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A NEW CONTRACTIVE CONDITION RELATED TO RHOADES'S OPEN QUESTION

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An open problem proposed by Rhoades is the following. Is there a contractive condition which guarantees the existence of a fixed point, but does not require the mapping to be continuous at the point? In this paper, we generalize a celebrated result of Eshaghi et al., [On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory*, 18 (2017), 569-578], which allows us to find a new solution to this open problem. Furthermore we show that a claim of the aforementioned paper, that Banach's fixed point theorem cannot be applied in their application, is incorrect. Finally, as an application, we prove that a multivalued function satisfying a general linear functional inclusion admits a unique selection fulfilling the corresponding functional equation.

Key words : Orthogonal set; fixed point; multivalued mapping; selection; Picard operator.

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1. Introduction and Preliminaries

Recently, Eshaghi *et al*. [1] introduced the notion of orthogonal sets and then gave an interesting extension of Banach's fixed point theorem. They proved, by means of an example, that their main theorem is a real generalization of the Banach fixed point theorem. Moreover, they studied the existence and uniqueness of solution for a first-order ordinary differential equation and claimed that Banach's fixed point theorem is not applicable in their application. The main result of [1] is the following theorem.

***Theorem* 1.1** — *Let* $(X, \perp, d)$ *be an O-complete metric space* (*not necessarily complete metric space*) *and* $0 < \lambda < 1$. *Let* $f : X \to X$ *be O-continuous,* $\perp$*-contraction with Lipschitz constant* λ

and $\perp$*-preserving. Then* f *has a unique fixed point* $x^* \in X$*. Also,* f *is a Picard operator, that is,* $\lim_{n\to\infty} f^n(x) = x^*$ *for all* $x \in X$*.*

At first, we recall some important definitions and notations.

Definition 1.2 — [1]. Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a nonempty binary relation. If " $\perp$ " satisfies the following condition:

$$\exists x_0\text{: } (\forall y, y\perp x_0) \;\; or \;\; (\forall y, x_0 \perp y),$$

then " $\perp$ " is called an orthogonality relation and the pair $(X, \perp)$ an orthogonal set(briefly *O-set*).

Note that in above definition, we say that x_0 is an orthogonal element. Also, an orthogonal element x_0 is called left orthogonal element if $x_0 \perp x$ for each $x \in X$. Finally, we say that elements $x, y \in X$ are $\perp$-comparable if either $x\perp y$ or $y \perp x$.

Definition 1.3 — [1]. Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called an *orthogonal sequence* (briefly, O-sequence) if

$$(\forall n : x_n \perp x_{n+1}) \;\text{ or }\; (\forall n : x_{n+1} \perp x_n).$$

Definition 1.4 — [3]. Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called a *strongly orthogonal sequence* (briefly, SO-sequence) if

$$(\forall n, k : \quad x_n \perp x_{n+k}) \quad \text{or} \quad (\forall n, k : \quad x_{n+k} \perp x_n).$$

It is obvious that every SO-sequence is an O-sequence. The following example shows that the converse is not true in general.

Example 1.5 : Let $X = \mathbb{N} \cup \{0\}$. Suppose $x \perp y$ iff $xy = 0$. Define the sequence $\{x_n\}$ as follows:

$$x_n = \begin{cases} 0 & n = 2k, \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ n & n = 2k+1, \text{ for some } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

Then for all $n \in \mathbb{N} \cup \{0\}$, $x_n \perp x_{n+1}$, but x_{2n+1} is not orthogonal to x_{4n+1}. Therefore $\{x_n\}$ is an O-sequence which is not SO-sequence.

Definition 1.6 — [1]. Let $(X, \perp, d)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). X is said to be *orthogonal complete* (briefly, O-complete) if every Cauchy O-sequence is convergent.

Definition 1.7 — [3]. Let $(X, \perp, d)$ be an orthogonal metric space. X is said to be *strongly orthogonal complete* (briefly, SO-complete) if every Cauchy SO-sequence is convergent.

Clearly, every complete metric space is O-complete and every O-complete metric space is SO-complete.

Definition 1.8 — [1]. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is orthogonal continuous (briefly, O-continuous at $a \in X$ if for each O-sequence $\{a_n\}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also, f is O-continuous on X if f is O-continuous at each $a \in X$.

Definition 1.9 — [3]. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is strongly orthogonal continuous ($briefly, SO - continuous$) at $a \in X$ if for each SO-sequence $\{a_n\}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also, f is SO-continuous on X if f is SO-continuous at each $a \in X$.

It is easy to see that every continuous mapping is O-continuous and every O-continuous mapping is SO-continuous. The following example shows that the converse is not true.

Example 1.10 : Let $X = \mathbb{R}$ with the Euclidean metric. Suppose $x \perp y$ iff $xy \in \{x, y\}$. Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ \\ \dfrac{1}{x} & x \in \mathbb{Q}. \end{cases}$$

Notice that f is not continuous but we can see that f is SO-continuous. If $\{x_n\}$ is an SO-sequence in X which converges to $x \in X$. Applying definition $\perp$ we obtain that for enough large n, $x_n \in \mathbb{Q}$. This implies that $f(x_n) = 1 \to x = 1$. To see that f is not O-continuous, consider the sequence

$$x_n = \begin{cases} 0 & n = 2k + 1, \text{ for some } k \in \mathbb{Z}, \\ \frac{\sqrt{2}}{k} & n = 2k, \text{ for some } k \in \mathbb{Z}. \end{cases}$$

It is clear that $x_n \to 0$ while the sequence $\{f(x_n)\}$ is not convergent to $f(0)$.

Definition 1.11 — [1]. Let $(X, \perp)$ be an O-set. A mapping $f : X \to X$ is said to be $\perp$-preserving if $x \perp y$ implies $f(x) \perp f(y)$.

Notation 1.12 : Let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(I) ψ is nondecreasing;

(II) $\lim_{n \to \infty} \psi^n(t) = 0$ for all $t > 0$; where ψ^n is the n-th iterate of ψ.

Remark 1.13 : For each $\psi \in \Psi$, the following assertions hold:

(1) $\psi(t) < t$ for all $t > 0$;

(2) $\psi(0) = 0$.

The question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point was ingeninated by Rhoades in [4] as an existing open problem. The question was settled in the affirmative by Pant [2].

In this paper, we introduce a new contractive definition which is a generalization of contractive definition introduced by Eshaghi *et al*. [1] and also provide yet new solution to the aforementioned open problem. Moreover, our results apply to continuous as well as discontinuous mappings. Furthermore, we show that the claim that Banach's fixed point theorem is ineffective in their application is incorrect. Finally, as an application, we prove that a multivalued function satisfying a general linear functional inclusion admits a unique selection fulfilling the corresponding functional equation.

2. The Main Results

In this section, we state and prove the main theorem of this paper which is a generalization of contractive definition introduced by Eshaghi *et al*. [1].

***Theorem* 2.1** — *Let $(X, \perp, d)$ be an SO-complete metric space (not necessarily a complete metric space) with orthogonal element $\hat{x}_0$. Let $f : X \to X$ be SO-continuous and $\perp$-preserving. Suppose that there exist $\psi \in \Psi$ and $m \in \mathbb{N}$ such that*

$$d(f^m x, f^m y) \leq \psi(\max\{d(x, y), d(fx, fy), \cdots, d(f^{m-1}x, f^{m-1}y)\}) \tag{1}$$

for each $\perp$-comparable elements $x, y \in X$. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$. Moreover, f is discontinuous at x^* if and only if $\lim_{x\to x^*} \max\{d(x, x^*), d(fx, fx^*), \cdots, d(f^{m-1}x, f^{m-1}x^*)\} \neq 0$.

PROOF : Firstly, for each $x, y \in X$ and $n \in \mathbb{N} \cup \{0\}$, we adopt the following notations:

(i) $x_n := f^n(x)$ and $y_n := f^n(y)$;

(ii) $M_n(x, y) := \max\{d(x_n, y_n), d(x_{n+1}, y_{n+1}), \cdots, d(x_{n+m-1}, y_{n+m-1})\}$.

Let $x, y \in X$ be $\perp$-comparable elements, since f is $\perp$-preserving, then, for each $n \in \mathbb{N} \cup \{0\}$, x_n and y_n are $\perp$-comparable elements.

For better readability, we divide the proof into several steps.

Step 1 : The sequence $\{M_n(x,y)\}$ is decreasing.

Justification of Step 1. Let $x = x_n$ and $y = y_n$ in the inequality (1), then we get that

$$d(x_{m+n}, y_{m+n}) \leq \psi(M_n(x,y)) < M_n(x,y),$$

so, $M_{n+1}(x,y) \leq M_n(x,y)$ for every $n \in \mathbb{N} \cup \{0\}$.

Step 2 : $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Justification of Step 2. For every $i \in \{0, 1, 2, \cdots, m-1\}$ and $n \in \mathbb{N} \cup \{0\}$, taking $x = x_{n+i}$ and $y = y_{n+i}$ into the inequality (1), then, by Step 1, we get that

$$d(x_{m+n+i}, y_{m+n+i}) \leq \psi(M_{n+i}(x,y)) \leq \psi(M_n(x,y)),$$

so, $M_{m+n}(x,y) \leq \psi(M_n(x,y))$. Using the mathematical induction method, we obtain that $M_{n+km}(x,y) \leq \psi^k(M_n(x,y))$ for all $n, k \in \mathbb{N} \cup \{0\}$. Then $\lim_{k\to\infty} M_{n+km}(x,y) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Using Step 1 we deduce that $\lim_{n\to\infty} M_n(x,y) = 0$ and since $d(x_n, y_n) \leq M_n(x,y)$ for very $n \in \mathbb{N} \cup \{0\}$, we conclude that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

By taking $x = \hat{x}_0$, the orthogonal element of $(X, \perp)$, and $y = f(\hat{x}_0)$, from Step 2, we obtain that

$$\lim_{n\to\infty} d(\hat{x}_n, \hat{x}_{n+1}) = 0, \tag{2}$$

where

$$\hat{x}_1 = f\hat{x}_0, \;\; \hat{x}_2 = f(\hat{x}_1) = f^2(\hat{x}_0), \ldots, \hat{x}_{n+1} = f(\hat{x}_n) = f^{n+1}(\hat{x}_0)$$

for all $n \in \mathbb{N}$.

Step 3 : The sequence $\{\hat{x}_n\}$ is a Cauchy SO-sequence.

Justification of Step 3. By definition of orthogonality we have

$$(\forall y \in X, \quad \hat{x}_0 \perp y) \qquad \text{or} \qquad (\forall y \in X, \;\; y \perp \hat{x}_0).$$

It follows that $\hat{x}_0 \perp f\hat{x}_0$ or $f\hat{x}_0 \perp \hat{x}_0$. It is clear that

$$(\forall n \in \mathbb{N}, \quad \hat{x}_0 \perp \hat{x}_n) \qquad \text{or} \qquad (\forall n \in \mathbb{N}, \;\; \hat{x}_n \perp \hat{x}_0).$$

Since f is $\perp$-preserving, we see that

$$(\forall n, k \in \mathbb{N} : \;\; \hat{x}_k = f^k(\hat{x}_0) \perp f^k(\hat{x}_n) = \hat{x}_{n+k}) \;\; \text{or} \quad (\forall n, k \in \mathbb{N} : \;\; \hat{x}_{n+k} = f^k(\hat{x}_n) \perp f^k(\hat{x}_0) = \hat{x}_k).$$

This implies that $\{\hat{x}_n\}$ is an SO-sequence.

Now, we show that $\{\hat{x}_n\}$ is a Cauchy sequence. Suppose that $\{\hat{x}_n\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that, for all positive integers k, we have

$$n(k) > m(k) > k, \quad d(\hat{x}_{m(k)}, \hat{x}_{n(k)}) \geq \varepsilon, \quad d(\hat{x}_{m(k)}, \hat{x}_{n(k)-1}) < \varepsilon. \tag{3}$$

To prove (3), suppose that

$$\sum\nolimits_k = \{m \in \mathbb{N} : \quad \exists m(k) \geq k \ , \ d(\hat{x}_m, \hat{x}_{m(k)}) \geq \epsilon \ , \ m > m(k) > k\}.$$

Obviously, $\sum_k \neq \emptyset$ and $\sum_k \subseteq \mathbb{N}$. Then by the well-ordering principle, the minimum element of $\sum_k$ exists and denoted by $n(k)$, and clearly (3) holds. Applying (3), we deduce that

$$\varepsilon \leq d(\hat{x}_{m(k)}, \hat{x}_{n(k)}) \leq d(\hat{x}_{m(k)}, \hat{x}_{n(k)-1}) + d(\hat{x}_{n(k)-1}, \hat{x}_{n(k)}) < \varepsilon + d(\hat{x}_{n(k)-1}, \hat{x}_{n(k)}).$$

Let $k \to \infty$ and using (2), we have

$$\lim_{k\to\infty} d(\hat{x}_{n(k)}, \hat{x}_{m(k)}) = \varepsilon. \tag{4}$$

Triangle inequality implies that

$$|d(\hat{x}_{n(k)+1}, \hat{x}_{m(k)}) - d(\hat{x}_{m(k)}, \hat{x}_{n(k)})| \leq d(\hat{x}_{n(k)+1}, \hat{x}_{n(k)}).$$

Applying (2) and (4), we have

$$\lim_{k\to\infty} d(\hat{x}_{n(k)+1}, \hat{x}_{m(k)}) = \varepsilon.$$

Similarly,

$$\lim_{k\to\infty} d(\hat{x}_{n(k)}, \hat{x}_{m(k)-1}) = \varepsilon,$$

and also

$$\lim_{k\to\infty} d(\hat{x}_{n(k)+1}, \hat{x}_{m(k)+1}) = \varepsilon. \tag{5}$$

Using the above method, we can show that

$$\lim_{k\to\infty} d(\hat{x}_{n(k)+i}, \hat{x}_{m(k)+i}) = \varepsilon \tag{6}$$

for each $i \in \{2, 3, \cdots, m-1\}$. The above equalities assure us that there exits $l \geq 1$ such that

$$\begin{aligned} \varepsilon \leq M_0(\hat{x}_{n(k)}, \hat{x}_{m(k)}) \quad &= \quad \max\Big\{d(\hat{x}_{n(k)}, \hat{x}_{m(k)}), d(\hat{x}_{n(k)+1}, \hat{x}_{m(k)+1}), \\ &\qquad \cdots, d(\hat{x}_{n(k)+m-1}, \hat{x}_{m(k)+m-1})\Big\} \\ &\qquad \leq l\varepsilon \end{aligned}$$

for all $k \in \mathbb{N}$. Also, we can find $\hat{p} \in \mathbb{N}$ such that $\psi^{\hat{p}}(l\varepsilon) < \frac{\varepsilon}{5}$. Moreover, by inequality (2), there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $d(\hat{x}_n, \hat{x}_{n+1}) < \frac{\varepsilon}{3\hat{p}m}$.

Then, for $k \in \mathbb{N}$ such that $n(k) > n_0$, we have:

$$\begin{aligned}
\varepsilon &\leq d(\hat{x}_{n(k)}, \hat{x}_{m(k)}) \leq d(\hat{x}_{n(k)}, \hat{x}_{n(k)+\hat{p}m}) + d(\hat{x}_{n(k)+\hat{p}m}, \hat{x}_{m(k)+\hat{p}m}) + d(\hat{x}_{m(k)+\hat{p}m}, \hat{x}_{m(k)}) \\
&\leq \sum_{i=1}^{\hat{p}m} d(\hat{x}_{n(k)+i-1}, x_{n(k)+i}) + \psi^{\hat{p}}(M_0(\hat{x}_{n(k)}, \hat{x}_{m(k)})) + \sum_{i=1}^{\hat{p}m} d(\hat{x}_{m(k)+i-1}, \hat{x}_{m(k)+i}) \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{5} = \frac{13\varepsilon}{15}.
\end{aligned}$$

This contradiction closes the justification of the claim.

Since $(X, \perp, d)$ is an SO-complete metric space, Step 3 assures us that there exists $x^* \in X$ such that $\lim_{n\to\infty} \hat{x}_n = x^*$. On the other hand, f is an SO-continuous function, then

$$f(x^*) = \lim_{n\to\infty} f(\hat{x}_n) = \lim_{n\to\infty} \hat{x}_{n+1} = x^*,$$

i.e. x^* is a fixed point of f.

Now, we show that f is a Picard operator. Let $x \in X$ be arbitrary. By our choice of $\hat{x}_0$, we have

$$x \perp \hat{x}_0 \ \text{ or } \ \hat{x}_0 \perp x,$$

$\perp$-preserving of f implies that

$$[\forall n \in \mathbb{N}: \ \ \hat{x}_n \perp f^n(x)] \quad \text{ or } \quad [\forall n \in \mathbb{N}: \ \ f^n(x) \perp \hat{x}_n].$$

Now by using Step 2, we have $\lim_{n\to\infty} d(\hat{x}_n, f^n(x)) = 0$. Hence, $\lim_{n\to\infty} f^n(x) = x^*$.

Finally, to prove the uniqueness of fixed point, let $y^* \in X$ be another fixed point of f. Then $f^n(y^*) = y^*$ for all $n \in \mathbb{N}$. It follows from f is a Picard operator that $x^* = y^*$. The rest of proof is obvious. □

Corollary 2.2 — [1]. Let $(X, \perp, d)$ be an O-complete metric space (not necessarily a complete metric space). Let $f : X \to X$ be O-continuous and $\perp$-preserving. Suppose that there exists $0 < \lambda < 1$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for each $\perp$-comparable elements $x, y \in X$. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

PROOF : Take $\psi(t) = \lambda t$ for all $t \in [0, \infty)$. Then all of the conditions of Theorem 2.1 are satisfied. $\square$

The following simple examples show that our theorem is a real extension of Theorem 3.11 of [1].

Example 2.3 : Let $X = [0, +\infty)$ with the Euclidean metric d. Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & x > 1, \\ \frac{1}{n+1} & \frac{1}{n+1} < x \leq \frac{1}{n}, \quad n \in \mathbb{N}, \\ 0 & x = 0. \end{cases}$$

Suppose that

$$x \perp y \Longleftrightarrow (x = 0).$$

It is easy to see that $(X, \perp, d)$ is SO-complete. Moreover, $f : X \to X$ is SO-continuous and $\perp$-preserving. Also, for each $\perp$-comparable elements $x, y \in X$, we have

$$d(f(x), f(y)) \leq \psi(d(x, y)), \tag{7}$$

where $\psi : [0, \infty) \to [0, \infty)$ is defined by

$$\psi(x) = \begin{cases} 1 & x > 1, \\ \frac{1}{n+1} & \frac{1}{n+1} < x \leq \frac{1}{n}, \quad n \in \mathbb{N}, \\ 0 & x = 0. \end{cases}$$

It is easy to see that ψ is nondecreasing and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \geq 0$. Therefore by Theorem 2.1, it has a unique fixed point $x = 0$.

Notice that equation (8) is not hold for each $x, y \in X$. For example, we consider $x = \frac{1}{2}$ and $y = \frac{3}{2}$. Then, $\frac{2}{3} = d(fx, fy) > \frac{1}{2} = \psi(d(x, y))$.

In below, we show that the main theorem [1] is not applicable for the mapping f. Let $\perp \subseteq X \times X$ be an arbitrary orthogonality relation such that $(X, \perp, d)$ is an SO-complete metric space with orthogonal element x_0 and f is $\perp$-preserving. The following cases are considered:

Case 1 : Let $x_0 = 0$. By definition of orthogonality, x_0 and x_n are $\perp$-comparable, where $x_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. In this case, $\lim_{n\to\infty} \frac{d(fx_0, fx_n)}{d(x_0, x_n)} = 1$.

Case 2 : Let $x_0 \in (0, 1)$. Then, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0+1} < x_0 \leq \frac{1}{n_0}$. By definition of orthogonality, x_0 and x_n are $\perp$-comparable, where $x_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. In this case, $\lim_{n\to\infty} \frac{d(fx_0, fx_n)}{d(x_0, x_n)} = \frac{1 - \frac{1}{n_0+1}}{1 - x_0} > 1$.

Case 3 : Let $x_0 = 1$. By definition of orthogonality, x_0 and x_n are $\perp$-comparable, where $x_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. In this case, $\lim_{n\to\infty} \frac{d(fx_0, fx_n)}{d(x_0, x_n)} = +\infty$.

Case 4 : Let $x_0 > 1$. By definition of orthogonality, x_0 and 0 are $\perp$-comparable. Now since f is $\perp$-preserving then 1 and 0 are $\perp$-comparable. Repeating this process, we find that $x_n = \frac{1}{n}$ and 0 are $\perp$-comparable, for all $n \in \mathbb{N}$. In this case, $\lim_{n\to\infty} \frac{d(f0, fx_n)}{d(0, x_n)} = 1$.

Thus the main theorem [1] is not applicable for the mapping f in the metric spase.

Example 2.4 : Let $X = [0, 2]$ with the Euclidean metric d. Define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & 1 < x \leq 2. \end{cases}$$

Suppose that

$$x \perp y \Longleftrightarrow (x = 1).$$

It is easy to see that $(X, \perp, d)$ is SO-complete. Moreover, $f : X \to X$ is SO-continuous and $\perp$-preserving. Also, for each $\perp$-comparable elements $x, y \in X$, we have

$$d(f^2(x), f^2(y)) \leq \frac{1}{2} d(x, y).$$

Then, f satisfies all the conditions of Theorem 2.1 and has a unique fixed point $x = 1$. It can also be easily seen that f is discontinuous at the fixed point $x = 1$.

In below, we show that the main theorem [1] is not applicable for the mapping f. Let $\perp \subseteq X \times X$ be an arbitrary orthogonality relation such that $(X, \perp, d)$ is an SO-complete metric space with orthogonal element x_0 and f is $\perp$-preserving. The following cases are considered:

Case 1 : Let $x_0 = 0$. By definition of orthogonality, x_0 and x_n are $\perp$-comparable, where $x_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. In this case, $\lim_{n\to\infty} \frac{d(fx_0, fx_n)}{d(x_0, x_n)} = 1$.

Case 2 : Let $x_0 \in (0, 1]$. By definition of orthogonality, x_0 and y_0 are $\perp$-comparable, where $y_0 = 1 + x_0$. In this case, $\frac{d(fx_0, fy_0)}{d(x_0, y_0)} = 1$.

Case 3 : Let $x_0 \in (1, 2]$. By definition of orthogonality, x_0 and y_0 are $\perp$-comparable, where $y_0 = 1 - x_0$. In this case, $\frac{d(fx_0, fy_0)}{d(x_0, y_0)} = 1$.

Thus the main theorem [1] is not applicable for the mapping f in the metric space.

3. A Remark on An Incorrect Claim

Our purpose here is to apply Banach's fixed point theorem to prove the existence of a solution for the following differential equation:

$$\begin{cases} u'(t) = f(t, u(t)), \quad a.e.\ t \in I = [0, T] \\ u(0) = a, \quad a \geq 1, \end{cases} \tag{8}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(c1) $f(s, x) \geq 0$ for all $x \geq 0$ and $s \in I$,

(c2) there exists $\alpha \in L^1(I)$ such that

$$|f(s, x) - f(s, y)| \leq \alpha(s)|x - y|$$

for all $t \in I$ and $x, y \geq 0$ with $xy \geq (x \vee y)$, where $x \vee y = x$ or y.

***Theorem* 3.1** — *Under above assumptions, the differential equation (8) has unique positive solution.*

Proof : Let $X = \{u \in C(I, \mathbb{R}) : u(t) \geq 1, \ \ \forall t \in I\}$. Let $A(t) = \int_0^t |\alpha(s)| ds$, $t \in I$. Then $A'(t) = |\alpha(t)|$ for almost every $t \in I$. Define

$$\|x\|_A := \sup_{t \in I} e^{-A(t)} |x(t)|, \quad d(x, y) := \|x - y\|_A$$

for all $x, y \in X$. It is easy to see that (X, d) is a complete metric space.

Define a mapping $\mathcal{F} : X \to X$ by

$$\mathcal{F}u(t) := a + \int_0^t f(s, u(s)) ds.$$

Note that the fixed points of $\mathcal{F}$ are the solutions of (8). Let $x, y \in X$, then, for each $t \in I$, $x(t).y(t) \geq (x(t) \vee y(t))$. Hence, by $(c2)$, we have

$$\begin{aligned} e^{-A(t)} |\mathcal{F}x(t) - \mathcal{F}y(t)| &\leq e^{-A(t)} \int_0^t |f(s, x(s)) - f(s, y(s))| ds \\ &\leq e^{-A(t)} \int_0^t |\alpha(s)| e^{A(s)} e^{-A(s)} \ |x(s) - y(s)| ds \\ &\leq e^{-A(t)} \Big(\int_0^t |\alpha(s)| e^{A(s)} ds \Big) \ \|x - y\|_A \\ &\leq e^{-A(t)} (e^{A(t)} - 1) \ \|x - y\|_A \\ &\leq (1 - e^{-\|\alpha\|_1}) \ \|x - y\|_A \end{aligned}$$

and so

$$\|\mathcal{F}x - \mathcal{F}y\|_A \leq (1 - e^{-\|\alpha\|_1})\, \|x - y\|_A$$

for each $x, y \in X$. Since $1 - e^{-\|\alpha\|_1} < 1$, $\mathcal{F}$ satisfies in the Banach contraction principle. Thus, the operator $\mathcal{F}$ has a unique fixed point in X, which is a unique positive solution of the differential equation (8). Therefore, the claim that the Banach fixed point theorem is ineffective in this application is incorrect. □

4. Selection of Multivalued Mappings in Incomplete Metric Spaces

Let (X, d) be a metric space. We denote by $n(X)$ the family of all nonempty subsets of X and by $B(X)$ and $CP(X)$ the collections of all bounded and complete members of $n(X)$, respectively.

The number

$$diam(A) := \sup\{d(a, b) : a, b \in A\}$$

is said to be the diameter of A, where $A \in n(X)$.

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a nonempty subset of X. Consider a multivalued mapping $F : K \to n(Y)$. A function $f : K \to Y$ is called a selection of the F if and only if $f(x) \in F(x), x \in K$. Let

$$Sel(F) := \{f : K \to Y : f(x) \in F(x), x \in K\}.$$

It is easy to check that if there exists a constant $M > 0$ such that $diam(F(x)) \leq M\|x\|$ for all $x \in K$, then the distance function

$$d(f, g) = \sup\left\{\frac{\|f(x) - g(x)\|}{\|x\|}, 0 \neq x \in K\right\}, f, g \in Sel(F),$$

is a metric in $Sel(F)$. Moreover, if $F(x)$ is complete for every $x \in K$, the metric space $(Sel(F), d)$ is complete. Obviously, the convergence in the space $(Sel(F), d)$ implies the point wise convergence on the set K.

***Theorem* 4.1** — *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a nonempty subset of X such that $0 \in K$. Suppose that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:*

1. $|\alpha| < p$ and $K \subseteq pK$,

2. $|\beta| < q$ and $K \subseteq qK$.

Consider a multivalued function $F : K \to B(Y)$ *such that* $0 \in F(0)$ *and*

$$diam(F(x)) \leq M\|x\|, x \in K,$$

for some positive constant M. *Also, for each* $x \in K$, *there exists* $\perp_x \subseteq F(x) \times F(x)$ *such that* $(F(x), \perp_x, \|.\|)$ *is an SO-complete metric space with left orthogonal element* x^*. *If*

$$\begin{aligned} &\alpha F(x) + \beta F(y) \subseteq F(px + qy), \\ &\alpha \perp_x + \beta \perp_y \subseteq \perp_{px+qy}, \end{aligned} \tag{9}$$

where $x, y \in K$ *and* $px + qy \in K$, *then there exists a unique selection* $f : K \to Y$ *of multivalued mapping* F *such that*

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K, px + qy \in K.$$

PROOF : Assume that $|\alpha| < p$ and $K \subseteq pK$. Since $diamF(0) = 0$ and $0 \in F(0)$, then $F(0) = 0$ and $\perp_0 = \{(0, 0)\}$. Putting $y = 0$ in (10), since $\perp_0 = \{(0, 0)\}$, we obtain

$$\begin{aligned} &\alpha F(\frac{x}{p}) \subseteq F(x), \\ &\alpha \perp_{\frac{x}{p}} \subseteq \perp_x, \end{aligned} \tag{10}$$

for each $x \in K$.

Consider the following orthogonality relation on $Sel(F)$:

$$f \ \perp_* \ g \iff \Big((f(x) \ \perp_x \ g(x)), \ \ x \in K\Big).$$

Let $f^* : K \to Y$ be defined by $f^*(x) = x^*$. It is easy to check that $(Sel(F), \perp_*)$ is an orthogonal set and f^* is an orthogonal element of $(Sel(F), \perp_*)$. Let $\mathcal{F}(g)(x) := \alpha g(\frac{x}{p})$ for each $x \in K$ and $g \in Sel(F)$. By (11), $\mathcal{F}(g) \in Sel(F)$ and $\mathcal{F}$ is $\perp_*$-preserving. Hence, $\mathcal{F} : Sel(F) \to Sel(F)$ is an $\perp_*$-preserving mapping. Moreover, for each $g_1, g_2 \in Sel(F)$, we obtain that

$$\begin{aligned} d(\mathcal{F}(g_1), \mathcal{F}(g_2)) &= |\alpha| \sup\Big\{\frac{\|g_1(\frac{x}{p}) - g_2(\frac{x}{p})\|}{\|x\|}, 0 \neq x \in K\Big\} \\ &= \frac{|\alpha|}{p} \sup\Big\{\frac{\|g_1(\frac{x}{p}) - g_2(\frac{x}{p})\|}{\frac{\|x\|}{p}}, 0 \neq x \in K\Big\} \\ &\leq \frac{|\alpha|}{p} d(g_1, g_2) \\ &= \psi(d(g_1, g_2)), \end{aligned}$$

where $\psi(t) = \frac{|\alpha|}{p}t$, $t \in [0, \infty)$. Since $|\alpha| < p$, then $\psi \in \Psi$. Now, according to the assumptions, since for each $x \in K$, $(F(x), \perp_x, \|.\|)$ is an SO-complete metric space, then $(Sel(F), \perp_*, d)$ is an SO-complete metric space. Therefore by Theorem 2.1, it has a unique fixed point f and $\lim_{n\to\infty} \mathcal{F}^n(g) = f$ for each $g \in Sel(F)$. Hence $f : K \to Y$ is the unique selection of F such that

$$f(x) = \alpha f(\frac{x}{p}), \quad x \in K.$$

Fix $g \in Sel(F)$ and $x, y \in K$ such that $px + qy \in K$. Then $\frac{x}{p}$, $\frac{y}{p}$ and $\frac{px+qy}{p}$ are belong to K. By (10), $\alpha g(\frac{x}{p}) + \beta g(\frac{y}{p})$ and $g(\frac{px+qy}{p})$ are elements of $F(\frac{px+qy}{p})$. Hence

$$\begin{aligned}\|\alpha g(\frac{x}{p}) + \beta g(\frac{y}{p}) - g(\frac{px+qy}{p})\| &\leq diamF(\frac{px+qy}{p})\\ &\leq M\|\frac{px+qy}{p}\|.\end{aligned}$$

Thus

$$\|\alpha\mathcal{F}(g)(x) + \beta\mathcal{F}(g)(y) - \mathcal{F}(g)(px + qy)\| \leq M\frac{|\alpha|}{p}\|px + qy\|$$

for each $x, y \in K$ such that $px + qy \in K$. Repeating this process, we get

$$\|\alpha\mathcal{F}^n(g)(x) + \beta\mathcal{F}^n(g)(y) - \mathcal{F}^n(g)(px + qy)\| \leq M\Big(\frac{|\alpha|}{p}\Big)^n\|px + qy\|$$

for each $n \in \mathbb{N}$ and all $x, y \in K$ with $px + qy \in K$. Letting $n \to \infty$, we obtain

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K, px + qy \in K.$$

□

Corollary 4.2 — [5]. Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a nonempty subset of X such that $0 \in K$. Suppose that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:

1. $|\alpha| < p$ and $K \subseteq pK$,
2. $|\beta| < q$ and $K \subseteq qK$.

Consider a multivalued mapping $F : K \to CP(Y)$ such that $0 \in F(0)$ and

$$diam(F(x)) \leq M\|x\|, x \in K,$$

for some positive constant M. If

$$\alpha F(x) + \beta F(y) \subseteq F(px + qy),$$

where $x, y \in K$ and $px + qy \in K$, then there exists a unique selection $f : K \to Y$ of multivalued mapping F such that

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K, px + qy \in K.$$

Corollary 4.3 — Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a convex cone in X. Suppose that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:

1. $|\alpha| < p$ and $K \subseteq pK$,
2. $|\beta| < q$ and $K \subseteq qK$.

Consider a multivalued mapping $F : K \to B(Y)$ such that $0 \in F(0)$ and

$$diam(F(x)) \leq M\|x\|, x \in K,$$

for some positive constant M. Also, for each $x \in K$, there exists $\perp_x \subseteq F(x) \times F(x)$ such that $(F(x), \perp_x, \|.\|)$ is an SO-complete metric space with left orthogonal element x^*. If

$$\alpha F(x) + \beta F(y) \subseteq F(px + qy),$$
$$\alpha \perp_x + \beta \perp_y \subseteq \perp_{px+qy},$$

where $x, y \in K$, then there exists a unique selection $f : K \to Y$ of multivalued mapping F such that

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K.$$

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