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APPROXIMATE CONTROLLABILITY OF NON-AUTONOMOUS SOBOLEV TYPE INTEGRO-DIFFERENTIAL EQUATIONS HAVING NONLOCAL AND NON-INSTANTANEOUS IMPULSIVE CONDITIONS

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The aim of this article is to study approximate controllability of a class of non-autonomous Sobolev type integro-differential equations having non-instantaneous impulses with nonlocal initial condition. The results will be proved with the help of evolution system and Krasnoselskii fixed point theorem. An example is presented to show how our abstract results can be applied.

Key words : Approximate controllability; Krasnoselskii fixed point theorem; evolution system; non-instantaneous impulsive condition; Sobolev type integro-differential equations.

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1. INTRODUCTION

The Sobolev type differential equations appear in several fields such as thermodynamics [8], fluid flow via fissured rocks [4] and mechanics of soil [29]. Brill [5] first established the existence of solution for a semilinear Sobolev differential equation in a Banach space. Lightbourne *et al*. [21] studied a partial differential equation of Sobolev type.

In recent years many researchers paid attention to study the differential equations with instantaneous impulses, which have been used to described abrupt changes such as shocks, harvesting and natural disasters. Particularly, the theory of instantaneous impulsive equations have wide applications in control, mechanics, electrical engineering, biological and medical fields.

It seems that models with instantaneous impulses could not explain certain dynamics of evolution process in pharmacotherapy. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in bloodstream and the consequent absorbtion for the body are gradual and continuous process, we can interpret the above situations as an impulsive action which starts abruptly and stays active on a finite time interval. Hernández and O'Regan [16] and Pierri *et al.* [26], initially studied Cauchy problems for first order evolution equations with non-instataneous impulses. Kumar *et al.* [20] established the existence and uniqueness of mild solutions for non-instantaneous impulsive fractional differential equations. Chen *et al.* [9] investigated the existence of mild solutions for first order semi-linear evolution equations with non-instantaneous impulses using noncompact semigroup. Kumar *et al.* [18] derived a set of sufficient conditions for the existence and uniqueness of mild solutions to fractional integro-differential equations with non-instantaneous impulses. The work on nonlocal initial value problem was first studied by Byszewski. In [6], Byszewski established the results about the existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem. The nonlocal condition, in many cases, has better effect than the classical initial condition.

Controllability is one of the most important issue in engineering and mathematical control theory. The problem of controllability of various kinds of differential, integro-differential equations and impulsive differential equations are studied. In case of controllability, the literature on abstract impulsive differential equations considers basically problems for which the impulses are abrupt and instantaneous. In [3], Balasubramaniam *et al.* derived sufficient conditions for approximate controllability of impulsive fractional integro-differential equations with nonlocal conditions by assuming the compactness of impulsive and nonlocal functions in Hilbert space. Zhang *et al.* [31] discussed the approximate controllability of fractional impulsive integro-differential equations in Hilbert space using Krasnoselskii fixed point theorem and comapact analytic semigroup theory. Dong *et al.* [11] studied approximate controllability of semilinear fractional evolution equations with nonlocal and impulsive conditions via approximate technique.

Yan [30] investigated the existence of mild solutions of non-autonomous integro-differential equations with nonlocal conditions by using the theory of evolution families, Banach contraction principle and Schauder's fixed point theorem. Haloi *et al.* [14] studied existence, uniqueness and asymptotic stability of non-autonomous differential equations with deviated arguments via Banach fixed point theorem and theory of analytic semigroup. In [7], Alka *et al.* established the existence and uniquenss of mild solutions for non-autonomous instantaneous impulsive differential equations with iterated deviating arguments by using analytic semigroup theory and Banach fixed point theorem. Hamdy [1] studied sufficient conditions for controllability of autonomous Sobolev type fractional integro-differential equations with the help of Schauder's fixed point theorem and the theory of compact semigroup. Radhakrishnan *et al.* [27] discussed controllability of autonomous second order Sobolev type neutral instantaneous impulsive integro-differential systems using strongly continuous cosine family of operators and Banach contraction principle. In [19], Kamalendra *et al.* investigated sufficient conditions for controllability of non-autonomous Sobolev type instantaneous impulsive integro-differential equations with nonlocal conditions via semigroup theory and Banach contraction principle. Mahmudov [23] studied the approximate controllability of autonomous fractional Sobolev type differential equations in Banach space with the help of Schauder's fixed point theorem. In [15], Haloi established sufficient conditions for approximate controllability of non-autonomous nonlocal delay differential systems with deviating arguments by using theory of compact semigroup and Krasnoselskii fixed point theorem.

To the best of our knowledge, there is no work yet reported on approximate controllability of non-autonomous Sobolev type differential equations. Motivated by this fact, in this article, we consider the following non-autonomous Sobolev type integro-differential system with nonlocal and non-instantaneous impulsive conditions to investigate the sufficient conditions for the approximate controllability in a separable Hilbert space X:

$$\begin{aligned}
\frac{d}{dt}[Ex(t)] + A(t)x(t) &= f\left(t, x(t), \int_0^t k(t,s,x(s))ds, \int_0^b h(t,s,x(s))ds\right) + Bu(t), \\
&\quad t \in \cup_{i=0}^{m}(s_i, t_{i+1}], \\
x(t) &= \gamma_i(t, x(t)), \quad t \in \cup_{i=1}^{m}(t_i, s_i], \\
x(0) + g(x) &= x_0, \quad x_0 \in D(E), \qquad (1.1)
\end{aligned}$$

where $A(t) : D(A(t)) \subset X \to X$, $E : D(E) \subset X \to X$ are linear operators, and $f : J \times X \times X \times X \to X$, $g : PC(J, X) \to D(E)$, $k, h : \triangle \times X \to X$, and non-instataneous impulsive functions $\gamma_i : (t_i, s_i] \times X \to D(E)$, $i = 1, 2, \ldots, m$ are suitable functions will be specified later, $\triangle := \{(t, s) : 0 \leqslant s \leqslant t \leqslant b\}$, $J = [0, b]$, $0 < t_1 < t_2 < \ldots < t_m < t_{m+1} := b$, $s_0 := 0$ and $s_i \in (t_i, t_{i+1})$ for each $i = 1, 2, \ldots, m$. The control function $u \in L^2(J, U)$, U is a Hilbert space and $B : U \to X$ is a bounded linear operator.

The article is organized as following. In section 2, we will recall some basic theory. We will establish the existence of mild solutions for the system (1.1), then we will show that the control system (1.1) is approximately controllable in section 3. We will discuss an example to illustrate our results in section 4.

2. PRELIMINARIES

First, we recall the definition and some basic properties of evolution system.

Definition 2.1 — [25]. Let X be a Banach space. A two parameter family of bounded linear operators $\mathcal{S}(t_1, t_2), 0 \leqslant t_2 \leqslant t_1 \leqslant b$ on X is known as evolution system, if :

1. $\mathcal{S}(s, s) = I$, where I is the identity operator.
2. $\mathcal{S}(t_1, t_2)\mathcal{S}(t_2, t_3) = \mathcal{S}(t_1, t_3)$ for $0 \leqslant t_3 \leqslant t_2 \leqslant t_1 \leqslant b$.
3. $(t_1, t_2) \to \mathcal{S}(t_1, t_2)$ is strongly continuous for $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

For the family of linear operators $\{A(t) : t \in J\}$ on X, we impose the following assumptions (see [5, 12]) :

(A1) $A(t)$ is closed operator, the domain of $A(t)$ is independent of t, and dense in X.

(A2) The resolvent of $A(t)$ exists for $Re(\vartheta) \leqslant 0$, $t \in J$, and there exists a positive constant ς such that $\|\mathcal{R}(\vartheta; \sqcup)\| \leqslant \frac{\varsigma}{|\vartheta|+\infty}$.

(A3) There are positive constants K, and $\rho \in (0, 1]$ such that $\|[A(\tau_1) - A(\tau_2)]A^{-1}(\tau_3)\| \leqslant K|\tau_1 - \tau_2|^\rho$ for any $\tau_1, \tau_2, \tau_3 \in J$.

(S1) E is closed, bijective operator, and $D(E) \subset D(A)$.

(S2) $E^{-1} : X \to D(E)$ is compact.

The assumptions $(A1), (A2)$ imply that $-A(t)$ generates an analytic semigroup in $B(X)$, where the symbol $B(X)$ stands for Banach space of all bounded linear operators on X. The closed graph theorem with the above assumptions imply that the linear operator $-A(t)E^{-1} : X \to X$ is bounded, and so for each $t \in J$, $-A(t)E^{-1}$ generates a semigroup of bounded linear operators and hence a unique evolution system $\{\mathcal{S}(t_1, t_2) : 0 \leqslant t_2 \leqslant t_1 \leqslant b\}$ on X, which satisfies (see [12, 25]) :

(i) There is a positive constant M such that $\|\mathcal{S}(t_1, t_2)\| \leqslant M$, $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

(ii) For each fixed t_2, $\{\mathcal{S}(t_1, t_2), t_2 < t_1\}$ is continuous in t_1 uniformly with respect to operator norm.

(iii) The derivative $\frac{\partial \mathcal{S}(t_1,t_2)}{\partial t_1}$ exists in strong operator topology for $0 \leqslant t_2 < t_1 \leqslant b$, is strongly continuous and belongs to $B(X)$. Moreover,

$$\frac{\partial \mathcal{S}(t_1, t_2)}{\partial t_1} + A(t_1)\mathcal{S}(t_1, t_2) = 0,\ 0 \leqslant t_2 < t_1 \leqslant b.$$

***Theorem* 2.2** — *[12, 5]. Assume that $(A1)$-$(A3)$, $(S1)$-$(S2)$ are satisfied and f satisfies uniform Hölder continuity on J with exponent $\beta \in (0,1]$, then the unique solution of following linear Cauchy problem*

$$\begin{aligned} \frac{d}{dt}[Ex(t)] &= A(t)x(t) + f(t), \quad t \in J, \\ x(0) &= x_0 \in D(E), \end{aligned} \tag{2.1}$$

is given as following

$$x(t) = E^{-1}\mathcal{S}(t,0)Ex_0 + \int_0^t E^{-1}\mathcal{S}(t,\eta)f(\eta)d\eta. \tag{2.2}$$

Let $PC(J,X) = \{x : J \to X : x$ is continuous at $t \neq t_i$, $x(t_{i-}) = x(t_i)$ and $x(t_{i+})$ exists for all $i = 1,2,\ldots.m\}$, which is a Banach space endowed with supremum norm $\|x\|_{PC} = \sup_{t\in J}\|x(t)\|$ and $L^p(J,X)(1 \leqslant p < \infty)$ be the Banach space of all X-valued Bochner integrable functions defined on J with norm $\|x\|_{L^p(J,X)} = (\int_0^b \|x(t)\|^p dt)^{\frac{1}{p}}$. Denote $\Omega_r = \{x \in PC(J,X) : \|x(t)\| \leqslant r, t \in J\}$ for $r > 0$, $Kx(t) := \int_0^t k(t,s,x(s))ds$ and $Hx(t) := \int_0^b h(t,s,x(s))ds$.

Definition 2.3 — [20]. A function $x \in PC(J,X)$ is said to be a mild solution of the problem (1.1) if for any $u \in L^2(J,U)$, x satisfies $x(0) = x_0 - g(x)$, $x(t) = \gamma_i(t,x(t))$ for all $t \in \cup_{i=1}^m (t_i, s_i]$ and

$$x(t) = \begin{cases} \mathcal{V}(t,0)E(x_0 - g(x)) + \int_0^t \mathcal{V}(t,s)[f(s,x(s),Kx(s),Hx(s)) + Bu(s)]ds, & t \in [0,t_1], \\ \mathcal{V}(t,s_i)E\gamma_i(s_i,x(s_i)) + \int_{s_i}^t \mathcal{V}(t,s)[f(s,x(s),Kx(s),Hx(s)) + Bu(s)]ds, & t \in \cup_{i=1}^m (s_i, t_{i+1}], \end{cases}$$

where $\mathcal{V}(t,s) := E^{-1}\mathcal{S}(t,s)$.

Consider the linear control system :

$$\begin{aligned} \frac{d}{dt}[Ex(t)] &= A(t)x(t) + Bu(t), \quad t \in J, \\ x(0) &= x_0. \end{aligned} \tag{2.3}$$

Corresponding to (2.3) the controllability and resolvent operators are given as :

$$\Gamma_0^b = \int_0^b \mathcal{V}(b,s)BB^*\mathcal{V}^*(b,s)ds, \tag{2.4}$$

$$R(\delta,\Gamma_0^b) = (\delta I + \Gamma_0^b)^{-1}, \ \delta > 0, \tag{2.5}$$

respectively, where $*$ denotes the adjoint of operator. Notice that Γ_0^b is a linear bounded operator.

***Theorem* 2.4** — *[22]. Suppose X be a separable reflexive Banach space and X^* stands for it's dual space. Let $\Gamma : X^* \to X$ is a symmetric map. Then the following are equivalent :*

(i) $\Gamma : X^* \to X$ *is positive map.*

(ii) *For all* $x \in X,\ \delta(\delta I + \Gamma\mathfrak{J})^{-1}(\mathfrak{x})$ *strongly converges to zero as* $\delta \to 0^+$*, here* $\mathfrak{J}$ *is the duality map from* $X \to X^*$*.*

Remark 2.5 [22, 23]. The necessary and sufficient conditions for the linear system (2.3) to be approximately controllable on J is that, $\delta R(\delta, \Gamma_0^b) \to 0$ as $\delta \to 0^+$ in the strong operator topology.

Now, we recall Krasnoselskii fixed point theorem.

***Theorem* 2.6** — *[13]. Let* S *is a convex closed bounded subset of a Banach space* X*. Suppose that* $F_1,\ F_2 : S \to X$ *be two operators such that such that* $F_1x + F_2y \in S$ *whenever* $x,\ y \in S$*,* F_1 *is continuous and compact, and* F_2 *is contraction map. Then* $F_1 + F_2$ *has a fixed point in* S*.*

3. Main Results

In this section, we prove the existence of mild solutions and approximate controllability of (1.1). For $x \in PC(J, X)$, consider the control function for nonlinear system (1.1) as following :

$$u(t) = u_\lambda(t, x) = B^*\mathcal{V}^*(b, t)R(\lambda, \Gamma_0^b)p(x), \tag{3.1}$$

where,

$$p(x) = \begin{cases} x_b - \mathcal{V}(b, 0)E(x_0 - g(x)) - \int_0^b \mathcal{V}(b, s)f(s, x(s), Kx(s), Hx(s))ds, & t \in [0, t_1], \\ x_b - \mathcal{V}(b, s_i)E\gamma_i(s_i, x(s_i)) - \int_{s_i}^b \mathcal{V}(b, s)f(s, x(s), Kx(s), Hx(s))ds, & t \in \cup_{i=1}^m (s_i, t_{i+1}]. \end{cases}$$

For any $\lambda > 0$, define the operator $F_\lambda : PC(J, X) \to PC(J, X)$ as following :

$$(F_\lambda x)(t) = (\Phi_\lambda x)(t) + (\Psi_\lambda x)(t), \tag{3.2}$$

where

$$(\Phi_\lambda x)(t) = \begin{cases} \mathcal{V}(t, 0)E(x_0 - g(x)) + \int_0^t \mathcal{V}(t, s)f(s, x(s), Kx(s), Hx(s))ds, & t \in [0, t_1], \\ \gamma_i(t, x(t)), & t \in \cup_{i=1}^m (t_i, s_i], \\ \mathcal{V}(t, s_i)E\gamma_i(s_i, x(s_i)) + \int_{s_i}^t \mathcal{V}(t, s)f(s, x(s), Kx(s), Hx(s))ds, & t \in \cup_{i=1}^m (s_i, t_{i+1}]. \end{cases} \tag{3.3}$$

$$(\Psi_\lambda x)(t) = \begin{cases} \int_{s_i}^t \mathcal{V}(t, s)Bu_\lambda(s, x)ds, & t \in \cup_{i=0}^m (s_i, t_{i+1}], \\ 0, & \text{otherwise}. \end{cases} \tag{3.4}$$

Now, we introduce the assumptions that are needed to prove our objective.

(H1) $\mathcal{S}(t,s)$ is a compact evolution system for $0 \leqslant s < t \leqslant b$.

(H2) The function $f(t,\cdot,\cdot,\cdot)$ from $X \times X \times X$ to X is continuous for each fixed $t \in J$ and the function $f(\cdot,x,y,z)$ from J to X is Lebesgue measurable for all fixed $(x,y,z) \in X \times X \times X$, and there exists a constant $L_1 > 0$ such that for all $\varrho \in J$ and $x_i, y_i, z_i \in X$ $(i = 1,2)$,

$$\|f(\varrho,x_1,y_1,z_1) - f(\varrho,x_2,y_2,z_2)\| \leqslant L_1\{\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|\}.$$

(H3) $k : \triangle \times X \to X$ is continuous, and there exists a positive constant L_2 such that

$$\|k(\varrho,s,x_1) - k(\varrho,s,x_2)\| \leqslant L_2\|x_1 - x_2\|, \quad \text{for } (\varrho,s) \in \triangle,\ x_1\ ,\ x_2 \in X.$$

(H4) $h : \triangle \times X \to X$ is continuous, and there exists a positive constant L_3 such that

$$\|h(\varrho,s,x_1) - h(\varrho,s,x_2)\| \leqslant L_3\|x_1 - x_2\|, \quad \text{for } (\varrho,s) \in \triangle,\ x_1\ ,\ x_2 \in X.$$

(H5) g is continuous function and there exists a positive constant $L4$ such that

$$\|E(g(x_1) - g(x_2))\| \quad \leqslant \quad L_4\|x_1 - x_2\|, \quad \forall\, x_1\ ,\ x_2 \in PC(J,X).$$

(H6) γ_i is continuous and there exist positive constants α_i, δ_i , $i = 1,2,\ldots,m$, such that

$$\begin{aligned} \|\gamma_i(\varrho,x_1) - \gamma_i(\varrho,x_2)\| &\leqslant \alpha_i\|x_1 - x_2\|, \quad \forall\, x_1\ ,\ x_2 \in X,\ \varrho \in [t_i,s_i], \\ \|E(\gamma_i(\varrho,x_1) - \gamma_i(\varrho,x_2))\| &\leqslant \delta_i\|x_1 - x_2\|, \quad \forall\, x_1\ ,\ x_2 \in X,\ \varrho \in [t_i,s_i]. \end{aligned}$$

(H7) (A_1)-(A_3) and (S_1)-(S_2) hold.

For convenience, we use the following notations :

$$\begin{aligned} N_1 &= \sup_{t\in J}\|f(t,0,0,0)\|, \quad N_2 = \sup_{(t,s)\in\triangle}\|k(t,s,0)\|, \quad N_3 = \sup_{(t,s)\in\triangle}\|h(t,s,0)\|, \\ N_4 &= \max_{i=1,2,\ldots,m}\{\sup_{t\in J}\|\gamma_i(t,0)\|\}, \quad \alpha = \max_{i=1,2,\ldots,m}\alpha_i, \quad \delta = \max_{i=1,2,\ldots,m}\delta_i, \\ K_1 &= \left(L_1\left[r + (L_2 r + N_2)b + (L_3 r + N_3)b\right] + N_1\right)b, \quad M_1 = \|B\|, \quad M_2 = \|E^{-1}\|. \end{aligned}$$

Lemma 3.1 — If the assumptions $(H2)$-$(H4)$ hold, then for $x \in \Omega_r$ we have

(i) $\int_0^t \|k(t,\eta,x(\eta))\|d\eta \leqslant (L_2 r + N_2)b$, for all $t \in J$.

(ii) $\int_0^b \|h(t,\eta,x(\eta))\|d\eta \leqslant (L_3 r + N_3)b$, for all $t \in J$

(iii) $\int_0^t \|f(\eta, x(\eta), Kx(\eta), Hx(\eta))\| d\eta \leqslant K_1$, for all $t \in J$.

PROOF : (i) For $t \in J$, by $(H3)$, we have

$$\begin{aligned}\int_0^t \|k(t,\eta,x(\eta))\| d\eta &\leqslant \int_0^t \Big(\|k(t,\eta,x(\eta)) - k(t,\eta,0)\| + \|k(t,\eta,0)\| \Big) d\eta \\ &\leqslant \int_0^t (L_2\|x\| + N_2) d\eta \\ &\leqslant (L_2 r + N_2) b\end{aligned}$$

(ii) The proof is similar as the proof of (i).

(iii) For $t \in J$, using $(H2)$-$(H4)$ and (i), (ii), we estimate

$$\begin{aligned}&\int_0^t \|f(\eta, x(\eta), Kx(\eta), Hx(\eta))\| d\eta \\ &\leqslant \int_0^t \Big(\|f(\eta, x(\eta), Kx(\eta), Hx(\eta)) - f(\eta,0,0,0)\| + \|f(\eta,0,0,0)\| \Big) d\eta \\ &\leqslant \int_0^t \Big(L_1 \Big[\|x\| + \int_0^\eta \|k(\eta,\varrho,x(\varrho))\| d\varrho + \int_0^b \|h(\eta,\varrho,x(\varrho))\| d\varrho \Big] + N_1 \Big) d\eta \\ &\leqslant \Big(L_1 \Big[r + (L_2 r + N_2) b + (L_3 r + N_3) b \Big] + N_1 \Big) b = K_1. \square\end{aligned}$$

***Theorem* 3.2** — *Suppose (H1)-(H7) hold and the functions $Eg(0)$, $\gamma_i(\cdot,0)$ and $E\gamma_i(\cdot,0)$ are bounded for $i = 1,2,\ldots,m$, then there exists a mild solution to the system (1.1), provided that*

$$\Lambda := \max\{\alpha, K_4, K_5\} < 1, \tag{3.5}$$

where $K_4 = M_2 M[L_4 + L_1 b(1 + L_2 b + L_3 b)]$ and $K_5 = M_2 M[\delta + L_1 b(1 + L_2 b + L_3 b)]$.

PROOF : To prove that the system (1.1) has a mild solution, we need to prove F_λ has a fixed point. For convenience, we divide the proof into following steps :

Step I : For any $\lambda > 0$, there exists a constant $R = R(\lambda) > 0$, such that $F_\lambda(\Omega_R) \subset \Omega_R$.

For any positive constant r and $x \in \Omega_r$, if $t \in [0, t_1]$, then by using (3.1), $(H5)$ and Lemma (3.1), we have

$$\begin{aligned}u_\lambda(t,x) &= B^*\mathcal{V}^*(b,t) R(\lambda, \Gamma_0^b) \\ &\quad \Big[x_b - \mathcal{V}(b,0)E(x_0 - g(x)) - \int_0^b \mathcal{V}(b,\eta) f(\eta, x(\eta), Kx(\eta), Hx(\eta)) d\eta \Big] \\ \|u_\lambda(t,x)\| &\leqslant \frac{M_1 M_2 M}{\lambda} \Big[\|x_b\| + M_2 M (\|Ex_0\| + \|E(g(x) - g(0))\| + \|Eg(0)\|) + M_2 M K_1 \Big] \\ &\leqslant \frac{M_1 M_2 M}{\lambda} \Big[\|x_b\| + M_2 M(\|Ex_0\| + L_4 r + \|Eg(0)\|) + M_2 M K_1 \Big] := K_2, \end{aligned} \tag{3.6}$$

and from (3.2), (3.6), we obtain

$$\begin{aligned}(F_\lambda x)(t) &= \mathcal{V}(t,0)E(x_0-g(x))+\int_0^t \mathcal{V}(t,\eta)f(\eta,x(\eta),Kx(\eta),Hx(\eta))d\eta \\ &\quad +\int_0^t \mathcal{V}(t,\eta)Bu_\lambda(\eta,x)d\eta \\ \|(F_\lambda x)(t)\| &\leqslant \|\mathcal{V}(t,0)\|(\|E(x_0)\|+\|Eg(x)\|)+\int_0^t \|\mathcal{V}(t,\eta)\|\|f(\eta,x(\eta),Kx(\eta),Hx(\eta))\|d\eta \\ &\quad +\int_0^t \|\mathcal{V}(t,s)\|\|B\|\|u_\lambda(\eta,x)\|d\eta \\ &\leqslant M_2M(\|Ex_0\|+L_4r+\|Eg(0)\|)+M_2MK_1+M_2MM_1K_2b. \qquad (3.7)\end{aligned}$$

If $t\in(t_i,s_i]$, $i=1,2,\dots,m$, then by (3.2) and $(H6)$, we obtain

$$\begin{aligned}\|(F_\lambda x)(t)\| &= \|\gamma_i(t,x(t))\| \\ &\leqslant \alpha\|x(t)\|+\|\gamma_i(t,0)\| \\ &\leqslant \alpha r+N_4. \qquad (3.8)\end{aligned}$$

If $t\in(s_i,t_{i+1}]$, $i=1,2,\dots,m$ then (3.1), (3.2), $(H6)$ and Lemma (3.1) yield the following estimations,

$$\begin{aligned}u_\lambda(t,x) &= B^*\mathcal{V}^*(b,t)R(\lambda,\Gamma_0^b) \\ &\quad \left[x_b-\mathcal{V}(b,s_i)E\gamma_i(s_i,x(s_i))-\int_{s_i}^b \mathcal{V}(b,\eta)f(\eta,x(\eta),Kx(\eta),Hx(\eta))d\eta\right] \\ \|u_\lambda(t,x)\| &\leqslant \frac{M_1M_2M}{\lambda}\left[\|x_b\|+M_2M(\delta r+\|E\gamma_i(s_i,0)\|)+M_2MK_1\right]:=K_3. \qquad (3.9) \\ \|(F_\lambda x)(t)\| &\leqslant M_2M(\delta r+\|E\gamma_i(s_i,0)\|)+M_2MK_1+M_2MM_1K_3b. \qquad (3.10)\end{aligned}$$

Combining (3.7), (3.8) and (3.10), we get that for large enough $R>0$, $F_\lambda(\Omega_R)\subset\Omega_R$ holds.

Step II : $\Phi_\lambda:\Omega_R\to\Omega_R$ is contraction.

If $t\in[0,t_1]$ and $x,y\in\Omega_R$, using $(H2)$-$(H5)$ we have

$$\begin{aligned}\|(\Phi_\lambda x)(t)-(\Phi_\lambda y)(t)\| &\leqslant \|\mathcal{V}(t,0)E(g(x)-g(y))\|+\int_0^t \|\mathcal{V}(t,\varrho)\|\|f(\varrho,x(\varrho),Kx(\varrho),Hx(\varrho)) \\ &\quad -f(\varrho,y(\varrho),Ky(\varrho),Hy(\varrho))\|d\varrho \\ &\leqslant M_2ML_4\|x-y\|+M_2M\int_0^t L_1 \\ &\quad \left[\|x-y\|+\|Kx-Ky\|+\|Hx-Hy\|\right]d\varrho\end{aligned}$$

$$\begin{aligned} &\leqslant M_2ML_4\|x-y\| + M_2ML_1b(1+L_2b+L_3b)\|x-y\| \\ &\leqslant M_2M[L_4 + L_1b(1+L_2b+L_3b)]\|x-y\| = K_4\|x-y\|. \end{aligned} \tag{3.11}$$

For $t \in (t_i, s_i], i = 1, 2, \ldots, m$ and $x, y \in \Omega_R$, using the assumption $(H6)$, we obtain

$$\|(\Phi_\lambda x)(t) - (\Phi_\lambda y)(t)\| \leqslant \alpha_i\|x(t) - y(t)\| \leqslant \alpha\|x-y\|. \tag{3.12}$$

For $t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m$ and $x, y \in \Omega_R$, we have

$$\|(\Phi_\lambda x)(t) - (\Phi_\lambda y)(t)\| \leqslant M_2M[\delta + L_1b(1+L_2b+L_3b)]\|x-y\| = K_5\|x-y\|. \tag{3.13}$$

From (3.11), (3.12), (3.13) and (3.5), we obtain

$$\|\Phi_\lambda x - \Phi_\lambda y\| \leqslant \Lambda\|x-y\| < \|x-y\|. \tag{3.14}$$

Hence Φ_λ is contraction.

Step III : Ψ_λ is continuous in Ω_R.

Let $\{x_n\}$ be a sequence in Ω_R such that $\lim_{n\to\infty} x_n = x$ in Ω_R. Since f is continuous with respect to state variables, for each $\varrho \in J$, we have

$$\lim_{n\to\infty} f(\varrho, x_n(\varrho), Kx_n(\varrho), Hx_n(\varrho)) = f(\varrho, x(\varrho), Kx(\varrho), Hx(\varrho)).$$

So, we can conclude that

$$\sup_{\varrho\in J}\|f(\varrho, x_n(\varrho), Kx_n(\varrho), Hx_n(\varrho)) - f(\varrho, x(\varrho), Kx(\varrho), Hx(\varrho))\| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.15}$$

For $t \in (0, t_1]$, $(S1)$, $(H5)$, and (3.15) yield the following

$$\begin{aligned} \|p(x_n) - p(x)\| &\leqslant M_2M\|Eg(x_n) - Eg(x)\| \\ &\quad + M_2M\int_{s_i}^{b}\|f(\varrho, x_n(\varrho), Kx_n(\varrho), Hx_n(\varrho)) - f(\varrho, x(\varrho), Kx(\varrho), Hx(\varrho))\|d\varrho \\ &\leqslant M_2M\|Eg(x_n) - Eg(x)\| \\ &\quad + M_2Mb\sup_{\varrho\in J}\|f(\varrho, x_n(\varrho), Kx_n(\varrho), Hx_n(\varrho)) - f(\varrho, x(\varrho), Kx(\varrho), Hx(\varrho))\| \\ &\to 0 \quad \text{as} \quad n\to\infty, \end{aligned} \tag{3.16}$$

For $t \in (s_i, t_{i+1}]$,$i = 1, 2, \dots, m$, using $(S1)$, $(H6)$ and (3.15) we get

$$\begin{aligned}
\|p(x_n) - p(x)\| &\leqslant M_2 M \|E\gamma_i(s_i, x_n(s_i)) - E\gamma_i(s_i, x(s_i))\| \\
&\quad + M_2 M \int_{s_i}^{b} \|f(\varrho, x_n(\varrho), Kx_n(\varrho), Hx_n(\varrho)) - f(\varrho, x(\varrho), Kx(\varrho), Hx(\varrho))\| d\varrho \\
&\leqslant M_2 M \|E\gamma_i(s_i, x_n(s_i)) - E\gamma_i(s_i, x(s_i))\| \\
&\quad + M_2 M b \sup_{\varrho \in J} \|f(\varrho, x_n(\varrho), Kx_n(\varrho), Hx_n(\varrho)) - f(\varrho, x(\varrho), Kx(\varrho), Hx(\varrho))\| \\
&\to 0 \quad \text{as} \quad n \to \infty, \qquad (3.17)
\end{aligned}$$

therefore (3.1), (3.16) and (3.17) imply that

$$\|u_\lambda(\varrho, x_n) - u_\lambda(\varrho, x)\| \to 0 \quad \text{as} \quad n \to \infty, \qquad (3.18)$$

and so

$$\begin{aligned}
\|(\Psi_\lambda x_n)(t) - (\Psi_\lambda x)(t)\| &\leqslant M_2 M M_1 b \sup_{\varrho \in J} \|u_\lambda(\varrho, x_n) - u_\lambda(\varrho, x)\| \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

which means that Ψ_λ is continuous in Ω_R.

Step IV : $\Psi_\lambda : \Omega_R \to \Omega_R$ is compact. For this we need to prove :

(i) For any $t \in J$ the set $\{(\Psi_\lambda x)(t) : x \in \Omega_R\}$ is relatively compact in X.

If $t \notin (s_i, t_{i+1}]$, $i = 0, 1, 2, \dots, m$, obviously the set $\{(\Psi_\lambda x)(t) : x \in \Omega_R\} = \{0\}$ is compact in X. Let $t \in (s_i, t_{i+1}]$, $i = 0, 1, 2, \dots, m$ be fixed. For any $\varepsilon \in (s_i, t)$, we define an operator Ψ_λ^ε on Ω_R as following

$$\begin{aligned}
(\Psi_\lambda^\varepsilon x)(t) &= \int_{s_i}^{t-\varepsilon} \mathcal{V}(t, s) B u_\lambda(s, x) ds \\
&= \int_{s_i}^{t-\varepsilon} E^{-1} \mathcal{S}(t, t-\varepsilon) \mathcal{S}(t-\varepsilon, s) B u_\lambda(s, x) ds \\
&= E^{-1} \mathcal{S}(t, t-\varepsilon) \int_{s_i}^{t-\varepsilon} \mathcal{S}(t-\varepsilon, s) B u_\lambda(s, x) ds \\
&= E^{-1} \mathcal{S}(t, t-\varepsilon) y(t, \varepsilon).
\end{aligned}$$

Since E^{-1} and $\mathcal{S}(t, t-\varepsilon)$ is compact in X, and $y(t, \varepsilon)$ is bounded on Ω_R, we obtain that the set $\{(\Psi_\lambda^\varepsilon x)(t) : x \in \Omega_R\}$ is precompact in X. On the other hand

$$\begin{aligned}
\|(\Psi_\lambda x)(t) - (\Psi_\lambda^\varepsilon x)(t)\| &\leqslant \int_{t-\varepsilon}^{t} \|\mathcal{V}(t, s) B u_\lambda(s, x)\| ds \\
&\leqslant M_2 M M_1 \|u_\lambda\| \varepsilon \\
&\to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

This implies that for $t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, m$ the set $\{(\Psi_\lambda x)(t) : x \in \Omega_R\}$ is precompact in X.

(ii) The family of functions $\{\Psi_\lambda x : x \in \Omega_R\}$ is equicontinuous. For any $x \in \Omega_R$ and $s_i \leqslant t' < t'' \leqslant t_{i+1}$ for $i = 0, 1, 2, \dots, m$, we have

$$\begin{aligned}
\|(\Psi_\lambda x)(t'') - (\Psi_\lambda x)(t')\| &= \left\| \int_{s_i}^{t''} E^{-1}\mathcal{S}(t'', s)Bu_\lambda(s, x)ds - \int_{s_i}^{t'} E^{-1}\mathcal{S}(t', s)Bu_\lambda(s, x)ds \right\| \\
&\leqslant \left\| \int_{s_i}^{t'} E^{-1}[\mathcal{S}(t'', s) - \mathcal{S}(t', s)]Bu_\lambda(s, x)ds \right\| \\
&\quad + \left\| \int_{t'}^{t''} E^{-1}\mathcal{S}(t'', s)Bu_\lambda(s, x)ds \right\| \\
&\leqslant J_1 + J_2.
\end{aligned}$$

For $t' = s_i$, it is easy to see that $J_1 = 0$. For $t' > s_i$ and $\epsilon > 0$ small enough, we obtain

$$\begin{aligned}
J_1 &\leqslant \left\| \int_{s_i}^{t'-\epsilon} E^{-1}[\mathcal{S}(t'', s) - \mathcal{S}(t', s)]Bu_\lambda(s, x)ds \right\| + \\
&\quad \left\| \int_{t'-\epsilon}^{t'} E^{-1}[\mathcal{S}(t'', s) - \mathcal{S}(t', s)]Bu_\lambda(s, x)ds \right\| \\
&\leqslant M_2 M_1 \|u_\lambda\| \sup_{s \in [s_i, t'-\epsilon]} \|\mathcal{S}(t'', s) - \mathcal{S}(t', s)\|(t' - \epsilon - s_i) + 2M_2 M M_1 \|u_\lambda\| \epsilon \\
&\to 0 \quad \text{as} \quad t'' \to t',\ \epsilon \to 0,
\end{aligned}$$

since the family of operators $\{\mathcal{S}(t, s) : t > s\}$ is continuous in t uniformly for s in the uniform operator topology. Now,

$$\begin{aligned}
J_2 &\leqslant M_2 M M_1 \|u_\lambda\|(t'' - t') \\
&\to 0 \quad \text{as} \quad t'' \to t'.
\end{aligned}$$

As a result $\|(\Psi_\lambda x)(t'') - (\Psi_\lambda x)(t')\| \to 0$ independently of $x \in \Omega_R$ as $t'' \to t'$, that means that $\{\Psi_\lambda x : x \in \Omega_R\}$ is equicontinuous. Thus, from Arzela-Ascoli theorem, Ψ_λ is compact on Ω_R.

Therefore, Krasnoselskii fixed point theorem implies that F_λ has a fixed point, which is a mild solution of the Cauchy problem (1.1). □

Now, we are ready to discuss the approximate controllability of the system (1.1). In order to prove it, the following hypotheses are also required :

(H8) $\delta R(\delta, \Gamma_0^b) \to 0$ as $\delta \to 0^+$ in the strong operator topology.

(H9) There exist constants $L_5 > 0,\ L_6 > 0$, and $\beta > 0$ such that

$$\begin{aligned}\|Eg(x)\| &\leqslant L_5, \quad \forall\, x \in PC(J,X),\\ \|f(t,x,y,z)\| &\leqslant L_6, \quad \forall (t,x,y,z) \in J \times X \times X \times X,\\ \|E\gamma_i(t,x)\| &\leqslant \beta, \quad \forall x \in X,\ t \in [t_i, s_i],\ i = 1,2,\dots,m.\end{aligned}$$

***Theorem* 3.3** — *Suppose that the hypotheses of Theorem 3.2 hold aa well as assumptions $(H8)$ and $(H9)$ are satisfied, then the system (1.1) is approximately controllable on J.*

PROOF : Theorem 3.2 guaranteed that F_λ has a fixed point in Ω_R. Let x_λ is a mild solution of (1.1) under the control $u_\lambda(t, x_\lambda)$ given by (3.1) and satisfies

$$\begin{aligned} x_\lambda(b) &= \mathcal{V}(b,s_m)E\gamma_m(s_m, x_\lambda(s_m)) + \int_{s_m}^b \mathcal{V}(b,s)[f(s,x_\lambda(s),Kx_\lambda(s),Hx_\lambda(s)) + Bu_\lambda(s,x_\lambda)]ds\\ &= x_b - p(x_\lambda) + \int_{s_m}^b \mathcal{V}(b,s)Bu_\lambda(s,x_\lambda)ds\\ &= x_b - p(x_\lambda) + \int_{s_m}^b \mathcal{V}(b,s)BB^*\mathcal{V}^*(b,s)R(\lambda,\Gamma_0^b)p(x_\lambda)ds\\ &= x_b - p(x_\lambda) + \Gamma_{s_m}^b R(\lambda,\Gamma_{s_m}^b)p(x_\lambda)\\ &= x_b - \lambda R(\lambda,\Gamma_{s_m}^b)p(x_\lambda). \end{aligned} \tag{3.19}$$

where,

$$p(x_\lambda) = \begin{cases} x_b - \mathcal{V}(b,0)E(x_0 - g(x_\lambda)) - \int_0^b \mathcal{V}(b,s)f(s,x_\lambda(s),Kx_\lambda(s),Hx_\lambda(s))ds, t \in [0,t_1],\\ x_b - \mathcal{V}(b,s_i)E\gamma_i(s_i,x_\lambda(s_i)) - \int_{s_i}^b \mathcal{V}(b,s)f(s,x_\lambda(s),Kx_\lambda(s),Hx_\lambda(s))ds,\\ \qquad t \in \cup_{i=1}^m (s_i, t_{i+1}]. \end{cases}$$

According to the compactness of E^{-1}, $\mathcal{S}(t,s)$ and the uniform boundedness of Eg, we see that there exists a subsequence of $\{\mathcal{V}(b,0)Eg(x_\lambda) : \lambda > 0\}$, still denoted by it, converges to some $x_g \in X$ as $\lambda \to 0$. Similarly there exists a subsequence of $\{\mathcal{V}(b,s_i)E\gamma_i(s_i,x_\lambda(s_i)) : \lambda > 0\}$, still denoted by it, converges to some $x_{\gamma_i} \in X$ as $\lambda \to 0$. By the assumption that f is uniformly bounded, we have

$$\int_0^b \|f(s,x_\lambda(s),Kx_\lambda(s),Hx_\lambda(s))\|^2 ds \leqslant L_6^2 b.$$

Hence the sequence $f(\cdot,x_\lambda(\cdot),Kx_\lambda(\cdot),Hx_\lambda(\cdot))$ is bounded in $L^2(J,X)$. Then there exists a subsequence of $\{f(\cdot,x_\lambda(\cdot),Kx_\lambda(\cdot),Hx_\lambda(\cdot)) : \lambda > 0\}$, still denoted by it, converges weakly to some $f(\cdot) \in L^2(J, X)$. Define

$$\omega = \begin{cases} x_b - \mathcal{V}(b,0)Ex_0 + x_g - \int_0^b \mathcal{V}(b,s)f(s)ds, & t \in [0, t_1], \\ x_b - x_{\gamma_i} - \int_{s_i}^b \mathcal{V}(b,s)f(s)ds, & t \in \cup_{i=1}^m (s_i, t_{i+1}]. \end{cases}$$

By the compactness of $\mathcal{V}(t,s)$, it follows that, for $t \in (0, t_1]$

$$\begin{aligned} \|p(x_\lambda) - \omega\| &\leqslant \|\mathcal{V}(b,0)Eg(x_\lambda) - x_g\| + M\int_0^b \|f(s, x_\lambda(s), Kx_\lambda(s), Hx_\lambda(s)) - f(s)\|ds \\ &\to 0 \quad \text{as} \quad \lambda \to 0^+, \end{aligned} \tag{3.20}$$

in the same way, for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we get

$$\|p(x_\lambda) - \omega\| \to 0 \quad \text{as} \quad \lambda \to 0^+. \tag{3.21}$$

Then, from (3.19), (3.20), (3.21) and $(H8)$, we obtain

$$\begin{aligned} \|x_\lambda(b) - x_b\| &\leqslant \|\lambda R(\lambda, \Gamma_{s_m}^b)p(x_\lambda)\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_{s_m}^b)\omega\| + \|\lambda R(\lambda, \Gamma_{s_m}^b)\|\|p(x_\lambda) - \omega\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_{s_m}^b)\omega\| + \|p(x_\lambda) - \omega\| \to 0 \text{ as } \lambda \to 0^+. \end{aligned}$$

Hence, (1.1) approximate controllability on J. □

4. Example

Consider a control system governed by the following partial differential equation :

$$\begin{aligned} \frac{\partial}{\partial t}[x(t,z) &- x_{zz}(t,z)] + [a(t,z) + \frac{\partial^2}{\partial z^2}]x(t,z) = \mu(t,z) + \sin x(t,z) \\ &+ \int_0^t \frac{1}{c_1} e^{-s} \frac{|x(s,z)|}{1 + |x(s,z)|} ds + \int_0^1 \frac{1}{s + c_2} \sin x(s,z) ds, \\ &z \in (0,\pi), \ \ t \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1] \\ x(t,0) &= x(t,1) = 0, \ \ t \in [0,1] \\ x(t,z) &= \frac{e^{-t}}{c_3(1 + e^{-t})} \sin x(t,z), \ \ z \in (0,\pi), t \in (\frac{1}{3}, \frac{2}{3}] \\ x(0,z) + \frac{e^t}{c_4(1 + e^t)} \cos x(t,z) &= x_0(z), \ \ z \in [0,\pi], \end{aligned} \tag{4.1}$$

where $X = U = L^2[0,\pi]$, $x_0(z) \in D(E)$, $a(t,z) \in C^1([0,1] \times [0,1], \mathbb{R})$, $J = [0,1]$, i.e. $b = 1$, and c_1, c_2, c_3, c_4 are positive constants. Define

$$A(t)x(t,z) = [a(t,z) + \frac{\partial^2}{\partial z^2}]x(t,z), \quad Ex = x - x_{zz}$$

where $\frac{\partial^2}{\partial z^2}$ is distributional derivative, and $D(A(t)),\ D(E)$ is given by $H^2(0,1)\cap H_0^1(0,1)$. It is well known that $-A(t)$ generates a compact evolution system of bounded linear operators $W(t,s)$ on X and is given by (see [12])

$$W(t,s)x = T(t-s)e^{\int_s^t a(\tau)d\tau}x, \quad x \in D(A(t)).$$

Here

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n,$$

with $e_n(z) = \sqrt{2}\sin(nz)$, $0 \leqslant z \leqslant \pi$, $n = 1, 2, \ldots$. The operator E can be written as following (see [21])

$$Ex = \sum_{n=1}^{\infty}(1+n^2) < x, e_n > e_n,\ x \in D(E).$$

Furthermore for $x \in X$, we have

$$E^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1+n^2} < x, e_n > e_n,$$

is compact since $\lim_{n\to\infty} \frac{1}{1+n^2} = 0$. So, $-A(t)E^{-1}$ generates a compact evolution system of bounded linear operators that is given as

$$\mathcal{S}(t,s)x = U(t-s)e^{\int_s^t a(\tau)d\tau}x,$$

where

$$U(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t} < x, e_n > e_n.$$

Hence assumptions $(H1),\ (H7)$ holds. Let $t_2 = 1, t_0 = s_0 = 0, t_1 = \frac{1}{3}, s_1 = \frac{2}{3}$. Put $x(t) = x(t,\cdot)$, that is $x(t)(z) = x(t,z)$, $t \in [0,1]$, $z \in [0,\pi]$ and $u(t) = \mu(t,\cdot)$ is continuous. Let the operator $B : U \to X$ is defined as $Bu(t)(z) = \mu(t,z)$. Further

$$\begin{aligned}
f(t,x(t),Kx(t),Hx(t))(z) &= \sin x(t,z) + \int_0^t \frac{1}{c_1}e^{-s}\frac{|x(s,z)|}{1+|x(s,z)|}ds + \int_0^1 \frac{1}{s+c_2}\sin x(s,z)ds,\\
Kx(t)(z) &= \int_0^t \frac{1}{c_1}e^{-s}\frac{|x(s,z)|}{1+|x(s,z)|}ds,\\
Hx(t)(z) &= \int_0^1 \frac{1}{s+c_2}\sin x(s,z)ds,\\
\gamma_1(t,x(t))(z) &= \frac{e^{-t}}{c_3(1+e^{-t})}\sin x(t,z),\\
g(x) &= \frac{e^t}{c_4(1+e^t)}\cos x(t),
\end{aligned}$$

then the system (4.1) can be rewritten into the abstract form of (1.1) for $m = 1$. Note that $E\gamma_1(t,x) = \frac{2e^{-t}}{c_3(1+e^{-t})}\sin x(t)$ and $Eg(x) = \frac{2e^{t}}{c_4(1+e^{t})}\cos x(t)$. It is easy to see that the functions f, k, h, g, γ_1 satisfies the hypotheses $(H2)$-$(H6)$, and also $f, Eg, E\gamma_1$ are uniformly bounded.

Now we check that associated linear system is approximately controllable, for this we need to show that

$$B^*\mathcal{V}^*(b,s)x = 0,\ 0 \leqslant s < b \Rightarrow x = 0, \tag{4.2}$$

where $\mathcal{V}(t,s) = E^{-1}\mathcal{S}(t,s)$. Notice that $\mathcal{S}$ and E^{-1} are self adjoint. Indeed,

$$\begin{aligned} B^*\mathcal{V}^*(b,s)x &= \mathcal{V}^*(b,s)x = \mathcal{S}^*(b,s)(E^{-1})^*x = \mathcal{S}(b,s)E^{-1}x \\ &= e^{\int_s^b a(\tau)d\tau}\sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}(b-s)} < E^{-1}x, e_n > e_n \\ &= e^{\int_s^b a(\tau)d\tau}\sum_{n=1}^{\infty} \frac{1}{1+n^2} e^{\frac{-n^2}{1+n^2}(b-s)} < x, e_n > e_n. \end{aligned}$$

Therefore the condition (4.2) holds, and hence the assumption $(H8)$. Thus by Theorem 3.3 the system (4.1) is approximately controllable on J.

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