*Indian J. Pure Appl. Math.*, **51**(2): 533-556, June 2020

DOI: 10.1007/s13226-020-0415-7

# DOMINATION IN GENERALIZED UNIT AND UNITARY CAYLEY GRAPHS OF FINITE RINGS

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(*Received* 29 *May* 2018; *accepted* 6 *March* 2019)

Let $R$ be a finite commutative ring with nonzero identity and $U(R)$ be the set of all units of $R$. The graph $\Gamma$ is the simple undirected graph with vertex set $R$ in which two distinct vertices $x$ and $y$ are adjacent if and only if there exists a unit element $u$ in $U(R)$ such that $x + uy$ is a unit in $R$. Also, $\overline{\Gamma}$ denotes the complement of $\Gamma$. In this paper, we find the domination number $\gamma$ of $\Gamma$ as well as $\overline{\Gamma}$ and characterize all $\gamma$-sets in $\Gamma$ and $\overline{\Gamma}$. Also, we obtain the bondage number of $\Gamma$. Further, we obtain the values of some domination parameters like independent, strong and weak domination numbers of $\overline{\Gamma}$.

**Key words** : Commutative rings; generalized unit and unitary graph; Cayley graphs; complement graph; domination number; independent number.

**2010 Mathematics Subject Classification :** 05C25, 05C69, 16P10.

## 1. INTRODUCTION

Throughout this paper $R$ denotes a finite commutative ring with nonzero identity. The subsets $Z(R)$ and $U(R)$ denote the set of all zero-divisors and the multiplicative group of units of $R$ respectively. Constructing graphs from commutative rings was initiated by Beck through his work on zero-divisor graphs and thereafter several graphs constructions were made by several authors. Through the construction of graphs from commutative rings, interplay between algebraic properties of commutative rings and graph theoretical properties of derived graphs are studied. Khashyarmanesh and Khorsandi [6] provided a generalization of the unit and unitary Cayley graphs as follows: Let $G$ be a multiplicative subgroup of $U(R)$ and $S$ be a non-empty subset of $G$ such that $S^{-1} = \{s^{-1} : s \in S\} \subseteq S$.

Then $\Gamma(R,G,S)$ is the simple graph with vertex set $R$ in which two distinct elements $x,y \in R$ are adjacent if and only if there exists $s \in S$ such that $x+sy \in G$. As a generalization of unit graphs, Kiani *et al*. [7] provided classification of all rings with unit graphs having domination number less than four. Various properties of $\Gamma(R,G,S)$ were investigated by several authors in [1, 2, 6, 8]. More specifically, Khashyarmanesh *et al*. [6] obtained a necessary and sufficient condition for $\Gamma(R,G,S)$ to be a complete graph or a planar graph where $R$ is an Artinian ring and $G = U(R)$. Subsequently Asir and Tamizh Chelvam [1, 2] obtained a characterization of all commutative Artinian rings whose $\Gamma(R,G,S)$ has one or two.

As a special case of $\Gamma(R,G,S)$, a graph $\Gamma = \Gamma(R,U(R),U(R))$ was first studied by Ali Reza Naghipour *et al*. [8]. Hereafter, we denote the graph $\Gamma(R,U(R),U(R))$ by $\Gamma$. Also, Ali Reza Naghipour *et al*. [8] provided a necessary and sufficient condition for $\Gamma$ to be Hamiltonian. Tamizh Chelvam *et al*. [9] obtained a necessary and sufficient condition for $\Gamma$ to be Eulerian. They have also obtained a necessary and sufficient condition for $\overline{\Gamma}$ to be Eulerian or Hamiltonian or planar.

The aim of this paper is to obtain the value of the domination number $\gamma$ and a characterization for $\gamma$-sets in $\Gamma$ as well as $\overline{\Gamma}$. Also we obtain the values of certain other domination parameters for $\Gamma$ and $\overline{\Gamma}$. In Section 2 of this paper, we obtain certain basic structural properties of $\Gamma$. In Section 3 of this paper, first we obtain the domination number and further characterize all $\gamma$-sets in $\Gamma$. Subsequently we obtain a characterization for $\Gamma$ to be domatically full. Also, we obtain the bondage number of $\Gamma$. In Section 4 of this paper, we characterize all the $\gamma$-sets in $\overline{\Gamma}$. In sequel, we obtain the values of other domination parameters like independent, strong and weak domination numbers of $\overline{\Gamma}$. At the end of the section, we obtain a characterization for $\overline{\Gamma}$ to be domatically full or well-covered.

Let $\Gamma = (V,E)$ be a graph. For a subset $S \subseteq V$, $< S >$ denotes the subgraph of $\Gamma$ induced by $S$ and for a vertex $v \in V$, $\deg_\Gamma(v)$ is the degree of the vertex $v$ in $\Gamma$, $N(v) = \{u \in V : u\, is\, adjacent\, to\, v\}$ and $N[v] = N(v) \cup \{v\}$. A set of vertices $S \subseteq V$ is said to be independent if no two vertices are adjacent in $\Gamma$. The complement $\overline{\Gamma}$ of $\Gamma$ is the graph whose vertex set is $V(\Gamma)$ and such that each pair of vertices $u,v$ of $\Gamma$, $uv$ is an edge of $\overline{\Gamma}$ if and only if $uv$ is not an edge of $\Gamma$. For a graph $\Gamma = (V,E)$, a subset $S \subseteq V$ is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. A subset $S \subseteq V$ is called a *total dominating set* if every vertex in $v \in V$ is adjacent to some vertex $u \in S$ and $v \neq u$. A domination set $S$ is called a *strong* (or *weak*) domination set if for every vertex $u \in V \setminus S$, there is a vertex $v \in S$ with $\deg_\Gamma(v) \geq \deg_\Gamma(u)$ (or $\deg_\Gamma(v) \leq \deg_\Gamma(u)$) and $u$ is adjacent to $v$. The domination number $\gamma$ of $\Gamma$ is defined to be minimum cardinality of a dominating set in $\Gamma$ and such a domination set is called $\gamma$-set in $\Gamma$. One can refer [5] for definitions of other domination parameters like *total dominating number* $\gamma_t$, *connected dominating number* $\gamma_c$, *clique dominating number* $\gamma_{cl}$, *independent dominating number* $i(\Gamma)$, *perfect dominating number* $\gamma_p$, *efficient dominating number* $\gamma_{eff}$, *strong dominating number* $\gamma_s$ and *weak dominating number* $\gamma_w$. A graph $\Gamma$ is called *excellent* if, for every vertex $v \in V$, there exists a $\gamma$-set $S$ containing $v$. A *domatic partition* of $\Gamma$ is a partition of $V$, into dominating sets in $\Gamma$. The maximum number of sets in a domatic partition is called a *domatic number* of $\Gamma$ and is denoted by $d(\Gamma)$. The maximum number of sets in a domatic partition in which each partition is a total dominating set is called a *total domatic number* of $\Gamma$ and is denoted by $d_t(\Gamma)$. A graph $\Gamma$ is called *domatically full* if $d(\Gamma) = \delta(\Gamma) + 1$. The *bondage number* $b(\Gamma)$ is the minimum number of edges whose removal increases the domination number. The *independent number* $\beta_0(\Gamma)$ is the maximum cardinality of an independent set in $\Gamma$. A graph $\Gamma$ is well-covered if $\beta_0(\Gamma) = i(\Gamma)$. For basic domination parameters, we refer the reader to [5].

## 2. Basic Properties of The Graph

In this section, we observe certain basic properties of the graph $\Gamma$. Especially the note given below is useful in the proofs of the results.

*Note* 1 : [9, Note 2.1]. Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \leq i \leq q$. Arrange the indices in such a way that $|R_j/M_j| = 2$ for $1 \leq j \leq t$ and $|R_k/M_k| > 2$ for $t+1 \leq k \leq t+s = q$. Throughout this paper, we make use of the following notations on the indexes of the components:

(i) If $|R_i/M_i| = 2$ for every $i,\ 1 \leq i \leq q$, then $t > 0$, $s = 0$ and $q = t$;

(ii) If $|R_i/M_i| > 2$ for every $i,\ 1 \leq i \leq q$, then $t = 0$, $s > 0$ and $q = s$;

(iii) If $|R_j/M_j| = 2$ for $1 \leq j \leq t$ and $|R_k/M_k| > 2$ for $t+1 \leq k \leq t+s$, then $t > 0, s > 0$ and $q = t + s$.

Further, we use the following notations with regard to subsets of indexes for components of elements of $R$. For $x = (x_1, x_2, \ldots, x_q) \in R$, let $\theta_x = \{j : x_j \in Z(R_j)\ for\ 1 \leq j \leq t\}$, $\overline{\theta}_x = \{1, 2, \ldots, t\} \setminus \theta_x$, and $\Omega_x = \{j : x_j \in Z(R_j)\ for\ t+1 \leq j \leq q\}$, $\overline{\Omega}_x = \{t+1, \ldots, q\} \setminus \Omega_x$.

With the above notations, we have the following special instances:

(iv) If $|R_i/M_i| = 2$ for every $i,\ 1 \leq i \leq t = q$, then $\theta_x = \{j : x_j \in Z(R_j), 1 \leq j \leq q\}$, $\overline{\theta}_x = \{1, 2, \ldots, q\} \setminus \theta_x = \{j : x_j \in U(R_j), 1 \leq j \leq q\}$, $\Omega_x = \overline{\Omega}_x = \phi$.

(v) If $|R_i/M_i| > 2$ for every $i,\ 1 \leq i \leq s = q$, then, $\theta_x = \overline{\theta}_x = \phi$ and $\Omega_x = \{j : x_j \in Z(R_j)$ for $1 \leq j \leq q\}$ and $\overline{\Omega}_x = \{1, 2, \ldots, q\} \setminus \Omega_x = \{j : x_j \in U(R_j)$ for $1 \leq j \leq q\}$.

(vi) If $|R_j/M_j| = 2$ for $1 \leq j \leq t$ and $|R_k/M_k| > 2$ for t+1 $\leq k \leq q$, then $\theta_x = \{j : x_j \in Z(R_j), 1 \leq j \leq t\}$, $\overline{\theta}_x = \{1, 2, \ldots, t\} \setminus \theta_x = \{j : x_j \in U(R_j), 1 \leq j \leq t\}$, $\Omega_x = \{j : x_j \in Z(R_j)$

for $t+1 \leq j \leq q\}$ and $\overline{\Omega}_x = \{t+1, \ldots, q\} \setminus \Omega_x = \{j : x_j \in U(R_j)$ for $t+1 \leq j \leq q\}$.

The following results are proved in the paper [9] and stated here for reference

*Lemma* 1 — [9, Lemma 2.3]. Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Let $x = (x_1, \ldots, x_q)$ and $y = (y_1, \ldots, y_q)$ be two distinct elements in $R$. Then $x$ and $y$ are adjacent in $\Gamma$ if and only if $\theta_x = \overline{\theta}_y$ and $\Omega_x \subseteq \overline{\Omega}_y$. In particular if $\theta_x = \overline{\theta}_y$ and $\Omega_x = \phi$, then $x$ is adjacent to $y$ in $\Gamma$.

*Corollary* 2 — [9, Corollary 2.4]. Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Let $x$,$y$ be two distinct elements in $R$. Then the following are true:

(i) Assume that $|R_i/M_i| = 2$ for every $i,\ 1 \leq i \leq q$. Then $x, y$ are adjacent in $\Gamma$ if and only if $\theta_x = \overline{\theta}_y$;

(ii) Assume that $\theta_x = \overline{\theta}_y$ and $\overline{\Omega}_x = \phi$. Then $x, y$ are adjacent in $\Gamma$ if and only if $\Omega_y = \phi$;

(iii) Assume that $|R_i/M_i| > 2$ for every $i,\ 1 \leq i \leq q$. Then $x, y$ are adjacent in $\Gamma$ if and only if $\Omega_x \subseteq \overline{\Omega}_y$; In particular if $\Omega_x = \phi$, then $x$ is adjacent to all other vertex $y$ in $\Gamma$;

(iv) Assume that $|R_i/M_i| > 2$ for every $i,\ 1 \leq i \leq q$ and $\overline{\Omega}_x = \phi$. Then $x, y$ are adjacent in $\Gamma$ if and only if $\Omega_y = \phi$.

*Lemma* 3 — [9, Lemma 2.5]. Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Let $n = |R_1||R_2|\cdots|R_t|$ and $x = (x_1, x_2, \ldots, x_q) \in R$. Then the following are true:

(i) If $|R_i/M_i| > 2$ for every $i$, then

$$deg_\Gamma(x) = \begin{cases} |R| - 1 & \text{if } x \in U(R); \\ \prod\limits_{\ell \in \Omega_x} |U(R_\ell)| \prod\limits_{\ell \in \overline{\Omega}_x} |R_\ell| & \text{if } x \in Z(R). \end{cases}$$

(ii) If $|R_i/M_i| = 2$ for some $i,\ 1 \leq i \leq q$, then

$$deg_\Gamma(x) = \frac{n}{2^t} \prod_{\ell \in \Omega_x} |U(R_\ell)| \prod_{\ell \in \overline{\Omega}_x} |R_\ell|.$$

Moreover, $\Gamma$ has no isolated vertex.

*Corollary* 4 — [9, Corollary 3.2]. Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Then $\Gamma$ is bipartite if and only if $|R_j/M_j| = 2$ for some $j,\ 1 \leq j \leq q$.

***Theorem* 5** — *[9, Theorem 3.1]. Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. If $|R_j/M_j| = 2$ for some $j,\ 1 \leq j \leq q$, then $\Gamma$ is the union of $2^{t-1}$ vertex disjoint connected components.*

The following observation follows from [9, Theorem 3.1] and is useful in the proofs of the results of this paper.

*Observation* 6 : Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Assume that $|R_j/M_j| = 2$ for some $j$. Let $L$ be the set of all $2^t$ sequences of $0's$ and $1's$ of length $t$. Corresponding to each sequence $a = (a_1, a_2, \ldots, a_t)$, associate a subset $X = X_1 \times \cdots \times X_t \times R_{t+1} \times \cdots \times R_q \subseteq R$ where $X_\ell = Z(R_\ell)$ if $a_\ell$ is 0 and $X_\ell = U(R_\ell)$ if $a_\ell$ is 1 for $1 \leq \ell \leq t$. These associated $2^t$ subsets of $R$ form a partition of $R$.

Let $a = (a_1, a_2, \ldots, a_t)$ and $b = (b_1, b_2, \ldots, b_t)$ be two sequences in $L$ such that $a_\ell + b_\ell = 1$ for every $\ell$, $1 \leq \ell \leq t$. Let $X = X_1 \times \cdots \times X_t \times R_{t+1} \times \cdots \times R_q$ and $Y = Y_1 \times \cdots \times Y_t \times R_{t+1} \times \cdots \times R_q$ denote the corresponding subsets of $(a_1, a_2, \ldots, a_t)$ and $(b_1, b_2, \ldots, b_t)$ respectively. Then $|X| = |Y| = \frac{|R|}{2^t} \geq 1$. Let $g, h \in R$ and $g \in X$. Suppose $h \in X$. Then $\theta_g = \theta_h$. Suppose $h \in Y$. Then $\theta_g = \overline{\theta}_h$. Suppose $h \in R \setminus (X \cup Y)$. Then $\theta_g \neq \theta_h$ and $\theta_g \neq \overline{\theta}_h$. For any $g, h \in R$, then exactly any one of the following is true:
(i) $\theta_g = \theta_h$; (ii) $\theta_g = \overline{\theta}_h$; (iii) $\theta_g \neq \theta_h$ and $\theta_g \neq \overline{\theta}_h$.

Let $X' = \{x \in X : \Omega_x = \phi\}$ and $Y' = \{y \in Y : \Omega_y = \phi\}$. Since $|U(R_k)| \geq 2$ for $t+1 \leq k \leq q$, we have $|X'| = |Y'| = \frac{n}{2^t} \prod_{\ell=t+1}^{q} |U(R_\ell)| \geq 2$ and $n = |R_1||R_2| \cdots |R_t|$.

In view of Theorem 3.1 [9], $\Gamma$ is the union of of $2^{t-1}$ connected components and $\Gamma_c = < X, Y >$ (where $X$ and $Y$ are as specified in Observation 6) is one of the connected components of $\Gamma$.

***Theorem* 7** — *[9, Theorem 3.2]. Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Assume that $|R_i/M_i| = 2$ for some $i$, $1 \leq i \leq q$. Let $\Gamma_c$ be one of the connected components of $\Gamma$. Then the following are true:*

(*i*) $\Gamma_c = K_{\frac{|R|}{2^t}, \frac{|R|}{2^t}}$ *if and only if* $|R_i/M_i| = 2$ *for every* $i, 1 \leq i \leq q$;

(*ii*) *If* $|R_i/M_i| > 2$ *for some* $i$, $1 \leq i \leq q$, *then* $\Gamma_c$ *contains* $K_{\alpha,\alpha}$ *as a subgraph where* $\alpha = \frac{n}{2^t} \prod_{\ell=t+1}^{q} |U(R_\ell)|$ *and* $n = |R_1||R_2| \cdots |R_t|$.

In view of Theorems 5 and 7(i), one can obtain the structure theorem for $\Gamma$ and the same is stated below.

*Corollary* 8 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Then the following are true:

(*i*) $\Gamma = 2^{t-1}\, K_{\frac{|R|}{2^t}, \frac{|R|}{2^t}}$ if and only if $|R_i/M_i| = 2$ for every $i$, $1 \leq i \leq t = q$;

(*ii*) If $|R_i/M_i| = 2$ and $|R_\ell/M_\ell| > 2$ for some $1 \le i, \ell \le q$, then $\Gamma$ contains $2^{t-1}\ K_{\alpha,\alpha}$ as a subgraph where $\alpha = \frac{n}{2^t} \prod_{\ell=t+1}^{q} |U(R_\ell)|$ and $n = |R_1||R_2| \cdots |R_t|$.

Since Lemma 3 gives degrees of vertices in $\Gamma$, we observe the following for later use in the paper.

*Note* 2 : Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \le i \le q$. Let $n = |R_1|\,|R_2| \cdots |R_t|$.

(i) Assume that $R = R_1$ is a field with $|R_1| > 2$. Then $\Gamma$ is a complete graph;

(ii) Assume that $R$ is not a field with $|R_i/M_i| > 2$ for every $i$. Then $|Z(R)| \ge 2$. Let $u \in U(R)$. By Lemma 11(i), $\deg_\Gamma(u) = |R| - 1$. When $q = 1$. Note that $Z(R) = Z(R_1)$. Let $z \in Z(R)$. Then $\Omega_z = \{1\}$ and by Lemma 1(i), $\deg_\Gamma(z) = |U(R)|$. Since $R$ is not a field, $\deg_\Gamma(z) = |U(R)| < |R| - 1 = \deg_\Gamma(u)$. When $q \ge 2$. Let $a, b$ be distinct elements in $Z(R)$ such that $\Omega_a = \{1, \ldots, q\}$, $\phi \ne \Omega_b \not\subseteq \{1, \ldots, q\}$. By Lemma 1(i), $|U(R)| = \deg_\Gamma(a) < \deg_\Gamma(b) \le |R| - 2$;

(iii) Let $R$ be with $|R_i/M_i| = 2$ for some $i$ and $|R_\ell/M_\ell| > 2$ for some $\ell$. Suppose $a, b, c \in R$ such that $\Omega_a = \phi$, $\phi \ne \Omega_b \not\subseteq \{t+1, \ldots, q\}$ and $\Omega_c = \{t+1, \ldots, q\}$.
By Lemma 11(ii), $\deg_\Gamma(a) = \frac{|R|}{2^t}$, $\deg_\Gamma(b) = \frac{n}{2^t} \prod_{\ell \in \Omega_b} |U(R_\ell)| \prod_{\ell \in \overline{\Omega}_b} |R_\ell|$ and $\deg_\Gamma(c) = \frac{n}{2^t} \prod_{\ell \in \Omega_c} |U(R_\ell)|$. From this, $\deg_\Gamma(a) > \deg_\Gamma(b) > \deg_\Gamma(c)$;

(iv) $$\Delta(\Gamma) = \begin{cases} |R| - 1 \text{ if } |R_i/M_i| > 2 \text{ for every } i; \\ \frac{|R|}{2^t} \text{ if } |R_i/M_i| = 2 \text{ for some } i,\ 1 \le i \le q; \end{cases}$$

(v) $$\delta(\Gamma) = \begin{cases} |U(R)| \text{ if } |R_i/M_i| > 2 \text{ for every } i; \\ \frac{|R|}{2^t} \text{ if } |R_i/M_i| = 2 \text{ for every } i; \\ \frac{n}{2^t} \prod_{\ell \in t+1}^{q} |U(R_\ell)| \text{ if } |R_i/M_i| = 2, |R_\ell/M_\ell| > 2 \text{ for some } 1 \le i, \ell \le q. \end{cases}$$

One can observe the following corollary from Lemma 3.

*Corollary* 9 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \le i \le q$. Assume that $|R_i/M_i| > 2$ for some $i$. Let $x = (x_1, x_2, \ldots, x_q)$ and $y = (y_1, y_2, \ldots, y_q)$ be two distinct elements in $R$. If $\Omega_x = \Omega_y$, then $\deg_\Gamma(x) = \deg_\Gamma(y)$.

## 3. Domination in Generalized Unit and Unitary Cayley Graphs

In this section, we obtain the domination number of $\Gamma$ and subsequently characterize all $\gamma$-sets in $\Gamma$. Also, we obtain the bondage number of $\Gamma$. In the following theorem, we obtain the domination number of $\Gamma$.

***Theorem* 10** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$ and $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$. Then*

$$\gamma(\Gamma) = \begin{cases} 1 & if\ |R_i/M_i| > 2\ for\ every\ i; \\ 2^{t-1} & if\ |R_i/M_i| = 2\ and\ \alpha_i = 1\ for\ every\ i; \\ 2^t & otherwise. \end{cases}$$

PROOF : Suppose $|R_i/M_i| > 2$ for every $i$. Then $|U(R_i)| \geq 2$ for every $i$ and so $|U(R)| \geq 2$. By Lemma 3(i), $\deg_\Gamma(x) = |R| - 1$ for $x \in U(R)$. This implies that for $x \in U(R)$, $\{x\}$ is a minimal dominating set in $\Gamma$ and so $\gamma(\Gamma) = 1$.

Suppose $|R_i/M_i| = 2$ and $\alpha_i = 1$ for every $i$. Then $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and so $|R| = 2^t$. By Corollary 8(i), $\Gamma = 2^{t-1}K_{1,1}$ and so $\gamma(\Gamma) = 2^{t-1}$.

For remaining cases, $R$ satisfies any one of the following:

(i) $|R_i/M_i| = 2$ for every $i$ and $\alpha_\ell \geq 2$ for some $\ell$;

(ii) $|R_i/M_i| = 2$ and $|R_\ell/M_\ell| > 2$ for some $1 \leq i, \ell \leq q$.

(i) When $|R_i/M_i| = 2$ for every $i$ and $\alpha_\ell \geq 2$ for some $\ell$. Consider the subsets $X$ and $Y$ as given in Observation 6. Then $\frac{|R|}{2^t} = |X| = |Y| \geq 2$. By Corollary 8(i), $\Gamma = 2^{t-1}K_{\frac{|R|}{2^t}, \frac{|R|}{2^t}}$. This implies $\gamma(\Gamma) = 2^t$.

(ii) When $|R_i/M_i| = 2$ and $|R_\ell/M_\ell| > 2$ for some $i, \ell$ and $1 \leq i, \ell \leq q$. Consider the subsets $X, X', Y$ and $Y'$ as given in Observation 6. Let $\Gamma_c =< X, Y >$ in $\Gamma$. Then $|X| = |Y| \geq 3$ and $|X'| = |Y'| \geq 2$. Choose $x \in X'$ and $y \in Y'$. Then $\theta_x = \overline{\theta}_y$, $\Omega_x = \phi$ and $\Omega_y = \phi$. Let $z \in V(\Gamma_c) \setminus \{x, y\}$. If $z \in X$, then $\theta_z = \overline{\theta}_y$. By in particular case of Lemma 1, $z$ and $y$ are adjacent in $\Gamma_c$. Similarly $z \in Y$, then $z$ and $x$ are adjacent in $\Gamma_c$. By $|R_i/M_i| = 2$ for some $i$, $1 \leq i \leq q$ and Corollary 4, $\Gamma_c =< X, Y >$ is bipartite in $\Gamma$. Since $|X| = |Y| \geq 3$, $\{x, y\}$ is a minimal dominating set in $\Gamma_c$. Then $\gamma(\Gamma_c) = 2$. By Theorem 5, $\gamma(\Gamma) = 2^t$.

Now we obtain, a characterization for $\gamma$-sets in connected components of $\Gamma$.

*Lemma* 11 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ and $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$. Assume that $|R_i/M_i| = 2$ for some $i$. Then the following are true for a connected component $\Gamma_c =< X, Y >$ of $\Gamma$:

(i) If $|R_i/M_i| = 2$ and $\alpha_i = 1$ for every $i$, then $S = \{x\}$ for any $x \in V(\Gamma_c)$ is a $\gamma$-set in $\Gamma_c$;

(ii) Let $|R_i/M_i| = 2$ for every $i$, $\alpha_\ell \geq 2$ for some $\ell$ and $|R| = 2^{t+1}$. Then any set $S$ with $|S| = 2$ is a $\gamma$-set in $\Gamma_c$;

(iii) Let $|R_i/M_i| = 2$ for every $i,\ \alpha_\ell \geq 2$ for some $\ell$ and $|R| > 2^{t+1}$. Then $S = \{a, b\}$ is a $\gamma$-set in $\Gamma_c$ if and only if $a \in X$ and $b \in Y$;

(iv) Let $|R_i/M_i| > 2$ for some $i$. Then $S = \{a, b\}$ is a $\gamma$-set in $\Gamma_c$ if and only if $a = (a_1, \ldots, a_q) \in X$ and $b = (b_1, \ldots, b_q) \in Y$ with $a_j, b_j \in U(R_j)$ for every $j,\ t+1 \leq j \leq q$.

PROOF : Assume that $|R_i/M_i| = 2$ for every $i$. Then $t = q$ and by Theorem 7(i), $\Gamma_c = K_{\frac{|R|}{2^t}, \frac{|R|}{2^t}}$.

(i) Assume that $|R_i/M_i| = 2$ and $\alpha_i = 1$ for every $i$. Note that $\frac{|R|}{2^t} = 1$. Then $\Gamma_c = K_{1,1}$ and so $S = \{x\}$ for any $x \in V(\Gamma_c)$ is a $\gamma$-set in $\Gamma_c$.

(ii) Assume that $|R_i/M_i| = 2$ for every $i$ and $|R| = 2^{t+1}$. Note that $\frac{|R|}{2^t} = 2$. Then $\Gamma_c = K_{2,2}$ and so any set $S$ with $|S| = 2$ is a $\gamma$-set in $\Gamma_c$.

(iii) Assume that $|R_i/M_i| = 2$ for every $i$ and $|R| > 2^{t+1}$. Note that $\frac{|R|}{2^t} > 2$. Since $\Gamma_c =< X, Y >= K_{\frac{|R|}{2^t}, \frac{|R|}{2^t}}$, we have $S = \{a, b\}$ is a $\gamma$-set in $\Gamma_c$ if and only if $a \in X$ and $b \in Y$.

(iv) Assume that $a = (a_1, \ldots, a_q) \in X$ and $b = (b_1, \ldots, b_q) \in Y$ with $a_j, b_j \in U(R_j)$ for every $j,\ t+1 \leq j \leq q$. Then $\theta_a = \overline{\theta}_b$, $\Omega_a = \phi$ and $\Omega_b = \phi$. As the same argument in Theorem 10(ii), we have $\{a, b\}$ is a $\gamma$-set in $\Gamma_c$.

Conversely $S = \{a, b\}$ is a $\gamma$-set in $\Gamma_c$. Suppose $a, b \in X$ or $a, b \in Y$. Without loss of generality one can take $a, b \in X$ and hence $\theta_a = \theta_b$. Since $|R_i/M_i| = 2$ for some $i$, by Corollary 4, $\Gamma$ is bipartite and so $\Gamma_c$ is bipartite. Note that $|X| = |Y| \geq 3,\ X$ and $Y$ are independent sets in $\Gamma_c$. This implies that $S$ is not a dominating set in $\Gamma_c$, a contradiction. Hence $a \in X$ and $b \in Y$.

Suppose $\Omega_a \neq \phi$ or $\Omega_b \neq \phi$. Without loss generality let us take, $\Omega_a \neq \phi$. Then there exists $a_j \in Z(R_j)$ for some $j,\ t+1 \leq j \leq q$. Let $d \in Y$ with all its components as zero-divisors. i.e., $\overline{\Omega}_d = \phi$. Note that $\theta_a = \overline{\theta}_d$. By Corollary 2(ii), $a, d$ are not adjacent in $\Gamma_c$. Since $b, d \in Y, \theta_b = \theta_d$. By Lemma 1, $b, d$ are not adjacent in $\Gamma_c$. Therefore $\{a, b\}$ is not a dominating set in $\Gamma_c$, a contradiction.

Now we obtain a characterization of all $\gamma$-sets in $\Gamma$.

*Lemma* 12 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \leq i \leq q$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$. Then the following are true:

(i) If $R = R_1$ is a field and $|R_1/M_1| > 2$, then $S = \{a \mid a = (a_1, \ldots, a_q) \in R\}$ is a $\gamma$-set in $\Gamma$;

(ii) If $R$ is not a field and $|R_i/M_i| > 2$ for every $i$, then $S = \{a\}$ is a $\gamma$-set in $\Gamma$ if and only if $a \in U(R)$;

(iii) Assume that $|R_i/M_i| = 2$ and $\alpha_i = 1$ for every $i$. Then $S \subseteq R$ with $|S| = 2^{t-1}$ is a $\gamma$-set of $\Gamma$ if and only if $S$ contains exactly one element from $X_i \cup Y_i$ for $1 \leq i \leq 2^{t-1}$;

(iv) Assume that $|R_i/M_i| = 2$ for every $i,\ \alpha_\ell \geq 2$ for some $\ell$ and $|R| = 2^{t+1}$. Then $S \subseteq R$ with $|S| = 2^t$ is a $\gamma$-set in $\Gamma$ if and only if $S$ contains exactly two elements from $X_i \cup Y_i$ for $1 \leq i \leq 2^{t-1}$;

(v) Assume that $|R_i/M_i| = 2$ for every $i,\ \alpha_\ell \geq 2$ for some $\ell$ and $|R| > 2^{t+1}$. Then $S \subseteq R$ with $|S| = 2^t$ is a $\gamma$-set in $\Gamma$ if and only if $S$ contains exactly one element from each subset $X$ ($2^t$ subsets);

(vi) Assume that $|R_i/M_i| = 2$ for some $i$ and $|R_\ell/M_\ell| > 2$ for some $\ell$. Then $S \subseteq R$ with $|S| = 2^t$ is a $\gamma$-set in $\Gamma$ if and only if $S$ contains exactly one element $a$ from each subset $X$ ($2^t$ subsets) with $a_i \in U(R_i)$ for every $i,\ t+1 \leq i \leq q$.

PROOF : (i) Proof follows from Note 2(i).

(ii) By the assumption on $R$, $|Z(R)| \geq 2$. Assume that $a \in U(R)$. By Lemma 3(i), $\deg_\Gamma(a) = |R| - 1$ and hence $\{a\}$ is a $\gamma$-set in $\Gamma$.

Conversely $S = \{a\}$ is a $\gamma$-set in $\Gamma$. Suppose $a \in Z(R)$. Then $\Omega_a \neq \phi$. As observed in Note 2(ii), $\deg_\Gamma(a) < |R| - 1$. This implies that $\{a\}$ is not a dominating set in $\Gamma$, a contradiction.

Proofs of (iii),(iv),(v) and (vi) follow from Lemma 11 and Theorem 5.

In view of Lemma 12, we have the following corollary.

*Corollary* 13 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \leq i \leq q$. Then $\Gamma$ is excellent if and only if $R$ satisfies any one of the following:

(i) $R = R_1$ is a field with $|R_1/M_1| > 2$;

(ii) $|R_i/M_i| = 2$ for every $i$.

In the following theorem, we obtain the domatic number of $\Gamma$ and hence we characterize when $\Gamma$ is domatically full.

***Theorem* 14** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \leq i \leq q$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$. Then the following are true:*

(*i*) $d(\Gamma) = \begin{cases} \delta(\Gamma)+1 & if\ |R_i/M_i| = 2\ and\ \alpha_i = 1\ for\ every\ i; \\ \delta(\Gamma) & if\ |R_i/M_i| = 2\ for\ every\ i\ and\ \alpha_\ell \geq 2\ for\ some\ \ell; \\ \delta(\Gamma)+1 & if\ |R_i/M_i| > 2\ for\ some\ i. \end{cases}$

(*ii*) $\Gamma$ *is domatically full if and only if either* $|R_i/M_i| = 2$ *and* $\alpha_i = 1$ *for every* $i$ *or* $|R_i/M_i| > 2$ *for some* $i$.

PROOF : (i) *Case* 1 : Assume that $|R_i/M_i| = 2$ and $\alpha_i = 1$ for every $i$. By Corollary 8(i), $\Gamma = 2^{t-1}K_{1,1}$. It is clear from this that $d(\Gamma) = 2 = \delta(\Gamma)+1$.

*Case* 2 : Assume that $|R_i/M_i| = 2$ for every $i$ and $\alpha_\ell \geq 2$ for some $\ell$. By Corollary 8(i), $\Gamma = 2^{t-1}K_{\frac{|R|}{2^t},\frac{|R|}{2^t}}$. As given in Observation 6, consider the subset $X_i$ in $R$ for $1 \leq i \leq 2^t$. Note that $|X_i| = \frac{|R|}{2^t}$ for $1 \leq i \leq 2^t$. Also, by Note 2(v), $\delta(\Gamma) = \frac{|R|}{2^t}$. For $1 \leq \ell \leq \frac{|R|}{2^t}$, let $S_\ell$ be the set containing exactly one element from $X_i$ for $1 \leq i \leq 2^t$. By Lemma 12(iv) and (v), $S_\ell$ is a $\gamma$-set in $\Gamma$ for $1 \leq \ell \leq \frac{|R|}{2^t}$. By the choice of subsets $S_\ell$, $V(\Gamma) = \bigcup\limits_{\ell=1}^{\frac{|R|}{2^t}} S_\ell$ is a maximal domatic partition in $\Gamma$ and so $d(\Gamma) = \frac{|R|}{2^t} = \delta(\Gamma)$.

*Case* 3 : Assume that $|R_\ell/M_\ell| > 2$ for some $\ell$.

*Subcase* 3.1 : Suppose $|R_\ell/M_\ell| > 2$ for every $\ell$.

*Subcase* 3.1.1 : When $R$ is a field. Then $\Gamma$ is complete and so $d(\Gamma) = \delta(\Gamma)+1$.

*Subcase* 3.1.2 : When $R$ is not a field. By Note 2(v), $\delta(\Gamma) = |U(R)|$

For $u \in U(R)$, we have $\deg_\Gamma(u) = |R|-1$ and so $S = \{u\}$ is a dominating set in $\Gamma$.

Let $S_1 \not\subseteq Z(R)$ and $S_2 = Z(R) \setminus S_1$.

*Claim* : Either $S_1$ or $S_2$ is not a dominating set in $\Gamma$.

Let $z \in Z(R_1) \times \cdots \times Z(R_q)$. Then $\overline{\Omega}_z = \phi$. Without loss of generality, one can take $z \in S_1$. Let $w \in S_2$. Then $\Omega_w \neq \phi$. By Corollary 2(iv), $z$ and $w$ are not adjacent in $\Gamma$. This implies $S_2$ is not a dominating set in $\Gamma$.

Hence $\bigcup\limits_{u \in U(R)} \{u\} \cup Z(R)$ is the maximal domatic partition of $V(\Gamma)$ and so $d(\Gamma) = |U(R)|+1$. and so $d(\Gamma) = \delta(\Gamma)+1$.

*Subcase* 3.2 : $|R_i/M_i| = 2$ for some $i$.

As given in Observation 6, consider the subsets $X', X, Y'$ and $Y$ in $R$.

Let $\Gamma_c =< X, Y >$ of $\Gamma$. Note that $|X'| = |Y'| = \frac{n}{2^t} \prod_{\ell=t+1}^{q} |U(R_\ell)|$ and $X' \subset X$ and $Y' \subset Y$. Let $X'' = X \setminus X'$ and $Y'' = Y \setminus Y'$.

*Claim* 1 : $L = X'' \cup Y''$ is a dominating set in $\Gamma_c =< X, Y >$ .

Let $a \in (X \cup Y) \setminus L$. Then $a \in X'$ or $a \in Y'$. Suppose $a \in X'$. By Lemma 1, $a$ is adjacent to every element in $Y''$. Suppose $a \in Y'$. By Lemma 1, $a$ is adjacent to every element in $X''$. Therefore $X'' \cup Y''$ is a dominating set in $< X, Y >$ .

Let $L_1 \not\subseteq X'' \cup Y''$ and $L_2 = (X'' \cup Y'') \setminus L_1$.

*Claim* 2 : Either $L_1$ or $L_2$ is not a dominating set in $< X, Y >$ .

Let $z \in X''$ with $\overline{\Omega}_z = \phi$. Then either $z \in L_1$ or $z \in L_2$. Without loss of generality one can take $z \in L_1$. Let $w \in L$ and $w \neq z$. Then $\Omega_w \neq \phi$. If $w \in X''$, then $\theta_w = \theta_z$ and so by Lemma 1, $w, z$ are not adjacent in $\Gamma$. If $w \in Y''$, then $\overline{\theta}_w = \theta_z$. By Corollary 2(ii), $w$ is not adjacent to $z$. This implies that $L_2$ is not a dominating set in $< X, Y >$ .

For $1 \leq \ell \leq |X_i'|$, let $S_\ell$ be the set containing exactly one element from $X_i'$ $(\subseteq X_i)$ for $1 \leq i \leq 2^t$. By Lemma 12(vi), $S_\ell$ is a $\gamma$-set in $\Gamma$ for $1 \leq \ell \leq |X_i'|$. Let $S = \bigcup_{X_i'' \subseteq X_i} X_i''$ (From $2^t$ subsets $X$ as in Observation 6). By above Claim 1, $S$ is a dominating set in $\Gamma$ and $V(\Gamma) = \bigcup_{\ell=1}^{|X_i'|} S_\ell \ \cup \ S$ is a maximal domatic partition in $\Gamma$. From this $d(\Gamma) = \frac{n}{2^t} \prod_{\ell=t+1}^{q} |U(R_\ell)| + 1$. By Note 2(v), we have $d(\Gamma) = \delta(\Gamma) + 1$.

(ii) Proof follows from (i).

In the following theorem, we obtain the bondage number of $\Gamma$.

***Theorem* 15** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Then*

$$b(\Gamma) = \begin{cases} \lceil \frac{|R|}{2} \rceil \ if\ R = R_1\ is\ a\ field\ and\ |R_1/M_1| > 2; \\ \lceil \frac{|U(R)|}{2} \rceil \ if\ R\ is\ not\ a\ field\ and\ |R_i/M_i| > 2\ for\ every\ i; \\ \delta(\Gamma)\ if\ |R_i/M_i| = 2\ for\ some\ i. \end{cases}$$

PROOF : *Case* 1 : When $R = R_1$ is a field with $|R_1/M_1| > 2$. Then $\Gamma$ is complete and so $\mathrm{b}(\Gamma) = \lceil \frac{|R|}{2} \rceil$.

*Case* 2 : When $R$ is not a field and $|R_i/M_i| > 2$ for every $i$. Let $u \in U(R)$ and $z \in Z(R)$. By Note 2(ii), $|U(R)| \leq \deg_\Gamma(z) < \deg_\Gamma(u) = |R| - 1$. Again, by Note 2(v), $\delta(\Gamma) = |U(R)|$. Since $|U(R)| \geq 2,\ b(\Gamma) \leq |U(R)|$. Note that $< U(R) >= K_{|U(R)|}$ in $\Gamma$ and by Lemma 12(ii), $\{u\}$ is a gamma set in $\Gamma$. To increase domination number by removing minimum number of edges, we have to remove at least one edge from each vertex in $< U(R) >$. Therefore $b(\Gamma) = \left\lceil \frac{|U(R)|}{2} \right\rceil$.

*Case* 3 : Assume that $|R_i/M_i| = 2$ for some $i$. As given in Observation 6, consider the subsets $X, Y$ in $R$. Let $x$ and $y$ be two distinct vertices in $\Gamma_c =< X, Y >$. Suppose $|R_i/M_i| = 2$ and $\alpha_i = 1$ for every $i$. By Corollary 8(i), $\Gamma = 2^{t-1}\ K_{1,1}$. Then $b(\Gamma) = \delta(\Gamma) = 1$.

Suppose $|R_i/M_i| = 2$ for every $i$ and $\alpha_\ell \geq 2$ for some $\ell$. By Corollary 8(i), $\Gamma = 2^{t-1}\ K_{\frac{|R|}{2^t}, \frac{|R|}{2^t}}$. Then $b(\Gamma) = \delta(\Gamma) = \frac{|R|}{2^t}$.

Suppose $|R_\ell/M_\ell| > 2$ for some $\ell$, $1 \leq \ell \leq q$. As observed in Note 2(v), $\delta(\Gamma) = \frac{n}{2^t} \prod\limits_{\ell=t+1}^{q} |U(R_\ell)|$. Let $E_1 \subseteq E(\Gamma)$ with $|E_1| = \delta(\Gamma) - 1$. Without loss of generality one can take $E_1 \subseteq E(\Gamma_c)$.

*Claim* 1 : $\gamma(\Gamma \setminus E_1) = \gamma(\Gamma)$.

It suffices to prove that $\gamma(\Gamma_c \setminus E_1) = \gamma(\Gamma_c)$. Note that $|X'| = |Y'| = \delta(\Gamma)$ and $\deg_\Gamma(a) = \frac{|R|}{2^t}$ for any $a \in X' \cup Y'$ (Lemma 11(ii)). Since $|X'| > \delta(\Gamma) - 1$, there exists $x \in X'$ such that $\deg_{\Gamma_c \setminus E_1}(x) = \frac{|R|}{2^t}$. Similarly, there exists $y \in Y'$ such that $\deg_{\Gamma_c \setminus E_1}(y) = \frac{|R|}{2^t}$. Since $|X| = |Y| = \frac{|R|}{2^t}$, $\{x, y\}$ is a minimal dominating set in $\Gamma_c \setminus E_1$. Therefore $\gamma(\Gamma_c \setminus E_1) = \gamma(\Gamma_c)$.

Let $z \in X$ with $\overline{\Omega}_z = \phi$ and $E_2$ be the set of all edges incident at $z$ in $G$. Then $|E_2| = \delta(\Gamma)$ and so $\Gamma_c \setminus E_2$ is disconnected.

*Claim* 2 : $\gamma(\Gamma \setminus E_2) > \gamma(\Gamma)$.

It suffices to prove that $\gamma(\Gamma_c \setminus E_2) > \gamma(\Gamma_c)$. Since $|X| = |Y| \geq 3$ and $|X'| = |Y'| \geq 2$, choose $x \in X$ with $\Omega_x = \phi$ and $y \in Y$ with $\Omega_y = \phi$. Then $\deg_{\Gamma_c \setminus E_2}(x) = \frac{|R|}{2^t}$ and $\deg_{\Gamma_c \setminus E_2}(y) = \frac{|R|}{2^t} - 1$. Now $\{x, y, z\}$ is a minimal dominating set in $\Gamma_c \setminus E_2$ and so $\gamma(\Gamma_c \setminus E_2) = 3$. As the argument of (ii) in Theorem 10, we have $\gamma(\Gamma_c) = 2$ and so $\gamma(\Gamma \setminus E_2) > \gamma(\Gamma)$. By Claim 1, $|E_2|$ is minimal and so b$(\Gamma) = \delta(\Gamma)$.

## 4. Domination Parameters of the Complement

In this section, first we obtain the domination number of $\overline{\Gamma}$ and subsequently characterize all $\gamma$-sets in $\overline{\Gamma}$. Also, we obtain the values of other domination parameters such as independent, strong and weak domination numbers of $\overline{\Gamma}$. In the following theorem, we obtain the domination number of $\overline{\Gamma}$.

***Theorem* 16** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Then*

$$\gamma(\overline{\Gamma}) = \begin{cases} |U(R)| + 1 & if\ |R_i/M_i| > 2\ for\ every\ i; \\ 2 & if\ |R_i/M_i| = 2\ for\ some\ i. \end{cases}$$

PROOF : *Case* 1 : When $|R_i/M_i| > 2$ for every $i$. For $u \in U(R)$, $\deg_{\Gamma}(u) = |R| - 1$ in $\Gamma$ and so $u$ is an isolated vertex in $\overline{\Gamma}$. From this $\overline{\Gamma} = |U(R)|\ K_1 \cup < Z(R) >$ . Let $z \in Z(R_1) \times \cdots \times Z(R_q)$. Then $\overline{\Omega}_z = \phi$. Suppose $v \in Z(R)$ and $z \neq v$. Then $\Omega_v \neq \phi$. By Corollary 2(iv), $z, v$ are adjacent in $\overline{\Gamma}$. Therefore $\{z\}$ is a minimal dominating set in $\langle Z(R) \rangle \subset \overline{\Gamma}$ and so $\gamma(\overline{\Gamma}) = |U(R)| + 1$.

*Case* 2 : When $|R_i/M_i| = 2$ for some $i$. Consider the subsets $X$ and $Y$ specified in Observation 6. Note that $|X| = |Y| \geq 1$. Suppose $a, b$ are two distinct elements in $X$. Then $\theta_a = \theta_b$. By Lemma 1, $a, b$ are adjacent in $\overline{\Gamma}$. This gives that $< X >$ is a complete subgraph in $\overline{\Gamma}$. Similarly $< Y >$ is also a complete subgraph of $\overline{\Gamma}$. Let $x \in X$ and $y \in Y$. Suppose $z \in R \setminus \{x, y\}$. Then any one of the following is true: (i) $z \in Y$; (ii) $z \in R \setminus Y$;

(i) If $z \in Y$, then $\theta_y = \theta_z$ and by Lemma 1, $y, z$ are adjacent in $\overline{\Gamma}$.

(ii) If $z \in R \setminus Y$, then either $z \in X$ or $z \in R \setminus (X \cup Y)$. When $z \in X$. Then $\theta_x = \theta_z$ and by Lemma 1, $x, z$ are adjacent in $\overline{\Gamma}$. When $z \in R \setminus (X \cup Y)$. Then $\theta_x \neq \overline{\theta}_z$ and so $x$ and $z$ are adjacent in $\overline{\Gamma}$. Therefore $\{x, y\}$ is a dominating set in $\overline{\Gamma}$. By Lemma 3, $\Gamma$ has no isolated vertices and hence $\gamma(\overline{\Gamma}) > 1$. This implies $\{x, y\}$ is a minimal domination set and so $\gamma(\overline{\Gamma}) = 2$.

Now we obtain a characterization of all $\gamma$-sets of $\overline{\Gamma}$.

*Lemma* 17 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \leq i \leq q$. Then the following are true.

(i) Let $|R_i/M_i| > 2$ for every $i$. Then $S \subseteq R$ is a $\gamma$-set in $\overline{\Gamma}$ if and only if $S = U(R) \cup \{z\}$ where $z \in Z(R)$ with $z_i \in Z(R_i)$ for every $i,\ 1 \leq i \leq q$;

(ii) Let $|R_i/M_i| = 2$ for some $i$. Then $S = \{a, b\} \subseteq R$ is a $\gamma$-set in $\overline{\Gamma}$ if and only if $a, b$ belong to two different subsets (from $2^t$ subsets) of $R$.

PROOF : (i) Assume that $S = U(R) \cup \{z\}$ for some $z \in Z(R)$ with $z_i \in Z(R_i)$ for every $i,\ 1 \leq i \leq q$. Then $\overline{\Omega}_z = \phi$ and by Corollary 2(iv), $z$ is adjacent to every other vertex in $< Z(R) > \subset \overline{\Gamma}$ and so $S$ is a $\gamma$-set in $\overline{\Gamma}$.

Conversely, assume that $S$ is a $\gamma$-set in $\overline{\Gamma}$. By Lemma 16, $|S| = |U(R)| + 1$. Since each vertex $u \in U(R)$ is an isolated vertex in $\overline{\Gamma}$, $U(R) \subset S$. Suppose $S = U(R) \cup \{z\}$ with $z_i \notin Z(R_i)$ for some $i,\ 1 \leq i \leq q$. Then $\Omega_z \not\subseteq \{1, \ldots, q\}$. Choose $w \in Z(R)$ such that $\Omega_w = \overline{\Omega}_z$. By Corollary 2(iii), $z, w$ are not adjacent in $\overline{\Gamma}$. Note that, for any $u \in U(R)$, $u$ is an isolated vertex in $\overline{\Gamma}$. This implies that $w$ is not adjacent to any element in $S$ and so $S$ is not a $\gamma$-set in $\overline{\Gamma}$, a contradiction.

(ii) Let $|R_i/M_i| = 2$ for some $i$.

Assume that $a, b$ belong to different subsets(from $2^t$ subsets) of $R$. Without loss of generality assume that $a \in X$. Note that $b \in R \setminus X$.

*Case* 1 : Assume that $b \in Y$. Suppose $c \in R \setminus \{a, b\}$. Then any one of the following is true:

(i) $c \in Y$; (ii) $c \in R \setminus Y$.

(ii) If $c \in Y$, then $\theta_b = \theta_c$ and by Lemma 1, $b, c$ are adjacent in $\overline{\Gamma}$. (ii) If $c \in R \setminus Y$, then either $c \in X$ or $c \in R \setminus (X \cup Y)$. When $c \in X$. Then $\theta_a = \theta_c$ and by Lemma 1, $a, c$ are adjacent in $\overline{\Gamma}$. When $c \in R \setminus (X \cup Y)$. Then $\theta_a \neq \overline{\theta}_c$ and by Lemma 1, $a$ and $c$ are adjacent in $\overline{\Gamma}$. Therefore $S = \{a, b\}$ is a $\gamma$-set in $\overline{\Gamma}$.

*Case* 2 : Assume that $b \notin Y$. Since $b \notin X,\ b \in R \setminus (X \cup Y)$. Let $c \in R \setminus \{a, b\}$. Then any one of the following is true: (i) $c \in Y$; (ii) $c \in R \setminus Y$.

(i) If $c \in Y$, then $\theta_b \neq \overline{\theta}_c$ and so $b$ and $c$ are adjacent in $\overline{\Gamma}$.

(ii) If $c \in R \setminus Y$, then $a$ and $c$ are adjacent in $\overline{\Gamma}$. Therefore $S = \{a, b\}$ is a $\gamma$-set in $\overline{\Gamma}$.

Conversely assume that $S$ is a $\gamma$-set in $\overline{\Gamma}$. By Lemma 16, $|S| = 2$ and so $S = \{a, b\}$. If $a$ and $b$ belong to the same subset (from $2^t$ subsets) of $R$, then $\theta_a = \theta_b$.

*Subcase* 2.1 : When $|R_i/M_i| = 2$ for every $i$. Choose $u \in R$ with $\theta_u = \overline{\theta}_a$. By Corollary 2(i), $u$ is not adjacent to both $a$ and $b$ in $\overline{\Gamma}$. Therefore $S$ is not $\gamma$-set in $\overline{\Gamma}$, a contradiction.

*Subcase* 2.2 : When $|R_i/M_i| > 2$ for some $i$. Choose $u \in R$ with $\theta_u = \overline{\theta}_a$ and $\Omega_u = \phi$. By Lemma 1, $u$ is not adjacent to both $a$ and $b$ in $\overline{\Gamma}$. Therefore $S$ is not $\gamma$-set in $\overline{\Gamma}$, a contradiction.

In the following corollary, we characterize when $\overline{\Gamma}$ is excellent.

*Corollary* 18 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \leq i \leq q$. Then $\overline{\Gamma}$ is excellent if and only if either $R = R_1$ with $|R_1/M_1| > 2$ or $|R_i/M_i| = 2$ for some $i$.

*Corollary* 19 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ for $1 \leq i \leq q$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$. Suppose $R$ is not a field. Then the following are true:

(i) $i(\overline{\Gamma}) = \gamma_p(\overline{\Gamma}) = \gamma_{eff}(\overline{\Gamma}) = \gamma_s(\overline{\Gamma}) = |U(R)| + 1\ if\ |R_i/M_i| > 2$ for every $i$;

(ii) $\gamma_w(\overline{\Gamma}) = |U(R)| + q$ if $|R_i/M_i| > 2$ for every $i$;

(iii) $i(\overline{\Gamma}) = \gamma_s(\overline{\Gamma}) = \gamma_w(\overline{\Gamma}) = 2$ if $|R_i/M_i| = 2$ for every $i$;

(iv) $\gamma_p(\overline{\Gamma}) = \gamma_{eff}(\overline{\Gamma}) = 2$ and $\gamma_t(\overline{\Gamma}) = 4$ if $R = R_1$ with $|R_1/M_1| = 2$;

(v) $\gamma_p(\overline{\Gamma}) = \gamma_{eff}(\overline{\Gamma}) = 2\ if\ q \geq 2$ with $|R_1/M_1| = 2$ and $|R_\ell/M_\ell| > 2$ for $2 \leq \ell \leq q$;

(vi) $i(\overline{\Gamma}) = \gamma_t(\overline{\Gamma}) = \gamma_c(\overline{\Gamma}) = \gamma_{cl}(\overline{\Gamma}) = \gamma_s(\overline{\Gamma}) = \gamma_w(\overline{\Gamma}) = 2$ if $q \geq 2$ with $|R_i/M_i| = 2$ and $|R_\ell/M_\ell| > 2$ for some $1 \leq i, \ell \leq q$.

PROOF : (i) Assume that $|R_i/M_i| > 2$ for every $i$. Then $\overline{\Gamma} = |U(R)|K_1 \cup < Z(R) >$. Since $\gamma(\overline{\Gamma}) = |U(R)| + 1$, we have $i(\overline{\Gamma}) = \gamma_p(\overline{\Gamma}) = \gamma_{eff}(\overline{\Gamma}) = \gamma_s(\overline{\Gamma}) = |U(R)| + 1$.

(ii) Assume that $|R_i/M_i| > 2$ for every $i$. Then $\overline{\Gamma} = |U(R)|K_1 \cup < Z(R) >$.

When $q = 1$. Note that $< Z(R) >$ is a complete subgraph in $\overline{\Gamma}$. This implies any gamma set in $\overline{\Gamma}$ is a $\gamma_w$-set in $\overline{\Gamma}$ and so $\gamma_w(\overline{\Gamma}) = |U(R)| + 1$.

When $q \geq 2$. Let $a$ and $b$ be two distinct elements in $Z(R)$ such that $\Omega_a \subseteq \Omega_b$. By Lemma 3(i), $\deg_\Gamma(a) \geq \deg_\Gamma(b)$. This implies that $\deg_{\overline{\Gamma}}(a) \leq \deg_{\overline{\Gamma}}(b)$ whenever $\Omega_a \subseteq \Omega_b$. Consider the set $T = \{x^i = (x_1, \ldots, x_q) \in Z(R)$ such that $x_i = 0$ and $x_\ell = 1$ for some $\ell$, $1 \leq i \neq \ell \leq q\}$. Then $|T| = q$. Let $z \in Z(R) \setminus T$. Then $\Omega_z \cap \Omega_{x^i} \neq \phi$ for some $i$ and so $\Omega_z \not\subseteq \overline{\Omega}_{x_i}$ for some $i$. By Corollary 2(iii), $z$ is adjacent to $x^i \in T$ for some $i$ in $\overline{\Gamma}$ and so $T \cup U(R)$ is a domination set in $\overline{\Gamma}$. For any $z \in Z(R) \setminus T$, there exists $x^i \in T$ such that $\Omega_{x^i} \subseteq \Omega_z$. Then $\deg_{\overline{\Gamma}}(x^i) \leq \deg_{\overline{\Gamma}}(z)$. Therefore $T \cup U(R)$ is a $\gamma_w$-set in $\overline{\Gamma}$ and so $\gamma_w(\overline{\Gamma}) = |U(R)| + q$.

(iii) Suppose $|R_i/M_i| = 2$ for every $i$. By Corollary 8(i), $\Gamma = 2^{t-1}K_{\frac{|R|}{2^t}, \frac{|R|}{2^t}}$ and so $\overline{\Gamma}$ is a regular graph. This implies any gamma set is both a strong dominating set as well as a weak dominating set in $\overline{\Gamma}$. From this $\gamma_s(\overline{\Gamma}) = \gamma_w(\overline{\Gamma}) = 2$. Consider the subsets $X, Y$ as specified in Observation 6. Let $a \in X,\ b \in Y$ and $S = \{a, b\}$. By Lemma 17(ii), $S$ is a $\gamma$-set in $\overline{\Gamma}$. Note that $\theta_a = \overline{\theta}_b$. By Corollary 8(i), $a$ and $b$ are not adjacent in $\overline{\Gamma}$. Therefore $i(\overline{\Gamma}) = 2$.

(iv) Let $R = R_1$ with $|R_1/M_1| = 2$. By Corollary 8(i), $\Gamma = K_{\frac{|R|}{2}, \frac{|R|}{2}}$ and so $\overline{\Gamma} = K_{\frac{|R|}{2}} \cup K_{\frac{|R|}{2}}$. From this we have that $\gamma_p(\overline{\Gamma}) = \gamma_{eff}(\overline{\Gamma}) = 2$. Since $R$ is not a field, $|R| = 2^{\alpha_1}$ where $\alpha_1 \geq 2$. Note that $\frac{|R|}{2} \geq 2$. Therefore $\gamma_t(\overline{\Gamma}) = 4$.

(v) Let $q \geq 2$ with $|R_1/M_1| = 2$ and $|R_\ell/M_\ell| > 2$ for every $\ell$, $2 \leq \ell \leq q$. Consider the subsets $X, Y$ as specified in Observation 6. Then $R = X \cup Y$ and $|X| = |Y| \geq 2$. Choose $x \in X$ such that $\Omega_x = \phi$ and $y \in Y$ such that $\Omega_y = \phi$. Then $\theta_x = \overline{\theta}_y$. Let $S = \{x, y\}$. By Lemma 17(ii), $S$ is a $\gamma$-set in $\overline{\Gamma}$. Let $a \in X \setminus \{x\}$ and $b \in Y \setminus \{y\}$. Then $\theta_x = \overline{\theta}_b$ and $\theta_y = \overline{\theta}_a$. By Lemma 1, $x$ and $b$ are adjacent in $\Gamma$ and so $x, b$ are not adjacent in $\overline{\Gamma}$. Similarly $y, a$ are not adjacent in $\overline{\Gamma}$. Therefore $\gamma_p(\overline{\Gamma}) = 2$. Since $x$ and $y$ are not adjacent in $\overline{\Gamma}$, $\gamma_{eff}(\overline{\Gamma}) = 2$.

(vi) Suppose $|R_i/M_i| = 2$ for some $i$ and $|R_\ell/M_\ell| > 2$ for some $\ell$. Choose $x \in X$ such that $\Omega_x = \phi$ and $y \in Y$ such that $\Omega_y = \phi$. Then $\theta_x = \overline{\theta}_y$. Let $S = \{x, y\}$. By Lemma 17(ii), $S$ is a $\gamma$-set in $\overline{\Gamma}$. Since $x$ and $y$ are not adjacent in $\overline{\Gamma}$, $i(\overline{\Gamma}) = 2$. By Corollary 9, $\deg_{\overline{\Gamma}}(x) = \deg_{\overline{\Gamma}}(y)$. Again, by Lemma 3(ii), $\deg_\Gamma(x) = \deg_\Gamma(y) = \frac{|R|}{2^t}$. Also, by Note 2(iv), $\Delta(\Gamma) = \deg_\Gamma(x) = \frac{|R|}{2^t}$. This implies $\deg_{\overline{\Gamma}}(x) = \delta(\overline{\Gamma})$ and so $\gamma_w(\overline{\Gamma}) = 2$.

Choose $x \in X$ such that $\overline{\Omega}_x = \phi$ and $y \in Y$ such that $\overline{\Omega}_y = \phi$. Then $\theta_x = \overline{\theta}_y$. Let $S = \{x, y\}$. By Lemma 17(ii), $S$ is a $\gamma$-set in $\overline{\Gamma}$. Since $x$ and $y$ are adjacent in $\overline{\Gamma}$, $\gamma_t(\overline{\Gamma}) = \gamma_c(\overline{\Gamma}) = \gamma_{cl}(\overline{\Gamma}) = 2$. By Corollary 9, $\deg_{\overline{\Gamma}}(x) = \deg_{\overline{\Gamma}}(y)$. As observed in Note 2(iii), $\deg_\Gamma(x) = \deg_\Gamma(y) < \deg_\Gamma(z)$ where $z \in R$ such that $\Omega_z \not\subseteq \{1, \ldots, q\}$. From this, $\deg_{\overline{\Gamma}}(x) = \deg_{\overline{\Gamma}}(y) > \deg_{\overline{\Gamma}}(z)$ and hence $\gamma_s(\overline{\Gamma}) = 2$.

In Corollary 19(i),(iv) and (v), we obtained the perfect domination number $\overline{\Gamma}$ for some classes of finite rings. For rest of the finite rings, in the following corollary, we obtain a characterization for $\overline{\Gamma}$ to have a perfect domination set and hence we discuss the nature of efficient dominating sets in $\overline{\Gamma}$.

*Corollary* 20 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \le i \le q$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \le i \le q$. Assume that $R$ with $q \ge 2$ with $|R_1/M_1| = 2$ and $|R_2/M_2| = 2$. Then the following are true:

(i) $\overline{\Gamma}$ has a proper perfect dominating set if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$;

(ii) There does not exist any efficient dominating set in $\overline{\Gamma}$.

PROOF : Since $R$ with $q \ge 2$ with $|R_1/M_1| = 2$ and $|R_2/M_2|$, we have $t \ge 2$. As given in Observation 6, consider the subsets $X_i, Y_i$ for $1 \le i \le 2^{t-1}$.

(i) Assume that $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\overline{\Gamma} = K_{2,2}$ and so $\overline{\Gamma}$ has a perfect dominating set with $\gamma_p(\overline{\Gamma}) = 2$.

Conversely assume that $\overline{\Gamma}$ has a proper perfect domination set. To complete the proof, it is enough to show that $\overline{\Gamma}$ does not have any proper perfect dominating set in the following two cases.

(1) $q = t \ge 3$ with $\alpha_i = 1$ for every $i$;

(2) $R$ satisfies either $|R_i/M_i| = 2$ with $\alpha_i \ge 2$ for some $i$ or $|R_i/M_i| > 2$ for some $i$.

(1) Assume that $q = t \ge 3$ with $\alpha_i = 1$ for every $i$. Note that $|R| = 2^t$ and $|X_i| = |Y_i| = 1$ for $1 \le i \le 2^{t-1}$. Suppose $P$ is a proper perfect dominating set in $\overline{\Gamma}$.

If $|P \cap X_i| = 1$ and $|P \cap Y_i| = 1$ for all $i$, $1 \le i \le 2^{t-1}$, then $P = R$, contradiction.

If $|P \cap (X_i \cup Y_i)| = 2$ and $|P \cap (X_\ell \cup Y_\ell)| \le 1$ for some $i, \ell$, $1 \le i, \ell \le 2^{t-1}$. Let $a \in P \cap X_i$

and $b \in P \cap Y_i$. Note that $\{a, b\} \subseteq P$ and $\theta_a = \overline{\theta}_b$. Let $z \in (X_\ell \cup Y_\ell) \setminus P$. Then $\theta_a \neq \overline{\theta}_z$ and $\theta_b \neq \overline{\theta}_z$. By Lemma 1, $z$ is adjacent to both $a$ and $b$ in $\overline{\Gamma}$, a contradiction.

If $|P \cap (X_i \cup Y_i)| \leq 1$ for all $i,\ 1 \leq i \leq 2^{t-1}$. Since $P$ is a dominating set in $\overline{\Gamma},\ |P| \geq 2$. Without loss of generality, one can take $|P \cap X_i| \leq 1$ for all $i,\ 1 \leq i \leq 2^{t-1}$. Also, $|P \cap X_1| = 1, |P \cap X_2| = 1$. Let $a \in P \cap X_1$ and $b \in P \cap X_2$. Since $q = t \geq 3$, consider the pair subsets $(X_3, Y_3)$. Let $z \in Y_3$. Then $z$ is adjacent to both $a$ and $b$ in $\overline{\Gamma}$, a contradiction. Hence $\overline{\Gamma}$ does not have any proper perfect dominating set.

(2) Assume that $R$ satisfies either $|R_i/M_i| = 2$ with $\alpha_i \geq 2$ for some $i$ or $|R_i/M_i| > 2$ for some $i$. Note that $|X_i| = |Y_i| \geq 2$ for $1 \leq i \leq 2^{t-1}$. Suppose $P$ is a proper perfect dominating set in $\overline{\Gamma}$. Then $P$ satisfies any one of the following cases:

(a) $|P \cap X_i| \geq 2$ for some $i,\ 1 \leq i \leq 2^t$;

(b) $|P \cap X_i| = 1$ and $|P \cap Y_i| = 1$ for some $i,\ 1 \leq i \leq 2^{t-1}$;

(c) $|P \cap (X_i \cup Y_i)| \leq 1$ for all $i,\ 1 \leq i \leq 2^{t-1}$.

(a) Assume that $|P \cap X_i| \geq 2$ for some $i,\ 1 \leq i \leq 2^t$.

Without loss of generality, one can take $|P \cap X_1| \geq 2$. Let $a, b \in P \cap X_1$ such that $a \neq b$. Suppose $R \setminus Y_1 \not\subset P$. Let $z \in R \setminus Y_1$ and $z \notin P$. Then $\theta_z \neq \overline{\theta}_a$ and $\theta_z \neq \overline{\theta}_b$. This implies $z$ is adjacent to both $a$ and $b$ in $\overline{\Gamma}$, a contradiction. Hence $R \setminus Y_1 \subseteq P$. Suppose $Y_1 \not\subset P$. Let $z \in Y_1$ and $z \notin P$. Note that $z$ is adjacent to every element from $R \setminus X_1$ in $\overline{\Gamma}$. This gives a contradiction. Hence $P = R$, a contradiction.

(b) Assume that $|P \cap X_i| = 1$ and $|P \cap Y_i| = 1$ for some $i,\ 1 \leq i \leq 2^{t-1}$. Let $a \in P \cap X_1$ and $b \in P \cap Y_1$. Suppose $|P \cap (X_\ell \cup Y_\ell)| \leq 1$ for all $\ell,\ 1 \leq \ell \neq i \leq 2^{t-1}$. Since $t \geq 2$, consider the pair subset $(X_2, Y_2)$ in $R$. Since $|X_1| = |Y_1| \geq 2$, choose $z \notin P \cap (X_2 \cup Y_2)$. Then $z$ is adjacent to both $a$ and $b$ in $\overline{\Gamma}$, a contradiction. Suppose $|P \cap X_\ell| = 1$ and $|P \cap Y_\ell| = 1$ for some $\ell,\ 1 \leq \ell \neq i \leq 2^{t-1}$. Let $d \in P \cap X_\ell$. Since $|X_1| \geq 2$, choose $z \in X_1 \setminus \{a\}$. Then $z$ is adjacent to both $a$ and $d$ in $\overline{\Gamma}$, a contradiction.

(c) Assume that $|P \cap (X_i \cup Y_i)| \leq 1$ for all $i,\ 1 \leq i \leq 2^{t-1}$. Since $P$ is a dominating set in $\overline{\Gamma},\ |P| \geq 2$. Without loss of generality, one can take $|P \cap X_i| \leq 1$ for all $i,\ 1 \leq i \leq 2^{t-1}$. Also, $|P \cap X_1| = 1, |P \cap X_2| = 1$. Since $|X_1| \geq 2$, choose $z \in X_1 \setminus \{a\}$. Then $z$ is adjacent to both $a$ and $b$ in $\overline{\Gamma}$, a contradiction. Hence $\overline{\Gamma}$ does not have any proper perfect dominating set.

(ii) By above (i), it is enough to show that $\overline{\Gamma}$ does not have any efficient dominating set when

$R = \mathbb{Z}_2 \times \mathbb{Z}_2$. If $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{\Gamma} = K_{2,2}$ and so $\overline{\Gamma}$ does not have any efficient dominating set.

PROOF : In the following theorem, we obtain the maximal independent number of $\overline{\Gamma}$ for some finite rings and for the remaining rings we give a lower bound.

***Theorem* 21** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$ $1 \le i \le q$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \le i \le q$. Then*

$$\beta_0(\overline{\Gamma}) = \begin{cases} 2 \; if \; |R_i/M_i| = 2 \; for \; some \; i; \\ |U(R)| + 1 \; if \; R = R_1 \; with \; |R_1/M_1| > 2; \\ > |U(R)| + 1 \; if \; q \ge 2 \; with \; |R_i/M_i| > 2 \; for \; every \; i. \end{cases}$$

PROOF : *Case* 1 : Assume that $|R_i/M_i| = 2$ for some $i$. Let $X^* = Z(R_1) \times R_2 \times \cdots \times R_q$ and $Y^* = U(R_1) \times R_2 \times \cdots \times R_q$. Then $R = X^* \cup Y^*$. Note that $< X^* >$ and $< Y^* >$ are complete subgraphs in $\overline{\Gamma}$.

If $|R_i/M_i| = 2$ for every $i$, then choose $x \in X^*$ and $y \in Y^*$ such that $\theta_x = \overline{\theta}_y$. If $|R_i/M_i| = 2$ for some $i$ and $|R_\ell/M_\ell| > 2$ for some $\ell$, then choose $x \in X^*$ such that $\Omega_x = \phi$ and $y \in Y^*$ such that $\Omega_y = \phi$. In both the situations, $x$ and $y$ are not adjacent in $\overline{\Gamma}$. Suppose $z \in R \setminus \{x, y\}$. If $z \in X^*$, then $x, z$ are adjacent in $\overline{\Gamma}$. Similarly, if $z \in Y^*$, then $y, z$ are adjacent in $\overline{\Gamma}$. Therefore $\{x, y\}$ is a maximal independent set in $\overline{\Gamma}$ and $\beta_0(\overline{\Gamma}) = 2$.

*Case* 2 : Suppose $R = R_1 \; with \; |R_1/M_1| > 2$. Note that $< Z(R) >$ is a complete subgraph in $\overline{\Gamma}$ and for any $u \in U(R)$, $u$ is an isolated vertex in $\overline{\Gamma}$. This implies $U(R) \cup \{z\}$ where $z \in Z(R)$ is a maximal independent set in $\overline{\Gamma}$ and $\beta_0(\overline{\Gamma}) = |U(R)| + 1$.

*Case* 3 : Suppose $q \ge 2 \; with \; |R_i/M_i| > 2 \; forevery \; i$. For any $u \in U(R)$, $u$ is an isolated vertex in $\overline{\Gamma}$. Since $q \ge 2$, choose $a \in Z(R)$ with $\Omega_a \not\subseteq \{1, \ldots, q\}$ and $b \in Z(R)$ with $\Omega_a = \overline{\Omega}_b$. Then $a, b$ are not adjacent in $\overline{\Gamma}$. Since $U(R) \cup \{a, b\}$ is an independent set in $\overline{\Gamma}$, $\beta_0(\overline{\Gamma}) > |U(R)| + 1$.

In the following theorem, we characterize when $\overline{\Gamma}$ is well-covered.

***Theorem* 22** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \le i \le q$. Then $\overline{\Gamma}$ is well-covered if and only if $R$ satisfies any one of the following:*

*(a) $q = 1$ and $R$ is a field; (b) $q \ge 1$, $R$ is not a field and $|R_1/M_1| = 2$; (c) $R = R_1 \; with \; |R_1/M_1| >$ 2.*

PROOF : If $R$ is a field, then $\overline{\Gamma}$ is complete and so $\overline{\Gamma}$ is well-covered.

Assume that $q \ge 1$, $R$ is not a field and $|R_1/M_1| = 2$. By Corollary 19(iii), (vi) and Theorem 21,

$i(\overline{\Gamma}) = 2 = \beta_0(\overline{\Gamma})$.

When $R = R_1$ is not a field and $|R_1/M_1| > 2$. By Corollary 19(vi) and Theorem 21, $i(\overline{\Gamma}) = 2 = \beta_0(\overline{\Gamma})$.

Assume that $R = R_1$ with $|R_1/M_1| > 2$. By Corollary 19(i) and Theorem 21, $i(\overline{\Gamma}) = |U(R)| + 1 = \beta_0(\overline{\Gamma})$.

In all the cases $\overline{\Gamma}$ is well-covered. Suppose either (a) or (b) or (c) is not true. Then we have $q \geq 2$ and $|R_i/M_i| > 2$ for every $i$. By Corollary 19(i) and Theorem 21, $i(\overline{\Gamma}) \neq \beta_0(\overline{\Gamma})$, a contradiction.

In the following theorem, we obtain the domatic number of $\overline{\Gamma}$.

***Theorem* 23** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$. Then*

$$d(\overline{\Gamma}) = \begin{cases} 1 & if\ |R_i/M_i| > 2\ for\ every\ i; \\ \frac{|R|}{2} & if\ |R_i/M_i| = 2\ for\ some\ i. \end{cases}$$

PROOF : Suppose $|R_i/M_i| > 2$ for every $i$. Then $\overline{\Gamma} = |U(R)|\ K_1 \cup < Z(R) >$ and so $d(\overline{\Gamma}) = 1$.

Suppose $|R_i/M_i| = 2$ for some $i$. As given in Observation 6, consider $2^{t-1}$ pairs of subsets $(X_\ell, Y_\ell)$ for $1 \leq \ell \leq 2^{t-1}$. Let $X^* = \bigcup\limits_{\ell=1}^{2^{t-1}} X_\ell$ and $Y^* = \bigcup\limits_{\ell=1}^{2^{t-1}} Y_\ell$. Then $|X^*| = |Y^*| = \frac{|R|}{2}$. Let $S = \{\{x, y\} : x \in X^* \text{ and } y \in Y^*\}$. Then $|S| = \frac{|R|}{2}$. Let $\{x, y\} \in S$. By Lemma 17(ii), $\{x, y\}$ is a $\gamma$-set in $\overline{\Gamma}$ and so $d(\overline{\Gamma}) = \frac{|R|}{2}$.

In the following corollary, we characterize when $\overline{\Gamma}$ is domatically full.

*Corollary* 24 — Let $R$ be the direct product of finite local rings $(R_i, M_i)$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$. Then $\overline{\Gamma}$ is domatically full if and only if $R$ satisfies any one of the following:

(a) $|R_i/M_i| > 2$ for every $i$; (b) $R = R_1$ with $|R_1/M_1| = 2$.

PROOF : Assume that $|R_i/M_i| > 2$ for every $i$. Then $\overline{\Gamma} = |U(R)|\ K_1 \cup < Z(R) >$. By Theorem 23, $d(\overline{\Gamma}) = \delta(\overline{\Gamma}) + 1 = 1$. Therefore $\overline{\Gamma}$ is domatically full. Suppose $|R_1/M_1| = 2$. Then $\overline{\Gamma} = K_{2^{\alpha_1 - 1}} \cup K_{2^{\alpha_1 - 1}}$ and so $\delta(\overline{\Gamma}) = 2^{\alpha_1 - 1} - 1 = \frac{|R|}{2} - 1$. By Theorem 23, $d(\overline{\Gamma}) = \delta(\overline{\Gamma}) + 1$ and so $\overline{\Gamma}$ is domatically full.

When $t = 1$ and $q \geq 2$. By Note 2(iv), $\Delta(\Gamma) = \frac{|R|}{2}$ and so $\delta(\overline{\Gamma}) = \frac{|R|}{2} - 1$. By Theorem 23, $d(\overline{\Gamma}) = \delta(\overline{\Gamma}) + 1$ and so $\overline{\Gamma}$ is domatically full.

Conversely assume that $\overline{\Gamma}$ is domatically full. Suppose $q \geq 2$ and $t \geq 2$. By Note 2(iv), $\Delta(\Gamma) =$

$\frac{|R|}{2^t}$. Note that $\frac{|R|}{2^t} < \frac{|R|}{2}$ as $t \geq 2$ and so $\frac{|R|}{2^t} \leq \frac{|R|}{2} - 1$. This implies $\delta(\overline{\Gamma}) \geq \frac{|R|}{2}$. By Theorem 23, $d(\overline{\Gamma}) = \frac{|R|}{2}$. Therefore $d(\overline{\Gamma}) \neq \delta(\overline{\Gamma}) + 1$, a contradiction.

In the following theorem, we obtain the total domatic number of $\overline{\Gamma}$.

***Theorem* 25** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$ and $R$ is not a field. Then*

$$d_t(\overline{\Gamma}) = \begin{cases} |R|/4 & if\ R = R_1\ with\ |R_1/M_1| = 2; \\ |R|/2 & if\ q \geq 2\ with\ |R_1/M_1| = 2\ and\ |R_2/M_2| = 2; \\ |R|/2 - (|R_1|/4 \prod_{\ell=2}^{q} |U(R_\ell)|) & if\ q \geq 2,\ |R_1/M_1| = 2,\ |R_\ell/M_\ell| > 2\ for\ 2 \leq \ell \leq q; \\ does\ not\ exist & if\ |R_i/M_i| > 2\ for\ every\ i. \end{cases}$$

PROOF : *Case* 1 : Suppose $R = R_1$ with $|R_1/M_1| = 2$. Since $R$ is not a field, $|R| = 2^{\alpha_1}$ where $\alpha_1 \geq 2$. By Corollary 8(i), $\overline{\Gamma} = K_{2^{\alpha_1 - 1}} \cup K_{2^{\alpha_1 - 1}}$ and so $d_t(\overline{\Gamma}) = \frac{|R|}{4}$.

*Case* 2 : By the assumption, $t \geq 2$. As given in Observation 6, there exist $2^{t-1}$ ($\geq 2$ and even) pair subsets $X_i, Y_i$ in $R$ with $|X_i| = |Y_i| = \frac{|R|}{2^t}$ for $1 \leq i \leq 2^{t-1}$.

Since $t \geq 2$, $2^{t-1} - 1 \geq 1$. Let $S_j = \{\{x, y\} : x \in X_j \text{ and } y \in Y_{j+1}\}$ for $1 \leq j \leq 2^{t-1} - 1$, $S_{2^{t-1}} = \{\{x, y\} : x \in X_{2^{t-1}} \text{ and } y \in Y_1\}$ and $S = \bigcup_{j=1}^{2^{t-1}-1} S_j \cup S_{2^{t-1}}$. Then $|S| = \frac{|R|}{2}$. For each $\{x, y\} \in S$, we have that $\theta_x \neq \overline{\theta}_y$ and so $x$ and $y$ are adjacent in $\overline{\Gamma}$. By Lemma 17(ii), $\{x, y\}$ is a $\gamma$-set in $\overline{\Gamma}$. From this $V(\overline{\Gamma}) = \bigcup_{\{x,y\} \in S} \{x, y\}$ is a maximal total domatic partition in $\overline{\Gamma}$ and so $d_t(\overline{\Gamma}) = \frac{|R|}{2}$.

*Case* 3 : Suppose $q \geq 2 \text{ with } |R_1/M_1| = 2 \text{ and } |R_\ell/M_\ell| > 2 \text{ for } 2 \leq \ell \leq q$. Let $X$ and $Y$ be subsets specified in Observation 6. Note that $R = X \cup Y$ and $|X| = |Y| \geq 3$.

As observed in Observation 6, consider the subsets $X'$ and $Y'$. Let $X'' = X \setminus X'$ and $Y'' = Y \setminus Y'$. Let $M_i = \{x = (x_1, \ldots, x_q) \in X'' : x_i \in Z(R_i) \text{ and } x_\ell \notin Z(R_\ell) \text{ for } \ell \in \{2, \ldots, i-1\}\}$ for $i \in \{2, \ldots, q\}$. Then $\bigcup_{i=2}^{q} M_i = X''$.

Let $N_i = \{y = (y_1, \ldots, y_q) \in Y'' : y_i \in Z(R_i) \text{ and } y_\ell \notin Z(R_\ell) \text{ for } \ell \in \{2, \ldots, i-1\}\}$ for $i \in \{2, \ldots, q\}$. Then $\bigcup_{i=2}^{q} N_i = Y''$ and so $|M_i| = |N_i|$ for all $i,\ 2 \leq i \leq q$.

Let $T_{i,j} = \{\{a, b\} : a \in M_i\ and\ b \in N_i\}$ for $i,\ 2 \leq i \leq q$ and $j,\ 1 \leq j \leq |M_i|$.

Let $a \in X''$ and $b \in Y''$. Then $\theta_a = \overline{\theta}_b$. By Lemma 17(ii), $\{a, b\}$ is a $\gamma$-set in $\overline{\Gamma}$. Let $a \in M_i$ and $b \in N_i$ for any $i,\ 2 \leq i \leq q$. Then $i \in \Omega_a \cap \Omega_b$ and so $\Omega_a \not\subseteq \overline{\Omega}_b$. By Lemma 1, $a, b$ are adjacent in $\overline{\Gamma}$. This implies each $\{a, b\} \in T_{i,j}$ is $\gamma_t$-set in $\overline{\Gamma}$ for $i,\ 2 \leq i \leq q$ and $j,\ 1 \leq j \leq |M_i|$.

Note that $|X'| = |Y'|$ and are even. Also, $< X' >$ and $< Y' >$ are complete subgraphs in $\overline{\Gamma}$. For $1 \leq \ell \leq \frac{|X'|}{2}$, let $S_\ell$ be the set containing four elements such that two elements from $X'$ and two elements from $Y'$. Then $S_\ell$ is a $\gamma_t$-set in $\overline{\Gamma}$ for $1 \leq \ell \leq \frac{|X'|}{2}$ and hence $V(\overline{\Gamma}) = \bigcup_{\ell=1}^{\frac{|X'|}{2}} S_\ell \bigcup_{\{a,b\}\in T_{i,j}} \{a,b\}$ is a maximal total domatic partition in $\overline{\Gamma}$ for $i,\ 2 \leq i \leq q$ and $j,\ 1 \leq j \leq |M_i|$. Note that $\frac{|X'|}{2}+|X''| = \frac{|R_1|}{4}\prod_{\ell=2}^{q}|U(R_\ell)| + (\frac{|R|}{2} - (\frac{|R_1|}{2}\prod_{\ell=2}^{q}|U(R_\ell)|))$. Therefore $d_t(\overline{\Gamma}) = \frac{|R|}{2} - (\frac{|R_1|}{4}\prod_{\ell=2}^{q}|U(R_\ell)|)$.

*Case* 4 : Suppose $|R_i/M_i| > 2$ for every $i$. Note that $\overline{\Gamma} = |U(R)|\ K_1 \cup < Z(R) >$ . Then $\overline{\Gamma}$ contains isolated vertices and so $d_t(\overline{\Gamma})$ does not exist.

In the following theorem, we obtain the bondage number of $\overline{\Gamma}$.

***Theorem* 26** — *Let $R$ be the direct product of finite local rings $(R_i, M_i)$. Assume that $|R_i| = p_i^{\alpha_i}$ where $p_i's$ are primes and $\alpha_i's$ are positive integers for $1 \leq i \leq q$ and $R$ is not a field. Then the following are true:*

(*i*) $$b(\overline{\Gamma}) = \begin{cases} |R|-1 & if\ R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2\ with\ q \geq 2; \\ \frac{|R|}{4}+1 & if\ R = \mathbb{Z}_2 \times \mathbb{F}_{2k}\ where\ k \geq 2; \\ \lceil \frac{|R|}{4} \rceil & if\ R = \mathbb{Z}_2 \times \mathbb{F}_{2k+1}\ where\ k \geq 1. \\ 2^{\alpha_1-2} & if\ R = R_1\ with\ |R_1/M_1| = 2\ and\ \alpha_1 \geq 2; \\ \lceil \frac{\prod_{\ell=1}^{q}|Z(R_\ell)|}{2} \rceil & if\ |R_i/M_i| > 2\ for\ every\ i; \end{cases}$$

(*ii*) $b(\overline{\Gamma}) = \frac{|R|}{2} - (\frac{|R_1|}{4}\prod_{\ell=2}^{q}|U(R_\ell)|)$ *if $R$ satisfies any one of the following:*

(a) $R = R_1 \times R_2$ *where* $|R_1/M_1| = 2$ *with* $\alpha_1 \geq 2$ *and* $|R_2/M_2| > 2$;

(b) $R = R_1 \times R_2$ *where* $|R_1/M_1| = 2$ *and* $|R_2/M_2| > 2$ *with* $R_2$ *is not a field*;

(c) $R = R_1 \times \cdots \times R_q\ with\ q \geq 3,\ |R_1/M_1| = 2\ and\ |R_k/M_k| > 2\ for\ 2 \leq k \leq q.$

PROOF : (i) Note that $R = R_1 \times R_2,\ R_1 = \mathbb{Z}_2, |R_2/M_2| > 2\ and\ R_2\ is a field.$ Let $X = \{0\} \times R_2$ and $Y = \{1\} \times R_2$. Then $R = X \cup Y$ and $|X| = |Y| = \frac{|R|}{2}$. Note that $< X >$ and $< Y >$ are complete subgraphs of $\overline{\Gamma}$. Also, $(0,0) \in X$ is the only element is adjacent to $(1,0) \in Y$ and vice-versa in $\overline{\Gamma}$.

*Case* 1 : Assume that $q \geq 2$ and $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. Then $|R_i/M_i| = 2$ with $\alpha_i = 1$ for every i. By Corollary 8(i), $\Gamma = 2^{q-1}K_{1,1}$ and so $\overline{\Gamma} = K_{2,\ldots,2}(2^{q-1}\ times)$. By Theorem 2 [3], b($\overline{\Gamma}$) $= 2(2^{q-1}) - 1 = |R| - 1$.

*Case* 2 : Assume that $R = \mathbb{Z}_2 \times \mathbb{F}_{2k}$ and $k \geq 2$. Note that $|X| = |Y| = \frac{|R|}{2}$. For $u_2 \in U(R_2)$, $(1,0)$ and $(1,u_2)$ are adjacent in $\overline{\Gamma}$ and $e_1$ is the corresponding edge in $E(\overline{\Gamma})$. Since $< Y >$ is complete, choose $E_1$ as the set of all edges in $< Y >$ such that $\gamma(< Y > -E_1) > \gamma(< Y >)$ with $|E_1| = \frac{|R|}{4}$ and $e_1 \in E_1$. Then $\{(0,0),(1,u_2)\}$ is a $\gamma$-set in $\overline{\Gamma} \setminus e_1$. Note that $(0,0)$ and $(1,0)$ are adjacent in $\overline{\Gamma}$ and $e_2$ is the corresponding edge in $E(\overline{\Gamma})$. Let $E_2 = E_1 \cup \{e_2\}$. Then $\gamma(\overline{\Gamma} - E_2) > \gamma(\overline{\Gamma})$ and so $b(\overline{\Gamma}) = \frac{|R|}{4} + 1$.

*Case* 3 : Assume that $R = \mathbb{Z}_2 \times \mathbb{F}_{2k+1}$ and $k \geq 1$. Consider the set $Y^* = Y \setminus \{(1,0)\}$. Then $|Y^*| = \frac{|R|}{2} - 1$ and is even. Let $E_3$ be the set of all edges in $< Y^* >$ such that $\gamma(< Y^* > -E_3) > \gamma(< Y^* >)$ with $|E_3| = \frac{|R|-2}{4}$. Note that $(1,0)$ and $(1,1)$ are adjacent in $\overline{\Gamma}$ and $e$ is the corresponding edge in $E(\overline{\Gamma})$. Let $E_4 = E_3 \cup \{e\}$. Note that $\frac{|R|-2}{4} + 1 = \left\lceil \frac{|R|}{4} \right\rceil$. Since $\gamma(\overline{\Gamma} - E_4) > \gamma(\overline{\Gamma}),\ b(\overline{\Gamma}) = \left\lceil \frac{|R|}{4} \right\rceil$.

*Case* 4 : Assume that $R = R_1$ with $|R_1/M_1| = 2$ and $\alpha_1 \geq 2$. By Corollary 8(i), $\overline{\Gamma} = K_{2^{\alpha_1-1}} \cup K_{2^{\alpha_1-1}}$. Since $b(K_{2^{\alpha_1-1}}) = 2^{\alpha_1-2},\ b(\overline{\Gamma}) = 2^{\alpha_1-2}$.

*Case* 5 : When $|R_i/M_i| > 2$ for every $i$. Then $\overline{\Gamma} = |U(R)|\ K_1 \cup < Z(R) >$.

Let $S_1 = Z(R_1) \times \cdots \times Z(R_q)$. Then $|S_1| = \prod_{\ell=1}^{q} |Z(R_\ell)|$. If $|S_1| = 1$, then $S_1 = \{0\}$. Since $R$ is not a field, $|Z(R)| \geq 2$. Let $w \in Z(R) \setminus \{0\}$. Then $\phi \neq \Omega_w \not\subseteq \{1,\ldots,q\}$. By Note 2(ii), $|U(R)| = \deg_\Gamma(0) < \deg_\Gamma(w) \leq |R| - 2$. This gives $\deg_{\overline{\Gamma}}(w) < \deg_{\overline{\Gamma}}(0) = |Z(R)| - 1$. Remove one edge $e$ which is incident at 0. Let $y \in Z(R)$. Then $\deg_{\overline{\Gamma}-e}(y) \leq |Z(R)| - 2$ and so $\{y\}$ is not a dominating set in $< Z(R) > -e$. This gives $\gamma(\overline{\Gamma} - e) > \gamma(\overline{\Gamma})$ and so $b(\overline{\Gamma}) = 1$.

Suppose $|S_1| \geq 2$. Note that $< S_1 >$ is a complete subgraph in $< Z(R) >$ (in $\overline{\Gamma}$). Suppose $w, z \in Z(R)$ such that $\Omega_w \not\subseteq \{1,\ldots,q\}$ and $\Omega_z = \{1,\ldots,q\}$. Note that for any $u \in U(R),\ u$ is an isolated vertex in $\overline{\Gamma}$ and $z$ is adjacent to every other vertex in $< Z(R) >$ (in $\overline{\Gamma}$). This implies, $|S_1| \leq \deg_{\overline{\Gamma}}(w) < \deg_{\overline{\Gamma}}(z) = |Z(R)| - 1$.

Let $E_2 \subseteq E(< S_1 >)$ with $|E_2| = \left\lceil \frac{\prod_{\ell=1}^{q} |Z(R_\ell)|}{2} \right\rceil$ such that $\{z\}$ is not a dominating set in $< S_1 >$ where $z \in S_1$. Therefore for any $y \in Z(R),\ \{y\}$ is not a dominating set in $< Z(R) > -E_2$. This implies $\gamma(\overline{\Gamma} \setminus E_2) > \gamma(\overline{\Gamma})$ and so $b(\overline{\Gamma}) = \left\lceil \frac{\prod_{\ell=1}^{q} |Z(R_\ell)|}{2} \right\rceil$.

(ii) Assume that $R$ satisfies any one of (a),(b) or (c). Then $t = 1$. As given in Observation 6, consider the sets $X, Y$. Then $|X| = |Y| \geq 3$. Let $x \in X$ and $y \in Y$. By Note 2(iv), $\Delta(\Gamma) = \frac{|R|}{2}$ and so $\delta(\overline{\Gamma}) = \frac{|R|}{2} - 1$. Let $x \in X'$. By Lemma 3(ii), $\deg_\Gamma(x) = \frac{|R|}{2}$ and so $\deg_{\overline{\Gamma}}(x) = \delta(\overline{\Gamma})$. Let $y \in Y$. Then $\theta_x = \overline{\theta}_y$ and so $x, y$ are not adjacent in $\overline{\Gamma}$. This implies in $\overline{\Gamma}$, any element in $X'$ is not adjacent to any element in $Y$. Similarly any element in $Y'$ is not adjacent to any element in $X$. Let $E_x$ be the set of all edges which is incident at $x$ in $\overline{\Gamma}$. Then $\gamma(\overline{\Gamma} \setminus E_x) = 3 > 2 = \gamma(\overline{\Gamma})$ and so $b(\overline{\Gamma}) \leq \delta(\overline{\Gamma})$.

Let $u \in X'$ and $v \in X'' = X \backslash X'$. Since $< X' >$ is a complete subgraph in $\overline{\Gamma}$, b$(< X' >) = \frac{|X'|}{2}$. Then $\{u, y\}$ is not a dominating set in $\overline{\Gamma} \setminus E_{X'}$ where $E_{X'}$ is the set of all edges in $< X' >$ such that b$(< X' >) = \frac{|X'|}{2}$ with $|E_{X'}| = \frac{|X'|}{2}$.

Suppose $a, b \in X''$. Note that $a$ and $b$ are adjacent in $\overline{\Gamma}$ and $e$ is the corresponding edge in $E(\overline{\Gamma})$. Choose $c \in Y''$ such that $\Omega_c = \Omega_b$. Then $b, c$ are adjacent in $\overline{\Gamma}$ and so $\{a, c\}$ is a $\gamma$-set in $\overline{\Gamma} - e$.

Let $E_2 = \{uv : \forall v \in X''\}$. Then $|E_2| = |X''| = \frac{|R|}{2} - (\frac{|R_1|}{2} \prod_{\ell=2}^{q} |U(R_\ell)|)$. Then $\{v, y\}$ is not dominating set in $\overline{\Gamma} \setminus E_2$. From this, $\{x, y\}$ is not a dominating set in $\overline{\Gamma} \setminus (E_1 \cup E_2)$. Then $\gamma(\overline{\Gamma} \setminus (E_1 \cup E_2)) \geq 3$.

Note that $\frac{|X'|}{2} + |X''| = \frac{|R_1|}{4} \prod_{\ell=2}^{q} |U(R_\ell)| + (\frac{|R|}{2} - \frac{|R_1|}{2} \prod_{\ell=2}^{q} |U(R_\ell)|) = \frac{|R|}{2} - (\frac{|R_1|}{4} \prod_{\ell=2}^{q} |U(R_\ell)|)$ and so $\frac{|X'|}{2} + |X''| \leq \delta(\overline{\Gamma})$.

Therefore $b(\overline{\Gamma}) = \frac{|R|}{2} - (\frac{|R_1|}{4} \prod_{\ell=2}^{q} |U(R_\ell)|)$.

## ACKNOWLEDGEMENT

This research work is supported by the SERB Project No. SR/S4/MS.806/13 of Science and Engineering Research Board, Government of India through the first author.

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