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FIXED POINT THEOREMS FOR φ-CONTRACTION MAPPINGS IN PROBABILISTIC GENERALIZED MENGER SPACE

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In this paper, we define the concepts of φ-contraction in probabilistic generalized Menger space. We prove that φ-contraction mappings in probabilistic generalized Menger space have a unique fixed point. Also we prove some fixed point theorems related this concept. Some examples are given to support the obtained results.

Key words : Probabilistic $\mathbb{G}$-Menger space; contraction mappings; continuous t-norm; fixed point; distribution function.

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1. Introduction and Preliminaries

In 1963 Gahler introduced 2-metric spaces [4] and Ha *et al*. announced 2-metric spaces and usual metric spaces aren't related to each other [5]. Dhage [3] presented the concept of generalized metric spaces. After it Mustafa and Sims expressed a new approach to this spaces [10, 11]. This concept, specially in related fixed point theory, has been further used in many works, [8, 9, 12]. Similar work can be found in [13, 15]. Altering distance function is defined by Khan *et al*. [6]. This concept is further extended to Menger spaces by Choudhury and Das in [2], where a generalized contractive condition has been defined with the help of such function and a unique fixed point result has been established. In 2014, Zhou *et al*. defined the Menger probabilistic G-metric space and defined some basic concepts in this space [16]. Afterward in 2015 Alsulami and his coworkers investigated φ-contraction mappings in PGM-spaces [1]. In this paper, we define the concepts of φ-contraction mappings in probabilistic generalized Menger space and prove that φ-contraction mappings in probabilistic generalized Menger space have a unique fixed point. We first bring notions, definitions and known results, which are related to our work. The reader may gain extra information referring to [7, 14] and [16].

Definition 1.1 — If a mapping $F : (0,\infty) \to [0,\infty)$ is non-decreasing, left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$, is called a distribution function.

We will refer by D the set of all the distribution functions and the certain distribution function of this set referred by

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Definition 1.2 — If a mapping $T : [0,1] \times [0,1] \to [0,1]$ is associative, commutative, non-decreasing and $T(a,1) = a$, is called t-norm.

Definition 1.3 — [16]. A Menger probabilistic $\mathbb{G}$-metric space (shortly, $P\mathbb{G}M$-space) is a triplet $(S, \mathbb{G}^*, T)$, where S is a non-empty set, T is a continuous $t-$norm and $\mathbb{G}^*$ is a function from S^3 to the D satisfying the following conditions:

(a) $\mathbb{G}^*_{x,y,z}(s) = 1,\ \ for\ all\ x,y,z \in S$ and $s > 0$ iff $x = y = z$;

(b) $\mathbb{G}^*_{x,x,y}(s) \geq \mathbb{G}^*_{x,y,z}(s),$ for all $x,y,z \in S$ with $z \neq y$ and $s > 0$;

(c) $\mathbb{G}^*_{x,y,z}$ is a symmetric function of its three variables;

(d) $\mathbb{G}^*_{x,y,z}(t+s) \geq T\left(\mathbb{G}^*_{x,a,a}(t), \mathbb{G}^*_{a,y,z}(s)\right)$, for all $x,y,z,a \in S$ and $s,t > 0$.

Definition 1.4 — [16]. Let $(S,\mathbb{G}^*,T)$ be a $P\mathbb{G}M-$space and x_0 be any point in S, for any $\varepsilon > 0$ and λ with $0 < \lambda < 1$, $an\ (\varepsilon,\lambda) - neighborhood$ of x_0 is the set of all points $y \in S$ for which $\mathbb{G}^*_{x_0,y,y}(\varepsilon) > 1-\lambda$ and $\mathbb{G}^*_{y,x_0,x_0}(\varepsilon) > 1-\lambda$. We write

$$N_{x_0}(\varepsilon,\lambda) = \left\{y \in X : \mathbb{G}^*_{x_0,y,y}(\varepsilon) > 1-\lambda \text{ and } \mathbb{G}^*_{y,x_0,x_0}(\varepsilon) > 1-\lambda\right\},$$

which means that $N_{x_0}(\varepsilon,\lambda)$ is the set of all points $y \in S$ for which the probability of the distance from x_0 to y being less than ϵ is greater than $1-\lambda$.

Definition 1.5 — [16]. Let $(S,\mathbb{G}^*,T)$ be a $P\mathbb{G}M-$space.

(1) A sequence $\{x_n\} \subset S$ is said to be convergent to $x \in S$ $(x_n \to x)$ if, for every $\varepsilon > 0$ and $0 < \lambda < 1$ we can find a positive integer $M_{\varepsilon,\lambda}$ such that for all $n > M_{\epsilon,\lambda}$,

$$\mathbb{G}^*_{x,x_n,x_n}(\epsilon) \geq 1-\lambda.$$

(2) A sequence $\{x_n\}$ is called Cauchy sequence if given $\varepsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer $M_{\varepsilon,\lambda}$ such that $\mathbb{G}^*_{x_n,x_m,x_l}(\varepsilon) > 1-\lambda$ for all $n, m, l > M_{\varepsilon,\lambda}$.

(3) A $PGM-$space $(S, \mathbb{G}^*, T)$ is said to be complete if every Cauchy sequence in S is convergent to a point x in S.

Definition 1.6 — A t-norm T is said to be a Had$\hat{z}ic$-type if family $\{T^n\}_{n>0}$ of its iterates defined for each $t \in [0,1]$ by $T^1(t) = T(t,t)$ and, in general, for all $n > 1, T^n(t) = T(t, T^{n-1}(t))$ is equi-continuous at $t = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0,1)$ such that

$$\eta(\lambda) < t \leq 1 \Rightarrow T^n(\eta(\lambda)) \geq 1-\lambda \;\; for\ all\ n > 0.$$

Remark 1.7 : [17]. Let $(S, \mathbb{G}^*, T)$ be a $P\mathbb{G}M$-space and $\{x_n\}$ be a sequence in S. Then the following are equivalent:

(i) $\{x_n\}$ is convergent to a point $x \in S$,

(ii) $\mathbb{G}^*_{x,x,x_n}(t) \to 1$ as $n \to \infty$, for all $t > 0$,

(iii) $\mathbb{G}^*_{x_n,x_n,x}(t) \to 1$ as $n \to \infty$, for all $t > 0$.

***Theorem* 1.8** — *[16]. Let $(S, \mathbb{G}^*, T)$ be a $P\mathbb{G}M-$space. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in S and $x, y, z \in S$. If $x_n \to x,\ y_n \to y$ and $z_n \to z$ as $n \to \infty$ then, for any $t > 0$, $\mathbb{G}^*_{x_n,y_n,z_n}(t) \to \mathbb{G}^*_{x,y,z}(t)$ as $n \to \infty$.*

Alsulami *et al*. [1] defined a function $\varphi : [0,\infty) \to [0,\infty)$ such that for all $t > 0$, there exists r with $r \geq t$ such that $\lim_{n\to\infty} \varphi^n(r) = 0$ and the collection of all φ-functions denoted by ϕ_w. While regarding above but we will use the definition of Choudhury, is presented in [2], that it is: $\varphi : [0,\infty) \to [0,\infty]$ is a strictly increasing, left continuous function such that φ is continuous at 0, $\varphi(t) = 0 \Leftrightarrow t = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Also we show the collection of such functions by ϕ. Furthermore, it should be notified that the collection of function ϕ_w is different from the collection ϕ, for example, we consider $\varphi(t) = t$, then, $\varphi \in \phi$ but φ is not belong to ϕ_w.

The collection of all continuous non-decreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$ and $\psi^n(a_n) \to 0$, whenever $a_n \to 0$ as $n \to \infty$, is shown by Ψ.

2. φ-Contraction Mappings and Fixed Point

In this section we introduce the notion of φ-contraction and prove some fixed point theorems about this type of contraction mappings in $P\mathbb{G}M$-space.

Definition 2.1 — Let $(S, \mathbb{G}^*, T)$ be a $P\mathbb{G}M$-space and f be a self map on S. The mapping $f : S \to S$ will be called φ-contraction if

$$\mathbb{G}^*_{fx,fy,fz}(\varphi(t)) \geq \mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c})), \tag{2.1}$$

where $\varphi \in \Phi,\ 0 < c < 1,\ t > 0\ and\ x, y, z \in S$.

Lemma 2.2 — Let $(S, \mathbb{G}^*, T)$ be a $P\mathbb{G}M$-space with T of Hadzic-type and $\{x_n\}$ be a sequence in S. If we have

$$\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})), \tag{2.2}$$

where $\varphi \in \Phi,\ 0 < c < 1,\ t > 0,\ n \geq 1\ and\ x, y, z \in S$, then $\{x_n\}$ is a Cauchy sequence in S.

PROOF : By induction on (2.2), we get

$$\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq \mathbb{G}^*_{x_0,x_1,x_1}(\varphi(\frac{t}{c^n})).$$

Letting $n \to \infty$, for any $t > 0$ we have

$$\lim_{n\to\infty} \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) = 1. \tag{2.3}$$

Now, let us prove that

$$\mathbb{G}^*_{x_n,x_{n+k},x_{n+k}}(\varphi(t)) \geq T^k(\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - \varphi(ct))), \quad for\ k > 0. \tag{2.4}$$

By induction, for $k = 1$, it is clear. Now assume (2.4) holds for some $k > 1$. Then, we have

$$\begin{aligned}
\mathbb{G}^*_{x_n,x_{n+k+1},x_{n+k+1}}(\varphi(t)) &= \mathbb{G}^*_{x_n,x_{n+k+1},x_{n+k+1}}(\varphi(t) - \varphi(ct) + \varphi(ct)) \\
&\geq T(\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - \varphi(ct)), \mathbb{G}^*_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\varphi(ct)) \\
&\geq T\big(\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - \varphi(ct)), \mathbb{G}^*_{x_n,x_{n+k},x_{n+k}}(\varphi(t))\big) \\
&\geq T\big(\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - \varphi(ct)), T^k(\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - \varphi(ct)))\big) \\
&= T^{k+1}(\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - \varphi(ct))),
\end{aligned} \tag{2.5}$$

so (2.4) holds. Let us now show that $\{x_n\}$ is a Cauchy sequence in S. At first we prove that $\lim_{m,n\to\infty} \mathbb{G}^*_{x_n,x_m,x_m}(\varphi(t)) = 1$ for any $t > 0$. Let $t > 0$ and $\epsilon > 0$ be given. Because of property of T, there exists $\delta > 0$ such that, for all $s \in (1 - \delta, 1]$,

$$T^n(s) > 1 - \epsilon, \quad for\ all\ n \geq 1. \tag{2.6}$$

From (2.3), we have $\lim_{n\to\infty} \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) - \varphi(ct)) = 1$. Thus there exists $n_0 \in \mathbb{N}$ such that $\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - \varphi(ct)) \in (1-\delta, 1]$ for any $n \geq n_0$. Hence by (2.4) and (2.6), we get $\mathbb{G}^*_{x_n,x_{n+k},x_{n+k}}(\varphi(t)) > 1-\epsilon$ for any $k > 0$. This shows $\lim_{m,n\to\infty} \mathbb{G}^*_{x_n,x_m,x_m}(\varphi(t)) = 1$, for any $t > 0$ and $\varphi \in \Phi$. From properties of $(P\mathbb{G}M)$-space, we have

$$G^*_{x_n,x_m,x_l}(\varphi(t)) \geq T(\mathbb{G}^*_{x_n,x_n,x_m}(\frac{\varphi(t)}{2}), \mathbb{G}^*_{x_n,x_n,x_l}(\frac{\varphi(t)}{2})). \tag{2.7}$$

On the other hand

$$\mathbb{G}^*_{x_n,x_n,x_m}(\frac{\varphi(t)}{2}) \geq T(\mathbb{G}^*_{x_n,x_m,x_m}(\frac{\varphi(t)}{4}), \mathbb{G}^*_{x_n,x_m,x_m}(\frac{\varphi(t)}{4})), \tag{2.8}$$

and

$$\mathbb{G}^*_{x_n,x_n,x_l}(\frac{\varphi(t)}{2}) \geq T(\mathbb{G}^*_{x_n,x_l,x_l}(\frac{\varphi(t)}{4}), \mathbb{G}^*_{x_n,x_l,x_l}(\frac{\varphi(t)}{4})). \tag{2.9}$$

Therefore, from (2.8), (2.9) and since T is continuous, in (2.7) we get

$$\lim_{m,n,l\to\infty} \mathbb{G}^*_{x_n,x_m,x_l}(\varphi(t)) = 1, \tag{2.10}$$

for any $t > 0$. On the other hand, for $s > 0$ we can choose $t > 0$ such that $\varphi(t) < s$. Hence from (2.10) we have $\lim_{m,n,l\to\infty} G^*_{x_n,x_m,x_l}(s) = 1$. This shows that $\{x_n\}$ is a Cauchy sequence in S. This completes the proof. □

Lemma 2.3 — Assume that $(S, \mathbb{G}^*, T)$ be a $P\mathbb{G}M-$ space and $\varphi \in \phi$. If there exists a $c \in (0, 1)$, such that

$$\mathbb{G}^*_{x,y,z}(\varphi(t)) \geq \mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c})), \quad t > 0$$

where $x, y, z \in S$, then $x = y = z$.

PROOF : By induction, we have

$$\mathbb{G}^*_{x,y,z}(\varphi(t)) \geq \mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c})) \geq \mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c^2})) \geq \cdots \geq \mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c^n})).$$

Letting $n \to \infty$ we obtain $\mathbb{G}^*_{x,y,z}(\varphi(t)) = 1$, that is, $x = y = z$. □

Corollary 2.4 — Let $(S, \mathbb{F}, T)$ be a PM-space with T of Had$\hat{z}$i$\acute{c}$-type and $\{x_n\}$ be a sequence in S. If there exists a constant $c \in (0, 1)$ such that

$$\mathbb{F}_{x_n,x_{n+1}}(\varphi(t)) \geq \mathbb{F}_{x_{n-1},x_n}(\varphi(\frac{t}{c})), \quad for\ all\ n \geq 1\ and\ t > 0,$$

then $\{x_n\}$ is a Cauchy sequence.

PROOF : Define $\mathbb{G}^*_{x,y,z}(\varphi(t)) = \min\{\mathbb{F}_{x,y}(\varphi(t)), \mathbb{F}_{y,z}(\varphi(t)), \mathbb{F}_{x,z}(\varphi(t))\}$ for all $x, y, z \in S$ and all $t > 0$. Examples 1.6 in [16] shows that $(X, \mathbb{G}^*, T)$ is a $P\mathbb{G}M$-space. Since $\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) = \mathbb{F}_{x_n,x_{n+1}}(\varphi(t))$ and $\mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})) = \mathbb{F}_{x_{n-1},x_n}(\varphi(\frac{t}{c}))$, hence $\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c}))$ for all $n \geq 1$ and $t > 0$. By Lemma 2.1 we conclude that $\{x_n\}$ is a Cauchy sequence in the sense of $P\mathbb{G}M$-space $(S, \mathbb{G}^*, T)$. By the definition of $\mathbb{G}^*$ we have

$$\min\{\mathbb{F}_{x_n,x_m}(\epsilon), \mathbb{F}_{x_m,x_l}(\epsilon), \mathbb{F}_{x_n,x_l}(\epsilon)\} > 1 - \delta, \quad m, l, n > M_{\epsilon,\delta}.$$

This shows that $\{x_n\}$ is a Cauchy sequence in the sense of PM-space $(S, \mathbb{F}, T)$. □

***Theorem* 2.5** — *Suppose $(S, \mathbb{G}^*, T)$ be a complete $P\mathbb{G}M$-space with T of Hadžić-type. Let $f : S \longrightarrow S$ be a φ-contraction. Then for any $x_0 \in S$, the sequence $\{f^n x_0\}$ converges to unique fixed point of f.*

PROOF : Let x_0 in S be an arbitary point. Let us consider a sequence $\{x_n\}$ by $x_{n+1} = f^n x_0$ for all $n \geq 0$. By (2.1), for any $t > 0$, we have

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &= G^*_{f_{x_{n-1}},f_{x_n},f_{x_n}}(\varphi(t)) \\ &\geq G^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})). \end{aligned}$$

Applying Lemma 2.2 it follows that $\{x_n\}$ is a Cauchy sequence in S. Since S is complete, there exists a point $x \in S$ such that $x_n \to x$ as $n \to \infty$, By (2.1), it follows that

$$G^*_{fx,f_{x_n},f_{x_n}}(\varphi(t)) \geq G^*_{x,x_n,x_n}(\varphi(\frac{t}{c})).$$

Taking lim as $n \to \infty$. Since $x_n \to x$ and $fx_n \to x$, we have

$$G^*_{fx,x,x}(\varphi(t)) = 1,$$

for any $t > 0$. Hence $x = fx$. It is sufficient to show that x is a unique fixed point of the mapping f. Therefore, suppose that f has another fixed point. Therefore for all $t > 0$, we obtain

$$\mathbb{G}^*_{x,y,y}(\varphi(t)) = \mathbb{G}^*_{fx,fy,fy}(\varphi(t)) \geq \mathbb{G}^*_{x,y,y}(\varphi(\frac{t}{c})),$$

so by Lemma 2.3, we conclude that $x = y$. Hence the fixed point of f is unique.

***Theorem* 2.6** — *Let $(S, \mathbb{G}^*, T)$ be a complete $P\mathbb{G}M$-space with T of Hadžić-type. Let $f : S \longrightarrow S$ be a mapping satisfying*

$$\mathbb{G}^*_{fx,fy,fz}(\varphi(t)) \geq \frac{1}{3}[\mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})) + \mathbb{G}^*_{y,fy,fy}(\varphi(\frac{t}{c})) + \mathbb{G}^*_{z,fz,fz}(\varphi(\frac{t}{c}))] \tag{2.11}$$

for all $x, y, z \in S$, where $c \in (0,1)$. Then, for any $x_0 \in S$, the sequence $\{f^n x_0\}$ converges to a unique fixed point of f.

PROOF : Take an arbitrary point x_0 in X. Construct a sequence $\{x_n\}$ by $x_{n+1} = f^n x_0$ for all $n \geq 0$. By (2.11), for any $t > 0$, we have

$$\begin{aligned}\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &= \mathbb{G}^*_{fx_{n-1},fx_n.fx_n}(\varphi(t)) \\ &\geq \frac{1}{3}\mathbb{G}^*_{x_{n-1},fx_{n-1},fx_{n-1}}(\varphi(\frac{t}{c})) + 2\mathbb{G}^*_{x_n,fx_n,fx_n}(\varphi(\frac{t}{c}))] \\ &\geq \frac{1}{3}[\mathbb{G}^*_{x_{n-1},fx_{n-1},fx_{n-1}}(\varphi(\frac{t}{c})) + 2\mathbb{G}^*_{x_n,fx_n,fx_n}(\varphi(t))] \\ &= \frac{1}{3}[\mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})) + 2\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t))]. \end{aligned} \tag{2.12}$$

From (2.12) we get

$$\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) - \frac{2}{3}\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq \frac{1}{3}[\mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})),$$

hence $\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c}))$. According to Lemma 2.2, the sequence $\{x_n\}$ is a Cauchy sequence in S. Since S is complete, there exists a point $x \in S$ such that $x_n \to x$ as $n \to \infty$. By (2.11), it follows that

$$\mathbb{G}^*_{fx,fx,fx_n}(\varphi(t)) \geq \frac{1}{3}[2\mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})) + \mathbb{G}^*_{x_n,fx_n,fx_n}(\varphi(\frac{t}{c}))].$$

Letting $n \to \infty$, since $x_n \to x$ and $fx_n \to x$ as $n \to \infty$, we have, for any $t > 0$,

$$\begin{aligned}\mathbb{G}^*_{fx,fx,x}(\varphi(t)) &\geq \frac{1}{3}[2G^*_{x,fx,fx}(\varphi(\frac{t}{c})) + \mathbb{G}^*_{x,x,x}(\varphi(\frac{t}{c}))] \\ &\geq \frac{1}{3}[2\mathbb{G}^*_{x,fx,fx}(\varphi(t)) + \mathbb{G}^*_{x,x,x}(\varphi(\frac{t}{c}))],\end{aligned}$$

hence, $\mathbb{G}^*_{fx,fx,x}(\varphi(t)) - \frac{2}{3}\mathbb{G}^*_{fx,fx,x}(\varphi(t)) \geq \frac{1}{3}\mathbb{G}^*_{x,x,x}(\varphi(\frac{t}{c}))$. This shows that

$$\mathbb{G}^*_{fx,fx,x}(\varphi(t)) \geq \mathbb{G}^*_{x,x,x}(\varphi(\frac{t}{c})) = 1.$$

Hence $x = fx$.

Next, suppose that y is an another fixed point of f. Then by (2.11), we have for any $y > 0$,

$$\begin{aligned}\mathbb{G}^*_{x,y,y}(\varphi(t)) &= \mathbb{G}^*_{fx,fy,fy}(\varphi(t)) \\ &\geq \frac{1}{3}[\mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})) + 2\mathbb{G}^*_{y,fy,fy}(\varphi(\frac{t}{c}))] \\ &= 1.\end{aligned}$$

This shows that $x = y$. Therefore, f has a unique fixed point in S. This complete the proof. $\square$

***Theorem* 2.7** — *Assume that $(S, \mathbb{G}^*, T)$ be a complete $P\mathbb{G}M$-space, T is continuous, $T(a, a) \geq a$ and let $A_1, A_2, \cdots, A_n$ be non-empty closed subsets in S. If $f : \cup_{i=1}^n A_i \to \cup_{i=1}^n A_i$ satisfies the following conditions:*

(i) $f(A_1) \subseteq A_2, f(A_2) \subseteq A_3, \cdots, f(A_{n-1}) \subseteq A_n$ and $f(A_n) \subseteq A_1$,

(ii) $\mathbb{G}^*_{fx,fy,fz}(\varphi(t)) \geq \mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c}))$,

for all $x \in A_i$ and $y, z \in A_{i+1}$ $(A_{n+1} = A_1 \; and \; i \in \mathbb{N})$ where $c \in (0, 1)$ and $t > 0$. Then f has a unique fixed point in $\cap_{i=1}^n A_i$.

PROOF : Let x_0 be any arbitrary point in $\cup_{i=1}^n A_i$. Define $x_n = fx_{n-1}, n \geq 1$. As $x_0 \in A_1, fx_0 \in A_2$, so in general there exists $i_n \in \{1, 2, \cdots, n\}$ such that $x_n \in A_{i_n}, x_{n+1} \in A_{i_{n+1}}$ for each $n \geq 0$. For given $t > 0$ we have

$$\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})).$$

According to the Lemma 2.2, it is easy to conclude that $\{x_n\}$ is Cauchy sequence in S. Since S is complete, there exists $x \in S$ such that $\lim_{n\to\infty} x_n = x$. On the other hand, the iterative sequence $\{x_n\}$ has an infinitive number of terms in A_i, for each $i = 1, 2, \cdots, n$. From each $A_i, i = 1, 2, \cdots, n$, we can extract a subsequence of $\{x_n\}$ that converges to x. Since A_i is closed, we conclude that $x \in \cap_{i=1}^n A_i$. We shall prove that x is a fixed point of f. For $\varepsilon > 0$, we can choose $\delta > 0$ with $0 < \varphi(\delta) < \varepsilon$. Therefore for each $n = 1, 2, 3, ...$

$$\begin{aligned} \mathbb{G}^*_{fx,x,x}(\varepsilon) &\geq T\left(\mathbb{G}^*_{fx,x_n,x_n}(\varphi(\delta)), \mathbb{G}^*_{x_n,x,x}(\varepsilon - \varphi(\delta))\right) \\ &\geq T\left(\mathbb{G}^*_{x,x_{n-1},x_{n-1}}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{x_n,x,x}(\varepsilon - \varphi(\delta))\right). \end{aligned}$$

With limiting the two side of relation, it follows that for each $\varepsilon > 0$, $\mathbb{G}^*_{fx,x,x}(\varepsilon) = 1$, that is, $fx = x$. Next, suppose that f has another fixed point y. Therefore, for all $t > 0$, we get

$$\mathbb{G}^*_{x,y,y}(\varphi(t)) = \mathbb{G}^*_{fx,fy,fy}(\varphi(t)) \geq \mathbb{G}^*_{x,y,y}(\varphi(\frac{t}{c})),$$

so by Lemma 2.3, we conclude that $x = y$. Hence f has a unique fixed point in $\cap_{i=1}^n A_i$.

***Theorem* 2.8** — *Assume that $(S, \mathbb{G}^*, T)$ be a complete $P\mathbb{G}M$-space, T is continuous and $T(a, a) \geq a$. Let $f : S \to S$ be a function such that*

$$\begin{aligned} \mathbb{G}^*_{fx,fy,fz}(\varphi(t)) \;\;\geq\;\; & \min\left\{\mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c})), \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})), \mathbb{G}^*_{y,fy,fy}(\varphi(\frac{t}{c})), \mathbb{G}^*_{z,fz,fz}(\varphi(\frac{t}{c})),\right. \\ & \left.\mathbb{G}^*_{x,fy,fy}(2\varphi(\frac{t}{c})), \mathbb{G}^*_{y,fz,fz}(2\varphi(\frac{t}{c})), \mathbb{G}^*_{x,fz,fz}(2\varphi(\frac{t}{c}))\right\}, \end{aligned} \quad (2.13)$$

for all $x, y, z \in S$ and for all $t > 0$, where $\varphi \in \Phi$ and $c \in (0,1)$. Then f has a unique fixed point in S.

Let $x_0 \in S$ and define $x_n = fx_{n-1}$, for all $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, it is clear that $f(x_n) = x_n$. Therefore we suppose $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. From (2.13), for all $t > 0$ we obtain

$$
\begin{aligned}
\mathbb{G}^*_{fx,fy,fz}(\varphi(t)) \geq\ & \min\big\{\mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c})), \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})), \mathbb{G}^*_{y,fy,fy}(\varphi(\frac{t}{c})),\\
& \mathbb{G}^*_{z,fz,fz}(\varphi(\frac{t}{c})), \mathbb{G}^*_{x,fy,fy}(2\varphi(\frac{t}{c})), \mathbb{G}^*_{y,fz,fz}(2\varphi(\frac{t}{c})),\\
& \mathbb{G}^*_{x,fz,fz}(2\varphi(\frac{t}{c}))\big\}\\
\geq\ & \min\big\{\mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c})), \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})), \mathbb{G}^*_{y,fy,fy}(\varphi(\frac{t}{c})),\\
& \mathbb{G}^*_{z,fz,fz}(\varphi(\frac{t}{c})), T(\mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})), \mathbb{G}^*_{fx,fy,fy}(\varphi(\frac{t}{c}))),\\
& T(\mathbb{G}^*_{y,fy,fy}(\varphi(\frac{t}{c})), \mathbb{G}^*_{fy,fz,fz}(\varphi(\frac{t}{c}))),\\
& T(\mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})), \mathbb{G}^*_{fx,fz,fz}(\varphi(\frac{t}{c})))\big\}\\
\geq\ & \min\big\{\mathbb{G}^*_{x,y,z}(\varphi(\frac{t}{c})), \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{t}{c})), \mathbb{G}^*_{y,fy,fy}(\varphi(\frac{t}{c})),\\
& \mathbb{G}^*_{z,fz,fz}(\varphi(\frac{t}{c})), \mathbb{G}^*_{fx,fy,fy}(\varphi(\frac{t}{c})), \mathbb{G}^*_{fy,fz,fz}(\varphi(\frac{t}{c})),\\
& \mathbb{G}^*_{fx,fz,fz}(\varphi(\frac{t}{c}))\big\}. \qquad (2.14)
\end{aligned}
$$

Now we want to show that $\{x_n\}$ is Cauchy. Therefore by using (2.14) we have

$$
\begin{aligned}
\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) =\ & \mathbb{G}^*_{fx_{n-1},fx_n,fx_n}(\varphi(t))\\
\geq\ & \min\big\{\mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(t/c)), \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(t/c)),\\
& \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t/c)), \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t/c)),\\
& \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t/c)), \mathbb{G}^*_{x_{n+1},x_{n+1},x_{n+1}}(\varphi(t/c)),\\
& \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t/c))\big\}\\
= \min\ & \big\{\mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(t/c)), \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t/c))\big\}.
\end{aligned}
$$

We shall prove that

$$
\mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})). \qquad (2.15)
$$

If we consider

$$\min\{\mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{t}{c})), \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(\frac{t}{c}))\} = \mathbb{G}^*_{x_n,x_{n+1},x_{n+1}}(\varphi(\frac{t}{c})),$$

then by Lemma 2.3, we get that $x_n = x_{n+1}$, which leads to contradiction. Therefore (2.15) holds true. So by Lemma 2.2, we deduce that $\{x_n\}$ is a Cauchy sequence in S. Therefore, $x_n \to x$ as $n \to \infty$ for some $x \in S$. We will show that x is a fixed point of f. By Definition 1.3(d), we have

$$\begin{aligned} \mathbb{G}^*_{x,fx,fx}(t) &\geq T(\mathbb{G}^*_{x,x_n,x_n}(t-\varphi(\delta)), \mathbb{G}^*_{x_n,fx,fx}(\varphi(\delta))) \\ &\geq \min\left\{\mathbb{G}^*_{x,x_n,x_n}(t-\varphi(\delta)), \mathbb{G}^*_{x_n,fx,fx}(\varphi(\delta))\right\}. \end{aligned}$$

If for only finitely number of n, we have $x_n \neq fx$, then $x = fx$ and hence the proof finishes. Therefore, let $x_n \neq fx$ for each $n \in \mathbb{N}$. Hence, since $\lim_{n\to\infty} x_n = x$, for any arbitrary $\epsilon \in (0,1)$, there exists $n_1 \in \mathbb{N}$ such that $\mathbb{G}^*_{x,x_n,x_n}(t-\varphi(\delta)) > 1-\epsilon$, whenever $n \geq n_1$. Hence, we have $\mathbb{G}^*_{x,fx,fx}(t) \geq \min\{1-\epsilon, \mathbb{G}^*_{x_n,fx,fx}(\varphi(\delta))\}$. Since $\epsilon > 0$ is arbitrary, for n large enough and from(2.14), we get

$$\begin{aligned} \mathbb{G}^*_{x,fx,fx}(t) &\geq \mathbb{G}^*_{x_n,fx,fx}(\varphi(\delta)) \\ &\geq \min\left\{\mathbb{G}^*_{x_{n-1},x,x}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{\delta}{c})),\right. \\ &\qquad \left.\mathbb{G}^*_{x_n,fx,fx}(\varphi(\frac{\delta}{c}))\right\}. \end{aligned} \tag{2.16}$$

If the minimum be $\mathbb{G}^*_{x_n,fx,fx}(\varphi(\frac{\delta}{c}))$, then by Lemma 2.3 we have $x_n = fx$, that is contradiction. So from (2.16) we get

$$\begin{aligned} \mathbb{G}^*_{x,fx,fx}(t) &\geq \liminf_{n\to\infty} \mathbb{G}^*_{x_n,fx,fx}(\varphi(\delta)) \\ &\geq \lim\ inf_{n\to\infty} \min\left\{\mathbb{G}^*_{x_{n-1},x,x}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{x_{n-1},x_n,x_n}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{\delta}{c}))\right\} \\ &\geq \min\{1-\epsilon, 1-\epsilon, \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{\delta}{c}))\}. \end{aligned} \tag{2.17}$$

Finally, since $\epsilon \in (0,1)$ is arbitrary, we have $\mathbb{G}^*_{x,fx,fx}(\varphi(\delta)) \geq \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{\delta}{c}))$, and so, by Lemma 2.3, we deduce that $x = fx$. Now we are going to prove that x is a unique fixed point of f. Assume that f has another fixed point $y \in S$. From the property of φ, there exists $\delta > 0$ such that for all $t > 0$, $\varphi(\delta) < t$. Hence by using (2.14) we get

$$\begin{aligned} \mathbb{G}^*_{x,y,y}(t) &\geq \mathbb{G}^*_{x,y,y}(\varphi(\delta)) \\ &\geq \min\left\{\mathbb{G}^*_{x,y,y}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{x,fx,fx}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{y,fy,fy}(\varphi(\frac{\delta}{c}))\right\} \\ &= \min\left\{\mathbb{G}^*_{x,y,y}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{x,x,x}(\varphi(\frac{\delta}{c})), \mathbb{G}^*_{y,y,y}(\varphi(\frac{\delta}{c}))\right\}. \end{aligned}$$

By Lemma 2.3 we conclude that $x = y$, that is x is a unique fixed point of f.

***Theorem* 2.9** — *Assume that $(S, \mathbb{G}^*, T)$ be a complete $P\mathbb{G}M$-space. Let $f : S \to S$ be a φ-contraction and $f = f_1 f_2 f_3 ... f_n$, where $\{f_i\}_1^n : X \to X$ is a finite family of mapping and $f_i f_j = f_j f_i$ whenever $i \neq j$. Then the family of $\{f_i\}_1^n$ has a unique common fixed point.*

Proof : Since f is a φ-contraction, according to the Theorem 2.5, has a unique fixed point u. Now we have

$$\begin{aligned} f(f_i u) &= ((f_1 f_2 f_3 .. f_n) f_i) u \\ &= (f_1 f_2 f_3 ... f_{n-1})((f_i f_n) u) \\ &= (f_1 f_2 f_3 ... f_{n-2})((f_i f_{n-1} f_n) u) \\ &= ... \\ &= f_1 f_i (f_2 f_3 ... f_{n-1} f_n u) \\ &= f_i f_1 (f_2 f_3 ... f_n u) \\ &= f_i (fu) = f_i u. \end{aligned}$$

Since f has unique fixed point, hence $f_i u = u$. This shows that u is a fixed point of f_i for $i \in \{1, 2, ...n\}$. □

Let $f_1 = f_2 = f_3 = ... f_n$, then we have the following results.

Corollary 2.10 — Assume that $(S, \mathbb{G}^*, T)$ be a complete $P\mathbb{G}M$-space and f be a self-map on X. Let f^m is a φ-contraction. Then f has a unique fixed point.

Example 2.11 : Assume that $S = [0, \infty)$ and $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $\mathbb{G}^*_{x,y,z}(t) = \dfrac{t}{t + \mathbb{G}(x, y, z)}$ for all $x, y, z \in S$, where $\mathbb{G}(x, y, z) = |x - y| + |y - z| + |z - x|$. Then $\mathbb{G}$ is a $G-$metric. It is clear that $\mathbb{G}^*$ is a $P\mathbb{G}M$-space. Let $c \in (0, 1)$ and $\varphi(t) = t$. Suppose that $f : S \to S$ is defined by $fx = \frac{x}{3}$ for each $x \in S$. Let $t > 0$ be arbitrary, therefore

$$\frac{t}{t + \frac{1}{3}(|x - y| + |y - z| + |z - x|)} \geq \frac{\frac{t}{c}}{\frac{t}{c} + (|x - y| + |y - z| + |z - x|)},$$

for all $c \in [\frac{1}{3}, 1)$. Hence f is a φ-contraction mapping and f has a fixed point $x = 0$ by Theorem 2.5.

Example 2.12 : Let $S = R$ and $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $\mathbb{G}^*_{x,y,z}(t) = \dfrac{t}{t + \mathbb{G}(x, y, z)}$ for all $x, y, z \in S$, where $\mathbb{G}(x, y, z) = |x - y| + |y - z| + |z - x|$. Assume $A_1 = A_2 =$

$\cdots = [0,1]$. We define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t$. Define the mapping $f : \cup_{i=1}^{n} A_i \to \cup_{i=1}^{n} A_i$ by

$$fy = \frac{y}{4}, \quad \forall y \in [0,1].$$

It is clear that the hypothesis of Theorem 2.7 hold for all $c \in [\frac{1}{4}, 1)$ and $x = 0$ is a fixed point of f.

Example 2.13 : Let $S = [0,1]$ and $T(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$ and $\mathbb{G}^*_{x,y,z}(t) = \dfrac{t}{t + \mathbb{G}(x,y,z)}$ for all $x, y, z \in S$, where $\mathbb{G}(x,y,z) = |x-y| + |y-z| + |z-x|$. Define the mapping $f : S \to S$ by

$$fx = \begin{cases} \frac{x}{4} & x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1], \\ 0 & x = \frac{1}{2}. \end{cases}$$

Consider $\varphi(t) = t$ for all $t > 0$ and let $c = \frac{1}{2}$. To show that f satisfies the condition (2.13) of Theorem 2.8, we will consider four cases:

(1) If $x = y = z$, then $\mathbb{G}^*_{fx,fy,fz}(t) = 1$, hence the left side of inequality (2.13) is equal to 1. Thus (2.13) is obviously true.

(2) If $x = y = \frac{1}{2}$ and $z \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, then we have

$$\begin{aligned} \frac{t}{t + \frac{z}{2}} &\geq \min\{\frac{2t}{2t + |2z-1|}, \frac{2t}{2t+1}, \frac{2t}{2t+1}, \frac{2t}{2t + \frac{3}{2}z}, \frac{4t}{4t+1}, \frac{4t}{4t+1-\frac{z}{2}}, \frac{4t}{4t+1-\frac{z}{2}}\} \\ &= \{\frac{2t}{2t + |2z-1|}, \frac{2t}{2t+1}, \frac{2t}{2t+\frac{3}{2}z}\}. \end{aligned}$$

Since $\frac{t}{t+\frac{z}{2}} \geq \frac{2t}{2t+1}$, hence above inequality is true.

(3) If $x = \frac{1}{2}$ and $y, z \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, then we have

$$\begin{aligned} \frac{t}{t + \frac{1}{4}(y + z + |y-z|)} \geq \min\{ &\frac{2t}{2t + |\frac{1}{2} - y| + |y - z| + |z - \frac{1}{2}|}, \frac{2t}{2t+1}, \frac{2t}{2t + \frac{3}{2}y}, \frac{2t}{2t+\frac{3}{2}z}, \\ &\frac{4t}{4t + 1 - \frac{y}{2}}, \frac{4t}{4t + |\frac{z}{4} - \frac{1}{2}| + |\frac{1}{2} - \frac{z}{4}|}, \frac{4t}{4t + |y - \frac{z}{4}| + |\frac{z}{4} - y|}\}. \end{aligned}$$

There is no loss of generality in assuming that $y > z$. Since $\frac{t}{t+\frac{1}{2}y} \geq \frac{2t}{2t+\frac{3}{2}y}$, hence above inequality is true.

(4) If $x, y, z \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, then we have

$$\frac{t}{t + \frac{1}{4}(|x-y| + |y-z| + |z-x|)} \geq \min\Big\{\frac{2t}{2t + (|x-y| + |y-z| + |z-x|)}, \frac{2t}{2t + \frac{3}{2}x}, \frac{2t}{2t + \frac{3}{2}y}, \frac{2t}{2t + \frac{3}{2}z}, \frac{4t}{4t + |2x - \frac{y}{2}|}, \frac{4t}{4t + |2y - \frac{z}{2}|}, \frac{4t}{4t + |2x - \frac{z}{2}|}\Big\}.$$

There is no loss of generality in assuming that $x > y > z$. Hence from above inequality we have

$$\frac{t}{t + \frac{1}{2}(x-z)} \geq \min\Big\{\frac{2t}{2t + 2(x-z)}, \frac{2t}{2t + \frac{3}{2}x}, \frac{4t}{4t + 2(x - \frac{z}{4})}\Big\},$$

thus the condition (2.13) is true too.

Thus f satisfies the hypotheses of Theorem 2.8 and has a fixed point $x = 0$. We can see for $x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $y = z = \frac{1}{2}$, function f, does not satisfy the hypotheses of Theorem 2.5.

Example 2.14 : Let $S = \mathbb{R}, T(a, b) = \min\{a, b\}$ and $\mathbb{G}^*_{x,y,z}(t) = \dfrac{t}{t + \mathbb{G}(x, y, z)}$ for all $x, y, z \in S$, where $\mathbb{G}(x, y, z) = |x-y| + |y-z| + |z-x|$.

Define the mapping $f : S \to S$ by

$$fx = \begin{cases} 4 & x \in [0, 1) \\ 1 & \text{otherwise.} \end{cases}$$

We can see that $f^2x = 1$ for all $x \in S$. Hence according to the Theorem 2.5, for every $\varphi \in \Phi$ and every constant $c \in (0, 1)$, f^2 has a unique fixed point $x = 1$. Now consider $x = y = 0$ and $z = 2$, then by applying inequality (2.1), we get $c \geq 2$, which gives a contradiction. This shows that f does not satisfy Theorem 2.5, but by using Corollary 2.10, we can say f has a unique fixed point $x = 1$.

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