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SOME PROPERTIES OF THE c-NILPOTENT MULTIPLIER OF A PAIR OF LIE ALGEBRAS

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The concept of the Schur multiplier of a Lie algebra was introduced by Batten *et al*. (1996). This concept was extended to that of a c-nilpotent multiplier of a Lie algebra, and then further extended to a theory of a c-nilpotent multiplier of a pair of Lie algebras, by the author and others. In this paper we present some new inequalities for the dimension of the c-nilpotent multiplier of a pair of Lie algebras. We also give a necessary and sufficient condition for the c-nilpotent multiplier of a pair of Lie algebras to embed into the c-nilpotent multiplier of their quotient. Finally we provide a sufficient condition for the c-nilpotent multiplier of a pair of Lie algebras to be finite dimensional.

Key words : Pair of Lie algebras; c-nilpotent multiplier.

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1. Introduction and Preliminary

The Schur multiplier was born in Schur's works [20] on projective representation of a group in 1904. For a given group G, the Schur multiplier of G is $\mathcal{M}(G) = (R \cap G')/[R, F]$, where $G \cong F/R$ and F is a free group (See [10] for more information).

Ellis [9] extended the theory of the Schur multiplier for a pair of groups. Let (N, G) be a pair of groups, in which N is a normal subgroup of G. The Schur multiplier of (N, G) is an abelian group $\mathcal{M}(N, G)$ appearing in the following exact sequence

$$H_3(G) \to H_3(G/N) \to \mathcal{M}(N, G) \to \mathcal{M}(G) \to$$
$$\mathcal{M}(G/N) \to \frac{N}{[N, G]} \to (G)^{ab} \to (G/N)^{ab} \to 1,$$

in which $H_3(G)$ is the third homology of G. The Lie algebra analogue of the Schur multiplier was introduced by Batten et al. (See [6, 12] for more information).

In [14], the authors defined the Schur multiplier of a pair of Lie algebras. Let (N, L) be a pair of Lie algebras in which N is an ideal in L. The Schur multiplier of the pair (N, L) is

$$\mathcal{M}(N, L) = (R \cap [S, F])/[R, F],$$

where S is an ideal in free Lie algebra F such that $N \cong S/R$ (See [3, 13] for more details).

In [1, 2, 4, 15] we studied the concept of the c-nilpotent multiplier of a pair of Lie algebras. The c-nilpotent multiplier of a pair (N, L) is $\mathcal{M}^{(c)}(N, L) = (R \cap [S,_c F])/[R,_c F]$, where $[X,_c Y] = [X, \underbrace{Y, \dots, Y}_{c-times}]$ and $c \geq 1$. In particular, if $N = L$, then $\mathcal{M}^{(c)}(L, L) = \mathcal{M}^{(c)}(L)$ is the c-nilpotent multiplier of L (See [18, 19] for more information).

In this paper, we prove some inequalities for the dimension of the c-nilpotent multiplier of a pair of Lie algebras (Proposition 2.2). Moreover, we present a necessary and sufficient condition in which the c-nilpotent multiplier of pair of Lie algebras can be embedded into the c-nilpotent multiplier of their factor Lie algebras (Theorem 2.3). In [1], we proved that if (N, L) is a pair of finite dimensional Lie algebras, then the Lie algebra $\mathcal{M}^{(c)}(N, L)$ is finite dimensional. Here, we extend this result to a stronger version (Corollary 2.7).

All Lie algebras are considered over a fixed field Λ and $[,]$ denotes the Lie bracket. Let L and M be two Lie algebras. An action of L on M is an Λ-bilinear map

$$L \times M \to M$$

$$(l, m) \mapsto^{l} m$$

satisfying

$$^{[l,l']}m =^{l} (^{l'}m) -^{l'} (^{l}m) \quad \textit{and} \quad ^{l}[m, m'] = [^{l}m, m'] + [m,^{l} m'],$$

for all $l, l' \in L$ and $m, m' \in M$.

Recall from [11] that a crossed module of pairs of Lie algebras (N, L) is a Lie homomorphism $\sigma : M \to L$ together with an action of L on M, which is denoted by ^{l}m for all $l \in L, m \in M$ satisfying the following conditions:

(i) $\sigma(^{l}m) = [l, \sigma(m)]$, for all $l \in L, m \in M$

(ii) $^{\sigma(m)}m' = [m, m']$, for all $m, m' \in M$

(iii) $\sigma(M) = N$.

Moreover, the subalgebras $Z_c(L)$ and $[N,_c L]$, for all $c \geq 1$, as follows:

$$Z_c(L) = \{n \in N \mid [n, l_1, \dots, l_c] = 0, \forall l_1, , l_c \in L\},$$

$$[N,_c L] = \langle [n, l_1, \dots, l_c] \mid n \in N, l_1, \dots, l_c \in L \rangle,$$

where $[n, l_1, \dots, l_c] = [\dots [n, l_1], l_2], \dots, l_c]$ (See [15]).

Also, let (N, L) be a pair of Lie algebras. Then the c-*epicenter* of the pair (N, L) defined to be

$$Z_c^*(N, L) = \bigcap \{\varphi(Z_c(N^*, L)) \mid \varphi : N^* \to L\},$$

where φ is a relative c-central extension of (N, L).

It is easy to see that $Z_c^*(L, L) = Z_c^*(L)$, which is the epicenter of a Lie algebra L, discussed in [5] (Also, see [17] for more information). Let $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of L such that $N \cong S/R$ for an ideal S in F. Define

$$\gamma_{c+1}^*(N, L) = [S,_c F]/[R,_c F].$$

It is easy to see that this definition is independent of the free presentation for L (See [16]).

We recall that $X^c \otimes Y = \underbrace{X \otimes \cdots \otimes X}_{c-times} \otimes Y \ (c \geq 1)$, is the abelian tensor product and the Lie algebra $\wedge^c X$ is the c-th exterior product of X, which is the free Λ-module generated by $x_1 \wedge \dots \wedge x_c$ with $x_i \in X$ (See [7] for more details).

2. Main Results

In this section, we present our main results. We first give some upper bounds for the dimension of the c-nilpotent multiplier of a pair of Lie algebras. The following proposition was proved by the author in [1], which is needed for proving our results.

Proposition 2.1 — (See [1], Proposition 2.3). Let L be a Lie algebra and K be an ideal in L contained in N; then the following sequences are exact

(a)

$$0 \longrightarrow \mathcal{M}^{(c)}(K, L) \longrightarrow \mathcal{M}^{(c)}(N, L) \xrightarrow{\alpha}$$
$$\mathcal{M}^{(c)}(\frac{N}{K}, \frac{L}{K}) \xrightarrow{\beta} \frac{K \cap [N,_c L]}{[K,_c L]} \longrightarrow 0;$$

(b)
$$\mathcal{M}^{(c)}(N,L) \longrightarrow \mathcal{M}^{(c)}(\frac{N}{K},\frac{L}{K}) \longrightarrow N$$
$$\longrightarrow \frac{L}{[N,_c L]} \longrightarrow \frac{L}{[N,_c L]+K} \longrightarrow 0.$$

Using Proposition 2.1, we prove the following result. In the following proposition we generalize Corollary 3.3 of [13].

Proposition 2.2 — Let (N,L) be a pair of finite dimensional Lie algebras such that N be a complement of L and $Z_c(L) \subseteq N$. Let $H = L/Z_c(L)$ and $P = N/Z_c(L)$. Then

(i) $\dim([N,_c L]) \leq \dim \mathcal{M}^{(c)}(P,H) + \dim([P,_c H]) \leq \dim \mathcal{M}^{(c)}(N,L) + \dim([N,_c L])$,

(ii) $\dim([N,_c L]) = \dim \mathcal{M}^{(c)}(P,H) + \dim([P,_c H]) \Leftrightarrow \mathcal{M}^{(c)}(P,H) \cong Z_c(L) \cap [N,_c L]$,

(iii) $\dim \mathcal{M}^{(c)}(P,H) + \dim([P,_c H]) = \dim \mathcal{M}^{(c)}(N,L) + \dim([N,_c L]) \Leftrightarrow \mathcal{M}^{(c)}$ $(P,H)/\mathcal{M}^{(c)}(L) \cong Z_c(L) \cap [N,_c L]$.

PROOF :

(i) Set $K = Z_c(L)$ and using Proposition 2.1(a), we obtain
$$\dim \mathcal{M}^{(c)}(P,H) = \dim(Z_c(L) \cap [N,_c L]) + \dim(\ker \beta).$$
On the other hand,
$$[P,_c H] = \frac{[N,_c L] + Z_c(L)}{Z_c(L)} \cong \frac{[N,_c L]}{Z_c(L) \cap [N,_c L]}.$$
So, we have
$$\dim([N,_c L]) + \dim(\ker \beta) = \dim \mathcal{M}^{(c)}(P,H) + \dim([P,_c H]).$$

(ii) By first part, we have $\dim(\ker \beta) = 0$ if and only if $\mathcal{M}^{(c)}(P,H) \cong Z_c(L) \cap [N,_c L]$ and $\dim(\ker \beta) = 0$ if and only if
$$\dim([N,_c L]) = \dim \mathcal{M}^{(c)}(P,H) + \dim([P,_c H]).$$

(iii) By Proposition 2.1(a), we obtain
$$\dim(\ker \alpha) + \dim \mathcal{M}^{(c)}(P,H) + \dim([P,_c H])$$

$$= \dim \mathcal{M}^{(c)}(L) + \dim([N,_c L]).$$

Also, $\dim(\ker \alpha) = 0$ if and only if

$$\mathcal{M}^{(c)}(P, H)/\mathcal{M}^{(c)}(L) \cong Z_c(L) \cap [N,_c L]$$

which completes the proof. □

Recall that the authors [16] proved that if (N, L) is a pair of Lie algebras and $K \subseteq Z^*(L)$. Then

(i) The natural homomorphism $\mathcal{M}(L) \to \mathcal{M}(L/K)$ is monomorphism.

(ii) $K \subseteq Z^*(L) \cap N$.

Here, we extend this result and prove that the parts (i) and (ii) are equivalent, without condition $K \subseteq Z^*(N, L)$. Indeed, the following theorem generalizes a result of Rismanchian and Araskhan (2012) and Araskhan (2016).

***Theorem* 2.3** — *Let (N, L) be a pair of Lie algebras and K and N be complements of L such that $K \subseteq N \cap Z_c(L)$. Then $K \subseteq Z_c^*(L)$ if and only if the natural map $\mathcal{M}^c(N, L) \to \mathcal{M}^c(N/K, L/K)$ is monomorphism.*

PROOF : Let $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of the Lie algebra L and let $0 \to M \to K \xrightarrow{\theta} \overline{L} \to 0$ be a c-central extension of another Lie algebra $\overline{L}$. Then by [5, Lemma 1.2] for each homomorphism $\alpha : L \to \overline{L}$, there exists a homomorphism $\beta : F/[R,_c F] \to K$ such that $\beta(R/[R,_c F]) \subseteq M$ and the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{R}{[R,_c F]} & \longrightarrow & \dfrac{F}{[R,_c F]} & \xrightarrow{\overline{\pi}} & L & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta_1} & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & \overline{L} & \longrightarrow & 0 \end{array} \qquad (1)$$

where, β_1 is the restriction of β to $R/[R,_c F]$. Now, let $K \cong T/R$ be as in Proposition 2.1. Then, we have

$$\mathrm{Ker}(\mathcal{M}^{(c)}(\mathrm{N}, \mathrm{L}) \to \mathcal{M}^{(c)}(\mathrm{N/K}, \mathrm{L/K})) = [\mathrm{T},_c \mathrm{F}]/[\mathrm{R},_c \mathrm{F}].$$

Thus, we only need to verify that $[T,_c F] = [R,_c F]$ if and only if $K \subseteq Z_c^*(L)$. Set $\overline{F} = F/[R,_c F], \overline{R} = R/[R,_c F]$ and $\overline{T} = T/[R,_c F]$. Then $[T,_c F] = [R,_c F]$ is equivalent to $\overline{T} \subseteq Z_c(\overline{F})$. Let $0 \to M \to P \xrightarrow{\theta} L \to 0$ be a c-central extension of L. Then, there exists a homomorphism $\beta : F/[R,_c F] \to P$ such that the corresponding diagram with the above c-central extension in

(1) is commutative. We can see that $P = M + Im\beta$ and hence $\beta(Z_c(F/[R,_c F])) \subseteq Z_c(P)$. So, $\overline{\pi}(Z_c(F/[R,_c F])) = \theta(\beta(Z_c(F/[R,_c F]))) \subseteq \theta(Z_c(P))$. Hence, $Z_c^*(L) = \overline{\pi}(Z_c(F/[R,_c F]))$ Consequently, we obtain $\overline{\pi}(\overline{T}) \subseteq Z_c^*(L)$ if and only if $\overline{T} \subseteq Z_c(\overline{F})$, now the result follows. □

By the above theorem, the following corollaries can be obtained immediately.

Corollary 2.4 — Let (N, L) be a pair of Lie algebras and $Z_c(L) \subseteq N$. Then $Z_c^*(L)$ is trivial if and only if the natural map $\mathcal{M}^{(c)}(N, L) \to \mathcal{M}^{(c)}(\frac{N}{\langle x \rangle}, \frac{L}{\langle x \rangle})$ has a non-trivial kernel for all non zero elements x in $Z_c(L)$.

Corollary 2.5 — Let (N, L) be a pair of finite dimensional Lie algebras and K be an ideal in L such that $K \subseteq N \cap Z_c(L)$. Then $K \subseteq Z_c^*(L)$ if and only if the map $\delta : \mathcal{M}^{(c)}(N, L) \to (L/[N,_c L])^c \otimes K$ is the trivial map.

In final, we state a sufficient condition for the c-nilpotent multiplier of a pair of Lie algebras to be finite dimensional. The next lemma is useful for the proof of our result, which is a generalization of Lemma 1.2 of [19].

Lemma 2.6 — Let (N, L) be a pair of Lie algebras. Then the Lie algebra $\gamma_{c+1}^*(N, L)$ is a homomorphic image of $\wedge^{c+1} N$.

PROOF : Let $0 \to R \to F \xrightarrow{\pi} L \to 0$ be a free presentation of L such that $N \cong S/R$ for an ideal S in free Lie algebra F and M be an ideal in L such that $M \subseteq N$ and $M \cong T/R$ for some an ideal T in S. We prove that $[T,_{i+1} S]/[R,_{i+1} F](i \geq 1)$ is a homomorphic image of $M \wedge^i N$. Using induction on i, we first show that $[K, [T,_i S]] \subseteq [K,_i S]$, for all ideals K in S. The result is true for $i = 1$. Suppose that the result holds for $i \geq 1$. Then for any ideal K in S, we have

$$[K, [T,_{i+1} S]] \subseteq [[T,_i S], [F, K]] + [S, [T,_i S]]$$

$$\subseteq [[K, S],_i S] + [S, [K,_i S]]] = [K,_{i+1} S].$$

So, the Lie algebras N and $[T,_i S]/[R,_i F]$ act compatibly on each other. Also, the map $\lambda : [T,_i S]/[R,_i F] \longrightarrow N$ together with the above action of N on $[T,_i S]/[R,_i F]$ and the identity map $Id : N \longrightarrow N$ are crossed modules and so, we obtain an epimorphism

$$\psi_i : [T,_i S]/[R,_i F] \wedge N \to [T,_{i+1} S]/[R,_{i+1} F].$$

Thus, [8, Proposition 9] gives an epimorphism

$$\overline{\psi_i} : ([T,_i S]/[R,_i F] \wedge N) \wedge N \to ([T,_{i+1} S]/[R,_{i+1} F]) \wedge N.$$

Therefore, the result is obtained by induction on i. □

Now, we prove the last result.

Corollary 2.7 — Let (N, L) be a pair of Lie algebras. If $N/Z_c^*(N, L)$ is finite dimensional, then dimension of $\mathcal{M}^{(c)}(N, L)$ is finite.

PROOF : Using Lemma 2.6 and a simple generalization of [16, Theorem 3.8], we obtain the following epimorphism

$$\wedge^{c+1}(N/Z_c^*(N, L)) \to \gamma_{c+1}^*(N/Z_c^*(N, L), L/Z_c^*(N, L)) \cong \gamma_{c+1}^*(N, L).$$

On the other hand; It is shown in [8] that the exterior product $X \wedge Y$ of two crossed modules is finite dimensional, if both X and Y are finite dimensional. Consequently, if $N/Z_c^*(N, L)$ is finite dimensional then, $\wedge^{c+1}(N/Z_c^*(N, L))$ is finite dimensional. This complete the proof of the corollary.□

REFERENCES

1. H. Arabyani, Bounds for the dimension of the c-nilpotent multiplier of a pair of Lie algebras, *Bull. Iranian Math. Soc.*, **43**(7) (2017), 2411-2418.
2. H. Arabyani, Some results on the c-nilpotent multiplier of a pair of Lie algebras, *Bull. Iranian Math. Soc.*, **45**(1) (2018), 205-212.
3. H. Arabyani, F. Saeedi, M. R. R. Moghaddam, and E. Khamseh, Characterization of nilpotent Lie algebras pair by their Schur multipliers, *Comm. Algebra*, **42** (2014), 5474-5483.
4. H. Arabyani and H. Safa, Some properties of c-covers of a pair of Lie algebras, *Quaest. Math.*, **42**(1) (2019), 37-45.
5. M. Araskhan, On the c-Covers and a special ideal of Lie algebras, *Iran. J. Sci. Technol. Trans. A Sci.*, **40**(3) (2016), 165-169.
6. P. Batten, K. Moneyhun and E. Stitzinger, On characterizing nilpotent Lie algebras by their multipliers, *Comm. Algebra*, **24**(14) (1996), 4319-4330.
7. G. Ellis, Nonabelian exterior products of Lie algebras and an exact sequence in the homology of Lie algebras, *J. pure Appl. Algebra*, **46** (1987), 111-115.
8. G. Ellis, A non-abelian tensor product of Lie algebras, *Glasg Math. J.*, **39** (1991), 101-120.
9. G. Ellis, The Schur multiplier of a pair of groups, *Appl. Categ. Structures*, **6**(3) (1998), 355-371.
10. G. Karpilovsky, The Schur Multiplier, Clarendon Press, Oxford, 1987.
11. C. Kassel and J. L. Loady, Extensions centrales d'algebras de Lie, *Ann. Inst. Fourier*, **33** (1982), 119-142.

12. K. Moneyhun, Isoclinisms in Lie algebras, *Algebras Groups Geom.*, **11** (1994), 9-22.

13. M. R. Rismanchian and M. Araskhan, Some properties on the Schur multiplier of a pair of Lie algebras, *J. Algebra Appl.*, **11** (2012), 1250011(9 pages).

14. F. Saeedi, A. R. Salemkar, and B. Edalatzadeh, The commutator subalgebra and Schur multiplier of a pair of nilpotent Lie algebras, *J. Lie Theory*, **21** (2011), 491-498.

15. H. Safa and H. Arabyani, On c-nilpotent multiplier and c-covers of a pair of Lie algebras, *Comm. Algebra*, **45**(10) (2017), 4429-4434.

16. H. Safa and H. Arabyani, Capable pairs of Lie algebras, *Math. Proc. R. Ir. Acad.*, **118**A (2018), 39-45.

17. A. R. Salemkar, V. Alamian, and H. Mohammadzadeh, Some properties of the Schur multiplier and covers of Lie algebras, *Comm. Algebra*, **36** (2008), 697-707.

18. A. R. Salemkar, B. Edalatzadeh, and M. Araskhan, Some inequalities for the dimension of the c-nilpotent multiplier of Lie algebras, *J. Algebra*, **322** (2009), 1575-1585.

19. A. R. Salemkar and Z. Riyahi, Some properties of the c-nilpotent multiplier of Lie algebras, *J. Algebra*, **370** (2012), 320-325.

20. I. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.*, **127** (1904), 20-50.