

Let for all $i = 1,2,3,4$,

$$d_i(x) := e^{-x/i}.$$

Then by differential mean value theorem, for all $x', x'' \in [0,1]$,

$$\begin{aligned} |d_i(x') - d_i(x'')| &\le \max_{\xi\in[0,1]} |\frac{1}{i} e^{-x/i}||x' - x''| \\ &\le L_{d_i}|x' - x''|, \end{aligned}$$

and

$$\max_{x\in[x_0,x_4]} |d_i(x)| = 1,$$

where $L_{d_i} = 1/i$.

Let for $z \in [0,+\infty)$,

$$\begin{aligned} &s_1(z) := \frac{1}{1+z},\ \ s_2(z) := \frac{z}{1+z}, \\ &s_3(z) := \frac{z}{1+2z},\ \ s_4(z) := \sin(z)(cf.\ [12],\ Example\ 1). \end{aligned}$$

Then, for $z', z'' \in [0,+\infty)$,

$$\begin{aligned} |s_1(z') - s_1(z'')| &= |\frac{1}{1+z'} - \frac{1}{1+z''}| \le \frac{|z'-z''|}{1+|z'-z''|} = \varphi_1(|z'-z''|) \le \varphi(|z'-z''|), \\ |s_2(z') - s_2(z'')| &= |\frac{z'}{1+z'} - \frac{z''}{1+z''}| \le \frac{|z'-z''|}{1+|z'-z''|} = \varphi_1(|z'-z''|) \le \varphi(|z'-z''|), \\ |s_3(z') - s_3(z'')| &= |\frac{z'}{1+2z'} - \frac{z''}{1+2z''}| \le \frac{|z'-z''|}{1+|z'-z''|} = \varphi_1(|z'-z''|) \le \varphi(|z'-z''|), \end{aligned}$$

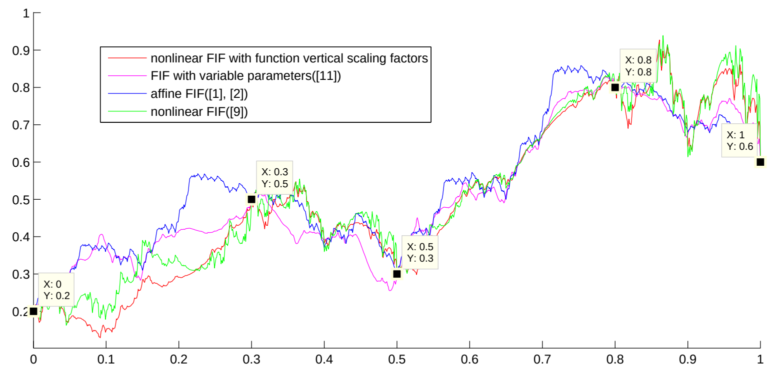

Figure 1: The graph of the nonlinear FIF with function vertical scaling factors

and, for $z', z'' \in [0, +\infty)$,

$$\begin{aligned} |s_4(z') - s_4(z'')| &= |\sin z' - \sin z''| \\ &= 2|\cos\frac{z'+z''}{2}\sin\frac{z'-z''}{2}| \\ &\leq 2|\sin\frac{z'-z''}{2}| \leq \varphi_2(|z'-z''|) \leq \varphi(|z'-z''|). \end{aligned}$$

That is, s_1, s_2, s_3, s_4 are Geraghty contractions (with the same function φ) that are not Banach contractions on $[0, +\infty)$ (cf. [7], p.262).

Let

$$w_i(x, y) := (a_i x + e_i, c_i x + d_i(x)s_i(y) + f_i).$$

By Theorem 3.2 and Theorem 3.3, there exists a continuous function $f : [0, 1] \to \mathbb{R}$ that interpolates the given data $\{(x_i, y_i) : i = 0, 1, 2, 3, 4\}$.

The graph of the nonlinear FIF with the above function vertical scaling factors is displayed in Figure 1.

Remark 3.6 : It is not necessary to find the examples of function φ corresponding to Geraghty II and Geraghty III contractions respectively, because Geraghty II and Geraghty III come down to Geraghty I(Rakotch) if the working space is such a subset of $\mathbb{R}$ with the Euclidean metric d_0 or equivalent metric to it.

In fact, let s be a Geraghty II contraction $i.e.$, $\frac{\varphi(t)}{t}$ is a non-decreasing function, and for $z', z'' \in$

$[0, +\infty)(z' \neq z'')$,

$$|s(z') - s(z'')| \leq \varphi(|z' - z''|).$$

Since $\varphi(t)/t$ is non-decreasing in $(0, +\infty)$, there exists $m := \liminf_{t \to 0+} \frac{\varphi(t)}{t}$. Let $\varepsilon > 0$ and $z' < z''$. Then there exists $t_0 > 0$ and a natural number N such that $\varphi(t_0) < m + \varepsilon$ and

$$z' = z_0 < z_1 < \cdots < z_N = z'',$$
$$z_1 - z_0 = z_2 - z_1 = \cdots = z_N - z_{N-1} < t_0.$$

Then

$$|s(z') - s(z'')| \leq \sum_{i=1}^{N} |s(z_i) - s(z_{i-1})|$$
$$\leq \sum_{i=1}^{N} \varphi(|z_i - z_{i-1}|)$$
$$\leq \sum_{i=1}^{N} \frac{\varphi(|z_i - z_{i-1}|)}{|z_i - z_{i-1}|} |z_i - z_{i-1}|$$
$$\leq (m + \varepsilon) \sum_{i=1}^{N} |z_i - z_{i-1}| = (m + \varepsilon)|z' - z''|.$$

Since ε is arbitrary,

$$|s(z') - s(z'')| \leq m|z' - z''|,\ m < 1.$$

So s is a Banach contraction.

Next, let s be a Geraghty III contraction i.e., $\varphi(t) < t$ for $t > 0$ and $\frac{\varphi(t)}{t}$ is a continuous function, and for $z', z'' \in [0, +\infty)(z' \neq z'')$,

$$|s(z') - s(z'')| \leq \varphi(|z' - z''|).$$

Since $\varphi(t)/t$ is continuous in $(0, +\infty)$, there exists $m := \max_{t \in [1/2, 1]} \frac{\varphi(t)}{t}$. Let $|z'' - z'| \geq 1$. Then there exists a natural number N such that $|z' - z''|/N \in [1/2, 1]$ and, similarly to the above, $|s(z') - s(z'')| \leq m|z' - z''|,\ m < 1$. So from Proposition 1 of [4], s is a Rakotch contraction.

Remark 3.7 : From Remark 3.6, Theorem 3.2 and Theorem 3.3 described by Geraghty contraction essentially come down to ones by Rakotch contraction. However, we intend to show the most expansive (previous) fixed point theorem by which the proof technique of Theorem 3.2 and Theorem 3.3 still hold. This is just the reason why we use Graghty contraction instead of Rakotch contraction in this paper.

4. Conclusion

The FIFs have been widely used in approximation theory, image compression, computer graphics and modeling of natural surfaces such as rocks, metals, terrains and so on. In order to ensure more flexibility in modeling natural shapes and phenomena or in image processing we introduced a new nonlinear FIF by a new nonlinear IFS. In this paper we have presented the principle and the method of nonlinear fractal interpolation in detail. Theorem 3.2 and Theorem 3.3 ensure that an attractor of constructed nonlinear IFS is a graph of some continuous function which interpolates the given data. In particular, Example 3.5 shows that our result remains still true under essentially weaker conditions on the maps of IFS.

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