Indian J. Pure Appl. Math., **51**(2): 589-610, June 2020

DOI: 10.1007/s13226-020-0419-3

DILATIONS OF DUAL *g*-FRAME GENERATORS FOR AN ABSTRACT WAVELET SYSTEM[1]

Liang Li* and Pengtong Li**

**Department of Mathematics and Statistics, Nanjing Audit University, Nanjing* 210029, *P.R. China*
***Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing* 210016, *P.R. China*
e-mails: liliang1100@126.com; pengtongli@nuaa.edu.cn

(*Received* 28 *September* 2018; *accepted* 22 *March* 2019)

In this paper, we study the dilation of *g*-frame generators and the dual *g*-frame generators for an abstract wavelet system. By the semi-orthogonality of a special unitary system, we prove the dual *g*-frame generators can be dilated to a pair of dual *g*-Riesz basis generators for some larger Hilbert space. We first show the existence of the dual *g*-frame generators with the same structure of a *g*-frame generator for a unitary system. We then get a sufficient and necessary condition for the existing of the dual *g*-frame generators for unitary groups on a subspace.

Key words : Frames; G-frames; G-frame generators; G-dual; dilations; unitary system.

2010 Mathematics Subject Classification : 42C15, 46L10, 42C40.

1. Introduction

In the past decades, the wavelet theory and the Gabor analysis have undergone a vast development and many different aspects of the theory were studied extensively in the literatures and many generalizations also appeared in frame theory. Frames with special structures are very important since most of the useful frames in theory and applications are of this kind, such as Gabor frames and wavelet frames. Recently, many researchers are interested in studying the generalized frames (cf. [19]) or the operator-valued frames (cf. [16]). A sequence $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$, where $\mathbb{J}$ is a finite or

[1]This work is supported by National Natural Science Foundation of China (Nos. 11671201 and 11771379).

countable set, is called a *g-frame* for H with respect to a sequence $\{H_i : i \in \mathbb{J}\}$ of closed subspaces of a Hilbert space K, if there exist two positive constants a_A and b_A such that for any $f \in H$,

$$a_A||f||^2 \leq \sum_{i\in\mathbb{J}} ||A_i f||^2 \leq b_A||f||^2.$$

Moreover, we call this sequence $\{A_i\}_{i\in\mathbb{J}}$ an *operator-valued frame*, if

$$a_A I \leq \sum_{i\in\mathbb{J}} A_i^* A_i \leq b_A I,$$

where the series converges in the strong operator topology. From the definition, an operator valued frame is obvious a g-frame. In [19], the author proved if $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ is a *g*-frame, then $\sum\limits_{i\in\mathbb{J}} A_i^* A_i$ is well defined in the strong operator topology. Thus, if $H_i \subset K$ for any $i \in \mathbb{J}$, where K is a Hilbert space, the concepts of a *g*-frame and an operator-valued frame are equivalent. If only the above right inequalities are satisfied, we call $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ a *g-Bessel sequence* or an *operator-valued Bessel sequence*. In the case, we can define the *analysis operator* of $\{A_i\}_{i\in\mathbb{J}}$ as

$$\theta : H \to \bigoplus_{i\in\mathbb{J}} H_i,\ \theta f = \{A_i f\}_{i\in\mathbb{J}},$$

where $\bigoplus\limits_{i\in\mathbb{J}} H_i$ is the orthogonal direct sum Hilbert of $\{H_i\}_{i\in\mathbb{J}}$.

A unitary system $\mathcal{U}$ is a set of unitary operators acting on a Hilbert space H which contains the identity operator I of H. A complete wandering vector for $\mathcal{U}$ is a unit vector $x \in H$ with the property that $\mathcal{U}x := \{Ux : U \in \mathcal{U}\}$ is an orthonormal basis for H (cf. [5]). A complete frame vector for $\mathcal{U}$ is a vector $x \in H$ with the property that $\mathcal{U}x$ is a frame for H (see [12]). In [12], the authors studied the complete frame vectors for a unitary system of a Hilbert space and the results are extended to projective unitary representations [8-10]. These researchers show that such abstract ways to study orthogonal wavelets and frames are very feasible and fruitful. In [16], the authors studied the operator-valued frame generators for unitary groups and operator-valued frames. The author in [17] studied the operator-valued frame generators for group-like unitary systems and extended some results in [6, 12].

Let $\mathcal{U}$ be a unitary system on a Hilbert space H such that

$$\mathcal{U} = \mathcal{U}_1\mathcal{U}_0 := \{U = U_1U_0 : U_1 \in \mathcal{U}_1, U_0 \in \mathcal{U}_0\},$$

where $\mathcal{U}_1$ and $\mathcal{U}_0$ are two unitary operator groups on H such that $\mathcal{U}_1 \cap \mathcal{U}_0 = \{I\}$. Such a $\mathcal{U}$ will be called *an abstract wavelet system*. If $\mathcal{U}_1$ is trivial, i.e., $\mathcal{U} = \mathcal{U}_0$, $\mathcal{U}$ is called *a trivial abstract wavelet system*.

The dilation property is very important in frame theory. In the paper, we are interested in the dilations of dual pairs of the *g*-frame generators for a unitary system, but not every *g*-frame generator for a unitary system has a dual with the same structure, or has a dilation property. We only consider a special abstract wavelet system.

We first revisit some basic definitions and results for *g*-frames. In Section 2, we study the existence of the canonical dual *g*-frame generator, the dilation results of *g*-frame generators and dual *g*-frame generators for a semi-orthogonal wavelet system. In Section 3, we study the case for unitary groups. We will prove that the dilation of dual *g*-frame generators for a subspace is not true in general, and also give a result about the existence of the dual Parseval *g*-frame generators, which is equivalent to another case of the dilation.

In this paper, H, K denote separable Hilbert spaces. Let $B(H, K)$ denote the set of all the bounded linear operators from H to K and write $B(H) := B(H, H)$. An operator $P \in B(H)$ is said to be a *projection* (or an oblique projection) if $P^2 = P$. Let U, V be closed subspaces of H. $H = U \dot{+} V$ denotes that $U + V = H$ and $U \cap V = \{0\}$. If $H = U \oplus V$, we write $V = H \ominus U$. For an operator $A \in B(H)$, we let $\ker A$, $\operatorname{ran} A$ denote the null space of A and the range space of A respectively. We use $U^\perp$ to denote the orthogonal complement of a closed subspace U in H.

The followings are some definitions and results of *g*-frames and *g*-frame generators for a unitary system $\mathcal{U}$ which were introduced in [15-17, 19].

Definition 1.1 — Suppose $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ satisfies

(1) $\langle A_i^* g_i, A_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \forall i, j \in \mathbb{J}, \forall ð_ℶ \in \mathbb{H}_ℶ, ð_ℷ \in \mathbb{H}_ℷ$.

(2) $\sum_{i \in \mathbb{J}} ||A_i f||^2 = ||f||^2, \forall f \in H$.

We call $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ *a g-orthonormal basis*.

In fact, by [18, Corollary 2.13], $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ is a *g*-orthonormal basis if and only if $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ is a *g*-frame and (1) holds.

Lemma 1.2 — [19]. Suppose $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ is a *g*-frame for H. Then the followings are equivalent:

(1) $\{A_i \in B(H, H_i)\}_{i \in \mathbb{J}}$ is a *g*-Riesz basis for H.

(2) $\{A_i \in B(H, H_i)\}_{i \in \mathbb{J}}$ has a unique dual *g*-frame.

(2) There is an invertible operator $T \in B(H)$ such that $\{A_i T \in B(H, H_i)\}_{i \in \mathbb{J}}$ is a *g*-orthonormal basis for H.

(3) The analysis operator of $\{A_i \in B(H, H_i)\}_{i\in\mathbb{J}}$ is invertible.

Lemma 1.3 — [19]. Suppose $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ and $\{B_i \in B(H, H_i) : i \in \mathbb{J}\}$ are *g*-frames for H. Then the followings are equivalent:

(1) There is an invertible operator $T \in B(H)$ such that $B_i = A_i T$ for $i \in \mathbb{J}$; in this case, we call these two *g*-frames are similar.

(2) The ranges of the analysis operator of $\{A_i \in B(H, H_i)\}_{i\in\mathbb{J}}$ and $\{B_i \in B(H, H_i)\}_{i\in\mathbb{J}}$ are the same.

Definition 1.4 — We say that $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$ is *g-complete*, if $\{f : A_i f = 0 : i \in \mathbb{J}\} = \nvdash$, i.e., $\overline{\operatorname{span}}_{i\in\mathbb{J}} A_i^* K = H$.

Definition 1.5 — Let $\{A_i \in B(H, H_i) : i \in \mathbb{J}\}$, $\{B_i \in B(H, H_i) : i \in \mathbb{J}\}$ be two *g*-frames on H such that $f = \sum_{i\in\mathbb{J}} B_i^* A_i f$ for any $f \in H$. Then $\{B_i\}_{i\in\mathbb{J}}$ is called *a dual g-frame* of $\{A_i\}_{i\in\mathbb{J}}$.

In this case, we call $\{A_i\}_{i\in\mathbb{J}}$ and $\{B_i\}_{i\in\mathbb{J}}$ *a pair of dual g-frames*.

Definition 1.6 — Suppose $\mathcal{U}$ is a unitary system on H, $A \in B(H, K)$.

(1) If $A\mathcal{U}^* = \{AU^* : U \in \mathcal{U}\}$ is a *g*-orthonormal basis for H, A is called *a complete wandering operator* for $\mathcal{U}$.

(2) If $A\mathcal{U}^* = \{AU^* : U \in \mathcal{U}\}$ is a *g*-Riesz basis for H, A is called *a complete g-Riesz basis generator* for $\mathcal{U}$.

(3) If $A\mathcal{U}^* = \{AU^* : U \in \mathcal{U}\}$ is a *g*-frame for H, A is called *a complete g-frame generator* for $\mathcal{U}$.

(4) If $A\mathcal{U}^* = \{AU^* : U \in \mathcal{U}\}$ is a Parseval *g*-frame for H, A is called *a complete Parseval g-frame generator* for $\mathcal{U}$.

(5) If $A\mathcal{U}^* = \{AU^* : U \in \mathcal{U}\}$ is a *g*-Bessel sequence for H, A is called *a g-Bessel generator* for $\mathcal{U}$.

Moreover, if $A\mathcal{U}^*$ and $B\mathcal{U}^*$ are a pair of dual *g*-frames, we also call A and B *a pair of dual g-frame generators* for $\mathcal{U}$.

Let $A \in B(H, K)$ be a *g*-Bessel generator for $\mathcal{U}$. For any $f \in H$, the *analysis operator* of A is defined as

$$\theta_A : H \to l^2(\mathcal{U}) \otimes K,\ \theta_A f = \sum_{U\in\mathcal{U}} \chi_U \otimes AU^* f,$$

where $\{\chi_U : U \in \mathcal{U}\}$ is the orthonormal basis for $l^2(\mathcal{U})$. And the *frame operator* of A is defined as

$$S_A : H \to H,\ S_A f = \sum_{U\in\mathcal{U}} UA^*AU^*f.$$

For an abstract wavelet system $\mathcal{U} = \mathcal{U}_1\mathcal{U}_0$ on H, if $M \subset H$ is *a complete wandering subspace* for $\mathcal{U}_1$, i.e., $[\mathcal{U}_1 M] := \overline{\text{span}}_{U_1\in\mathcal{U}_1} U_1 M = H$ and $U_1M \perp V_1M$ for any $U_1, V_1 \in \mathcal{U}_1$ with $U_1 \neq V_1$, then $\mathcal{U}$ is called *semi-orthogonal*. If $A \in B(H,K)$ is a *g*-frame generator for $\mathcal{U}$ and $[\mathcal{U}_0A^*K] = M$, A is called *a semi-orthogonal g-frame generator* for $\mathcal{U}$.

2. Dilations of G-Frame Generators for Wavelet Systems

Different from the *g*-frame generators for a unitary operator group, not every *g*-frame generator for a unitary system has the dilation property. For a *g*-frame generator with respect to a special unitary system, we have the following result. For any $U \in \mathcal{U}$, where $\mathcal{U}$ is an abstract wavelet system, we denote $U = U_1U_0$, where $U_1 \in \mathcal{U}_1$, $U_0 \in \mathcal{U}_0$.

Proposition 2.1 — Let $\mathcal{U}$ be an abstract wavelet system on H, $T \in B(H,K)$ be a complete wandering operator for $\mathcal{U}$. If $A \in B(H,K)$ is a semi-orthogonal complete Parseval *g*-frame generator for $\mathcal{U}$, then there are a Hilbert space $\widetilde{H} \supset H$, a wavelet system $\sigma(\mathcal{U}) =: \{\sigma_U : U \in \mathcal{U}\}$ on $\widetilde{H}$, and a complete wandering operator $C \in B(\widetilde{H}, K)$ for $\sigma(\mathcal{U})$, such that $A = CP$, $U = \sigma_U P$ and H is $\sigma(\mathcal{U})$-invariant, where P is the orthogonal projection from $\widetilde{H}$ onto H.

Proof : For any $f \in H$, we define $\widetilde{\theta}_A f = \sum_{U\in\mathcal{U}} UT^*AU^*f$. Then

$$\widetilde{\theta}_A^* VT^*k = \sum_{U\in\mathcal{U}} UA^*TU^*VT^*k = VA^*k,\ \forall\, V \in \mathcal{U},\ k \in K.$$

Since $\widetilde{\theta}_A^* T^*k = A^*k$, we have $V\widetilde{\theta}_A^*T^*k = \widetilde{\theta}_A^*VT^*k$. As $A \in B(H,K)$ is a semi-orthogonal *g*-frame generator, we get

$$\widetilde{\theta}_A VA^*k = \sum_{U\in\mathcal{U}} UT^*AU^*VA^*k = \sum_{U_0\in\mathcal{U}_0} V_1U_0T^*AU_0^*V_0A^*k = V\widetilde{\theta}_A A^*k.$$

Let $P_A = \widetilde{\theta}_A\widetilde{\theta}_A^*$. Then P_A is the orthogonal projection from H onto $\text{ran}\widetilde{\theta}_A$. Therefore,

$$\begin{aligned} P_A VT^*k &= \widetilde{\theta}_A\widetilde{\theta}_A^*VT^*k = \widetilde{\theta}_A V\widetilde{\theta}_A^*T^*k \\ &= \widetilde{\theta}_A VA^*k = V\widetilde{\theta}_A A^*k = V\widetilde{\theta}_A\widetilde{\theta}_A^*T^*k \\ &= VP_AT^*k. \end{aligned}$$

Moreover, $P_A^{\perp}VT^*k = VP_A^{\perp}T^*k$. Let $B = TP_A^{\perp} \in B(H,K)$. Then B is a *g*-Bessel generator for $\mathcal{U}$. Let $\widetilde{H} = H \oplus (\text{ran}\theta_A)^{\perp}$. We claim that $A \oplus B$ is a complete wandering operator for $\sigma(\mathcal{U})$, where $\sigma(\mathcal{U}) := \{\sigma_U = U \oplus U : U \in \mathcal{U}\}$ is an abstract wavelet system on $\widetilde{H}$ obviously.

In fact,

$$\begin{aligned} UA^* \oplus UB^* &= \theta_A^* UT^* \oplus UP_A^{\perp}T^* = \theta_A^* UT^* \oplus P_A^{\perp}UT^* \\ &= (\theta_A^* \oplus P_A^{\perp})(P_A UT^* \oplus P_A^{\perp}UT^*), \end{aligned}$$

where $\begin{pmatrix} \theta_A^* & 0 \\ 0 & P_A^{\perp} \end{pmatrix} : \begin{pmatrix} \text{ran}P_A \\ \text{ran}P_A^{\perp} \end{pmatrix} \to \begin{pmatrix} H \\ \text{ran}P_A^{\perp} \end{pmatrix}$ is unitary since A is a complete Parseval *g*-frame generator.

Because $TU^*P_A \oplus TU^*P_A^{\perp} = TU^*P_A + TU^*P_A^{\perp} = TU^*$ for any $U \in \mathcal{U}$, we have $A \oplus B$ is a complete wandering operator for $\sigma(\mathcal{U})$. □

For an abstract wavelet system $\mathcal{U}$, the dual *g*-frame generator of a semi-orthogonal complete *g*-frame generator $A \in B(H,K)$ with the same structure exists, but it may not have the semi-orthogonal property. We give a result about the existence of the dual *g*-frame generator with the same structure.

***Theorem* 2.2** — *Let $\mathcal{U}$ be an abstract wavelet system on H, $A \in B(H,K)$ be a semi-orthogonal complete g-frame generator for $\mathcal{U}$ and $M = [\mathcal{U}_0 A^* K]$. Then*

(1) the frame operator S_A satisfies $S_A^t Uf = US_A^t f$ for any $f \in M, t \in \mathbb{Q}$.

(2) AS_A^t is a g-frame generator for $\mathcal{U}$.

(3) $A(I - S_A^t)^p$ is a semi-orthogonal g-Bessel generator for $\mathcal{U}$, $p \in \mathbb{Q}$, when $||S_A|| \leq 1$.

PROOF : Let $M_{U_1} = [U_1 M] = [U_1 \mathcal{U}_0 A^* K]$ for any $U_1 \in \mathcal{U}_1$. For every $g \in M_{U_1}$, we define

$$S_{U_1,A}g = \sum_{U_0 \in \mathcal{U}_0} U_1 U_0 A^* A U_0^* U_1^* g.$$

Since $M \subset H$ is a complete wandering subspace for $\mathcal{U}_1$, for any $f \in [U_1 M]$, we obtain

$$S_A f = \sum_{U \in \mathcal{U}} UA^*AU^* f = \bigoplus_{U_1 \in \mathcal{U}_1} S_{U_1,A} f = S_{U_1,A} f.$$

For any $k \in K$,

$$\begin{aligned} S_{U_1,A}U_1U_0A^*k &= \sum_{V_0 \in \mathcal{U}_0} U_1V_0A^*AV_0^*U_1^*U_1U_0A^*k \\ &= \sum_{V_0 \in \mathcal{U}_0} U_1U_0U_0^*V_0A^*AV_0^*U_0A^*k \\ &= U_1U_0 \sum_{V_0 \in \mathcal{U}_0} V_0A^*AV_0^*A^*k \\ &= U_1U_0S_{I,A}A^*k, \ \forall\, U_1 \in \mathcal{U}_1, U_0 \in \mathcal{U}_0. \end{aligned}$$

On the other hand, for $U_0 \in \mathcal{U}_0$,

$$\begin{aligned} S_{I,A}U_0A^*k &= \sum_{V_0 \in \mathcal{U}_0} V_0A^*AV_0^*U_0A^*k \\ &= \sum_{V_0 \in \mathcal{U}_0} U_0U_0^*V_0A^*AV_0^*U_0A^*k \\ &= U_0 \sum_{V_0 \in \mathcal{U}_0} V_0A^*AV_0^*A^*k \\ &= U_0S_{I,A}A^*k, \ \forall\, U_0 \in \mathcal{U}_0. \end{aligned}$$

Noting that $\mathcal{U}_0$ is a unitary group, then $S_{I,A}U_0f = U_0S_{I,A}f$ for every $f \in M$, $U_0 \in \mathcal{U}_0$. So for each $U_1 \in \mathcal{U}_1$,

$$S_AU_1U_0A^*k = S_{U_1,A}U_1U_0A^*k = U_1U_0S_{I,A}A^*k = U_1S_{I,A}U_0A^*k.$$

Hence, for any $f \in M$,

$$S_AU_1f = U_1S_{I,A}f, \ U_1^*S_AU_1f = S_{I,A}f.$$

So $U_1^*S_A^tU_1f = S_{I,A}^tf$, $t \in \mathbb{Q}$. Since $U_0A^*k \in M$ for $U_0 \in \mathcal{U}_0$, we obtain

$$S_A^tUA^*k = U_1S_{I,A}^tU_0A^*k = U_1U_0S_{I,A}^tA^*k = US_A^tA^*k.$$

Therefore, AS_A^t is a semi-orthogonal complete *g*-frame generator for $\mathcal{U}$, as S_A is invertible.

Moreover, for every $f \in M$, $(I - U_1^*S_A^tU_1)f = (I - S_{I,A}^t)f$. If $||S_A|| \leq 1$, we get

$$U_1^*(I - S_A^t)^pU_1f = (I - S_{I,A}^t)^pf.$$

Because $S_Af = S_{I,A}f$, we have

$$(I - S_A^t)f = (I - S_{I,A}^t)f, \ (I - S_A^t)^pf = (I - S_{I,A}^t)^pf.$$

Then

$$(I - S_A^t)^p U A^* k = U_1(I - S_{I,A}^t)^p U_0 A^* k$$
$$= U_1 U_0 (I - S_{I,A}^t)^p A^* k$$
$$= U(I - S_A^t)^p A^* k.$$

Hence, $A(I - S_A^t)^p$ is a semi-orthogonal *g*-Bessel generator for $\mathcal{U}$.

In the following we will explain the relations between *g*-frame generators for $\mathcal{U}_0$ on M and *g*-frame generators for $\mathcal{U}$ on H.

Lemma 2.3 — Let $\mathcal{U}$ be an abstract wavelet system on H, M be a complete wandering subspace for $\mathcal{U}_1$. Suppose $A \in B(H, K)$ such that $[\mathcal{U}_0 A^* K] \subset M$. Then $A\mathcal{U}_0^*$ is a *g*-Bessel sequence (respectively, *g*-frame, *g*-Riesz basis, *g*-orthonormal basis) on M if and only if $A\mathcal{U}^*$ a *g*-Bessel sequence (respectively, *g*-frame, *g*-Riesz basis, *g*-orthonormal basis) on H.

PROOF : For any $f_1 \in [U_1 M],\ U_1 \in \mathcal{U}_1$, we define

$$\theta_{A,U_1} f = \sum_{U_0 \in \mathcal{U}_0} \chi_{U_1 U_0} \otimes A U_0^* U_1^* f.$$

For $U_1 \in \mathcal{U}_1$, let P_{U_1} be the orthogonal projection from H onto $[U_1 M]$. Let $P_{U_1} f = U_1 g_1, g_1 \in M$, $\forall\ f \in H$. Then

$$||\theta_A f||^2 = \sum_{U \in \mathcal{U}} ||AU^* f||^2 = \sum_{U_1 \in \mathcal{U}_1} \sum_{U_0 \in \mathcal{U}_0} \langle AU^* f, AU^* f\rangle$$
$$= \sum_{U_1 \in \mathcal{U}_1} \sum_{U_0 \in \mathcal{U}_0} \langle U A^* A U^* f, f\rangle$$
$$= \sum_{U_1 \in \mathcal{U}_1} \sum_{U_0 \in \mathcal{U}_0} \langle U A^* A U^* U_1 g_1, U_1 g_1\rangle$$
$$= \sum_{U_1 \in \mathcal{U}_1} \sum_{U_0 \in \mathcal{U}_0} \langle U_0 A^* A U_0^* g_1, g_1\rangle$$
$$= \sum_{U_1 \in \mathcal{U}_1} \sum_{U_0 \in \mathcal{U}_0} ||A U_0^* g_1||^2 = \sum_{U_1 \in \mathcal{U}_1} ||\theta_{I,A} g_1||^2.$$

If $A\mathcal{U}_0^*$ has an upper *g*-Bessel bound b_A, we get

$$\sum_{U_1 \in \mathcal{U}_1} ||\theta_{I,A} g_{U_1}||^2 \le b_A \sum_{U_1 \in \mathcal{U}_1} ||g_1||^2 = b_A \sum_{U_1 \in \mathcal{U}_1} ||U_1 g_1||^2$$
$$= b_A \sum_{U_1 \in \mathcal{U}_1} ||P_{U_1} f||^2 = b_A ||f||^2,$$

which implies b_A is an upper bound for $A\mathcal{U}^*$. Similarly, if $A\mathcal{U}_0^*$ has a lower g-frame bound a_A, then

$$\begin{aligned} a_A||f||^2 &= a_A \sum_{U_1\in\mathcal{U}_1} ||P_{U_1}f||^2 = a_A \sum_{U_1\in\mathcal{U}_1} ||U_1 g_{U_1}||^2 \\ &= a_A \sum_{U_1\in\mathcal{U}_1} ||g_{U_1}||^2 \leq \sum_{U_1\in\mathcal{U}_1} ||\theta_{I,A} g_{U_1}||^2. \end{aligned}$$

It means that a_A is a lower frame bound of $A\mathcal{U}^*$.

The converse is obvious since $\theta_A f = \theta_{I,A} f$ for any $f \in M$, by the semi-orthogonality of A.

We only consider the case of g-Riesz basis and the other cases are similar.

In fact, for $f \in H$,

$$\begin{aligned} \theta_A f &= \sum_{U\in\mathcal{U}} \chi_U \otimes AU^* f = \sum_{U_1\in\mathcal{U}_1} \sum_{U_0\in\mathcal{U}_0} \chi_{U_1 U_0} \otimes AU_0^* U_1^* f \\ &= \sum_{U_1\in\mathcal{U}_1} \theta_{A,U_1} P_{U_1} f. \end{aligned}$$

For any $U_1 \in \mathcal{U}_1$, there is a unitary operator $\lambda_{U_1} : l^2(\mathcal{U}_0) \to l^2(U_1\mathcal{U}_0)$ such that $\chi_{U_1U_0} = \lambda_{U_1} \chi_{U_0}$ for each $U_0 \in \mathcal{U}_0$. Then we have

$$\chi_{U_1U_0} \otimes I = (\lambda_{U_1} \otimes I)(\chi_{U_0} \otimes I).$$

Hence, for any $f \in M$, there exists an $f_1 \in [U_1 M]$ such that $U_1 f = f_1$, then we have

$$(\lambda_{U_1} \otimes I)\theta_{A,I} f = \sum_{U_0\in\mathcal{U}_0} \chi_{U_1U_0} \otimes AU_0^* U_1^* f_1 = \theta_{A,U_1} U_1 f.$$

Therefore, θ_A is surjective if and only if $\theta_{A,I}$ is surjective.

Suppose $B\mathcal{U}^*$ is a dual g-frame of $A\mathcal{U}^*$. For any $f \in M$, we have

$$f = \sum_{U\in\mathcal{U}} UB^*AU^* f = \sum_{U_0\in\mathcal{U}_0} U_0 B^* A U_0^* f.$$

Hence, $[\mathcal{U}_0 A^* K] \subset [\mathcal{U}_0 B^* K]$. It is easy to get that a dual g-frame generator B satisfies $[\mathcal{U}_0 B^* K] \subset M$ if and only if B is semi-orthogonal. And in this case we have $[\mathcal{U}_0 A^* K] = [\mathcal{U}_0 B^* K]$. By Theorem 2.2, the canonical dual g-frame generator AS_A^{-1} has the property that $[\mathcal{U}_0 A^* K] = [\mathcal{U}_0 (AS_A^{-1})^* K]$.

Lemma 2.4 — Let $\mathcal{U}$ be an abstract wavelet system on H, M be a complete wandering subspace for $\mathcal{U}_1$. If $A, B \in B(H, K)$ are g-Bessel generators for $\mathcal{U}$ such that $[\mathcal{U}_0 A^* K], [\mathcal{U}_0 B^* K] \subset M$. Then $A, B \in B(H, K)$ are dual g-frame generators for $\mathcal{U}$ on H if and only if $A, B \in B(H, K)$ are dual g-frame generators for $\mathcal{U}_0$ on M.

PROOF : Suppose $A, B \in B(H,K)$ are dual *g*-frame generators for $\mathcal{U}_0$ on M. For every $f, g \in H$, let $P_{U_1} f = U_1 f_1$, $P_{U_1} g = U_1 g_1$, where P_{U_1} is the orthogonal projection from H onto $[U_1 M]$ and $f_1, g_1 \in M$. We get

$$\begin{aligned}
\langle \sum_{U\in\mathcal{U}} UB^*AU^*f, g\rangle &= \langle \sum_{U\in\mathcal{U}} UB^*AU^*f, P_{U_1}g\rangle \\
&= \langle \sum_{U_1\in\mathcal{U}_1}\sum_{U_0\in\mathcal{U}_0} U_1U_0B^*AU^*f, U_1g_1\rangle \\
&= \sum_{U_1\in\mathcal{U}_1}\sum_{U_0\in\mathcal{U}_0} \langle f, UA^*BU_0^*g_1\rangle \\
&= \sum_{U_1\in\mathcal{U}_1}\sum_{U_0\in\mathcal{U}_0} \langle f_1, U_0A^*BU_0^*g_1\rangle \\
&= \sum_{U_1\in\mathcal{U}_1} \langle f_1, g_1\rangle = \sum_{U_1\in\mathcal{U}_1} \langle U_1f_1, U_1g_1\rangle \\
&= \sum_{U_1\in\mathcal{U}_1} \langle P_{U_1}f, P_{U_1}g\rangle = \langle f, g\rangle.
\end{aligned}$$

On the other hand, if $A, B \in B(H,K)$ are dual *g*-frame generators for $\mathcal{U}$ on H, for any $f \in M, g \in H$, we obtain

$$\begin{aligned}
\langle f, g\rangle &= \langle \sum_{U\in\mathcal{U}} UB^*AU^*f, g\rangle \\
&= \langle \sum_{U_1\in\mathcal{U}_1}\sum_{U_0\in\mathcal{U}_0} U_1U_0B^*AU_0^*U_1^*f, g\rangle \\
&= \sum_{U_0\in\mathcal{U}_0} \langle f, U_0A^*BU_0^*g\rangle.
\end{aligned}$$

Hence, $f = \sum_{U_0\in\mathcal{U}_0} U_0B^*AU_0^*f$, and so $A, B \in B(H,K)$ are dual *g*-frame generators for $\mathcal{U}_0$ on M. □

We can also obtain a dilation result of the dual *g*-generators.

Proposition 2.5 — Let $\mathcal{U}$ be an abstract wavelet system on H, $T \in B(H,K)$ be a complete wandering operator for $\mathcal{U}$. If $A \in B(H,K)$ is a semi-orthogonal complete *g*-frame generator for $\mathcal{U}$, and $B \in B(H,K)$ is a dual *g*-frame generator such that $[\mathcal{U}_0A^*K] = [\mathcal{U}_0B^*K]$, then there are a Hilbert space $\widetilde{H} \supset H$, a wavelet system $\sigma(\mathcal{U}) =: \{\sigma_U : U \in \mathcal{U}\}$ on $\widetilde{H}$ and dual *g*-Riesz basis generators $C, D \in B(\widetilde{H}, K)$ for $\sigma(\mathcal{U})$, such that $A = CP, B = DP$, $U = \sigma_U P$ and H is $\sigma(\mathcal{U})$-invariant, where P is the orthogonal projection from $\widetilde{H}$ onto H.

PROOF : For every $f \in H$ and g-Bessel generator $\Gamma \in B(H,K)$ for $\mathcal{U}$. We define

$$\widetilde{\theta}_\Gamma f = \sum_{U \in \mathcal{U}} UT^*\Gamma U^* f.$$

Let $\Omega = [\mathcal{U}_0 T^* K]$, $M = [\mathcal{U}_0 A^* K] = [\mathcal{U}_0 B^* K]$. Obviously, Ω, M are complete wandering subspaces for $\mathcal{U}_1$. For each $U_1 \in \mathcal{U}_1$, $f \in [U_1 M]$, we define

$$\widetilde{\theta}_{U_1,A} f = \sum_{V_0 \in \mathcal{U}_0} U_1 V_0 T^* A V_0^* U_1^* f.$$

So $\mathrm{ran}\widetilde{\theta}_{U_1,A} \subset [U_1\Omega]$. Since $P_{U_1} f \in [U_1 M]$ for every $f \in H$, where P_{U_1} denotes the orthogonal projection from H onto $[U_1 M]$, we can easily get

$$\widetilde{\theta}_A P_{U_1} f = \sum_{V \in \mathcal{U}} VT^*AV^* P_{U_1} f = \widetilde{\theta}_{U_1,A} P_{U_1} f.$$

We only need to consider the dilation with respect to M. By Lemma 2.4, A, B are dual g-frame generators for $\mathcal{U}_0$ on M. Obviously, we get $\mathrm{ran}\widetilde{\theta}_{I,A}, \mathrm{ran}\widetilde{\theta}_{I,B} \subset \Omega$. Let $P_{I,A}$, $P_{I,A}^\perp$, $P_{I,B}$ and $P_{I,B}^\perp$ be the orthogonal projections from Ω onto $\mathrm{ran}\widetilde{\theta}_{I,A}$, $N_A := \Omega \ominus \mathrm{ran}\widetilde{\theta}_{I,A}$, $\mathrm{ran}\widetilde{\theta}_{I,B}$ and $N_B := \Omega \ominus \mathrm{ran}\widetilde{\theta}_{I,B}$ respectively.

We can directly obtain

$$P_{I,A} = \widetilde{\theta}_{I,AS_{I,A}^{-\frac{1}{2}}} \widetilde{\theta}^*_{I,AS_{I,A}^{-\frac{1}{2}}} = \widetilde{\theta}_{I,A} S_{I,A}^{-1} \widetilde{\theta}^*_{I,A},$$

where $S_{I,A} = \widetilde{\theta}^*_{I,A}\widetilde{\theta}_{I,A}$ is invertible on M. Moreover, as $P_{I,A}\widetilde{\theta}_{I,B} = \widetilde{\theta}_{I,A} S_{I,A}^{-1}$, we have $P_{I,A}P_{I,B}$ is invertible from $\mathrm{ran}\widetilde{\theta}_{I,B}$ onto $\mathrm{ran}\widetilde{\theta}_{I,A}$. Hence, $P_{I,A}^\perp P_{I,B}^\perp$ is invertible from N_B onto N_A.

Let $H_0 := M \oplus N_B$, $\sigma_{U_0} := U_0 \oplus U_0$ for every $U_0 \in \mathcal{U}_0$. Let $C = A \oplus TP_{I,B}^\perp$. Then

$$\sigma_{U_0} C^* = U_0 A^* \oplus U_0 P_B^\perp T^* = U_0 A^* \oplus P_B^\perp U_0 T^*,$$

as $\mathcal{U}_0$ is a unitary group. Therefore, for any $x \in M, y \in N_B$, we obtain

$$\begin{aligned}
\widetilde{\theta}_{I,C}(x \oplus y) &= \sum_{U_0 \in \mathcal{U}_0} U_0 T^* C \sigma^*_{U_0}(x \oplus y) \\
&= \sum_{U_0 \in \mathcal{U}_0} U_0 T^* (AU_0^* x + TU_0{}^* P_B^\perp y) \\
&= \widetilde{\theta}_{I,A} x + P_B^\perp y.
\end{aligned}$$

Let $Q = \widetilde{\theta}_{I,A}\widetilde{\theta}^*_{I,B}$. Then $Q^2 = Q$ by the dualities of A and B, which means

$$\Omega = \mathrm{ran}\widetilde{\theta}_{I,A} \dotplus N_B = \mathrm{ran}\widetilde{\theta}_{I,B} \dotplus N_A.$$

Thus $\widetilde{\theta}_{I,C}$ is invertible from H_0 onto Ω, which implies C is a *g*-Riesz basis generator for $\sigma(\mathcal{U}_0) := \{\sigma_{U_0} : U_0 \in \mathcal{U}_0\}$ on H_0.

Moreover, let $\sigma_{U_1} := U_1 \oplus U_1$ for each $U_1 \in \mathcal{U}_1$ and

$$\sigma(\mathcal{U}) = \sigma(\mathcal{U}_1)\sigma(\mathcal{U}_0) := \{\sigma_U = \sigma_{U_1}\sigma_{U_0} : U_1 \in \mathcal{U}_1, U_0 \in \mathcal{U}_0\}.$$

Let $\widetilde{H} = [\sigma(\mathcal{U})_1 H_0]$ and P be the orthogonal projection from $\widetilde{H}$ onto H. So H is $\sigma(\mathcal{U})$-invariant and $C \in B(\widetilde{H}, K)$ is a *g*-Riesz basis generator for $\sigma(\mathcal{U})$ on $\widetilde{H}$ by Lemma 2.3, as H_0 is a complete wandering subspace for $\sigma(\mathcal{U}_1)$ and $A = CP$.

Let $D = B \oplus TP^{\perp}_{I,A}\tau^*$ and $\rho = P^{\perp}_{I,A}P^{\perp}_{I,B}$, where $\tau : N_A \to N_B$ such that $\tau P^{\perp}_{I,A}P^{\perp}_{I,B} = P^{\perp}_{I,B}$. Then ρ is invertible from N_B onto N_A. For every $h \in N_B$, $U_0 \in \mathcal{U}_0$,

$$\rho U_0 h = P^{\perp}_{I,A}P^{\perp}_{I,B}U_0 h = U_0 \rho h.$$

Thus $\tau U_0 h = U_0 \tau h$ for each $h \in N_A$. And then

$$\sigma_{U_0} D^* = U_0 B^* \oplus U_0 \tau P^{\perp}_A T^* = U_0 B^* \oplus \tau P^{\perp}_A U_0 T^*.$$

Obviously, $T\mathcal{U}_0 P^{\perp}_A \tau^*$ is a *g*-frame for N_B. Therefore, $TP^{\perp}_A\tau^*$ is a *g*-frame generator for $\mathcal{U}_0$ on N_B since $\mathcal{U}_0$ is a unitary group.

For any $x \in M, y \in N_B$, we get

$$\begin{aligned}\widetilde{\theta}_{I,D}(x \oplus y) &= \sum_{U_0 \in \mathcal{U}_0} U_0 T^* D \sigma^*_{U_0}(x \oplus y)\\ &= \sum_{U_0 \in \mathcal{U}_0} U_0 T^*(BU_0^* x + TP^{\perp}_A \tau^* U_0^* y)\\ &= \widetilde{\theta}_{I,B} x + P^{\perp}_A \tau^* y.\end{aligned}$$

Similarly, $\widetilde{\theta}_{I,D}$ is invertible from H_0 onto Ω by $\Omega = \mathrm{ran}\widetilde{\theta}_{I,B} \dotplus N_A$, which implies D is a *g*-Riesz basis generator for $\sigma(\mathcal{U}_0)$ on H_0. Moreover, $D \in B(\widetilde{H}, K)$ is a *g*-Riesz basis generator for $\sigma(\mathcal{U})$ on $\widetilde{H}$ by Lemma 2.3 and $B = DP$.

Finally, we will show the dualities of C and D. In fact, for every $x, x_1 \in M, y, y_1 \in H_0$,

$$\begin{aligned}
& \sum_{U_0\in\mathcal{U}_0} \langle \sigma_{U_0} C^* D \sigma_{U_0}^*(x\oplus y), x_1 \oplus y_1\rangle \\
= & \sum_{U_0\in\mathcal{U}_0} \langle (U_0A^* \oplus P_B^{\perp} U_0 T^*)(BU_0^* \oplus TU_0^* P_A^{\perp}\tau^*)(x\oplus y), x_1\oplus y_1\rangle \\
= & \sum_{U_0\in\mathcal{U}_0} \langle (BU_0^* \oplus TU_0^* P_A^{\perp}\tau^*)(x\oplus y), (AU_0^* \oplus TU_0^* P_B^{\perp})(x_1\oplus y_1)\rangle \\
= & \sum_{U_0\in\mathcal{U}_0} \langle BU_0^* x + TU_0^* P_A^{\perp}\tau^* y, AU_0^* x_1 + TU_0^* P_B^{\perp} y_1\rangle \\
= & \sum_{U_0\in\mathcal{U}_0} \langle BU_0^* x, AU_0^* x_1\rangle + \sum_{U_0\in\mathcal{U}_0} \langle BU_0^* x, TU_0^* P_B^{\perp} y_1\rangle \\
& + \sum_{U_0\in\mathcal{U}_0} \langle TU_0^* P_A^{\perp}\tau^* y, AU_0^* x_1\rangle + \sum_{U_0\in\mathcal{U}_0} \langle TU_0^* P_A^{\perp}\tau^* y, TU_0^* P_B^{\perp} y_1\rangle. \\
= & \langle \widetilde{\theta}_B x, \widetilde{\theta}_A x_1\rangle + \langle \widetilde{\theta}_B x, P_B^{\perp} y_1\rangle + \langle P_A^{\perp}\tau^* y, \widetilde{\theta}_A x_1\rangle + \langle P_A^{\perp}\tau^* y, P_B^{\perp} y_1\rangle \\
= & \langle x, x_1\rangle + \langle y, y_1\rangle = \langle x\oplus y, x_1\oplus y_1\rangle.
\end{aligned}$$

Then $C, D \in B(H_0, K)$ are dual *g*-frame generators for $\sigma(\mathcal{U}_0)$ on H_0. Moreover, they are dual *g*-frame generators for $\sigma(\mathcal{U})$ on $\widetilde{H}$ by Lemma 2.3.

3. Dilation of Dual G-Frame Generators for Unitary Groups

In this section, we study the dilation properties of the dual *g*-frame generators for $\mathcal{U}$, when $\mathcal{U}_1$ is trivial, which implies $\mathcal{U} = \mathcal{U}_0$ is a unitary operator group.

The following is a result similar to Proposition 2.5, but the condition is weaker since it is for a unitary group.

Proposition 3.1 — Let $\mathcal{U} = \mathcal{U}_0$ be a trivial abstract wavelet system on H, the followings are equivalent:

(1) $A \in B(H, K)$ is a complete *g*-frame generator for $\mathcal{U}$, and $B \in B(H, K)$ is a dual *g*-frame generator for A.

(2) There are a Hilbert space $\widetilde{H} \supset H$, a unitary operator group $\sigma(\mathcal{U}) := \{\sigma_U : U \in \mathcal{U}\}$ on $\widetilde{H}$ and a pair of dual *g*-Riesz basis generators $C, D \in B(\widetilde{H}, K)$ for $\sigma(\mathcal{U})$, such that H is σ-invariant, $A = CP, B = DP$ and $U = \sigma_U P$, where P is the orthogonal projection from $\widetilde{H}$ onto H.

PROOF : Let $\Lambda(\mathcal{U}) := \{\Lambda_U := \lambda_U \otimes I_K \in B(l^2(\mathcal{U})\otimes K), U \in \mathcal{U}\}$, where $\lambda(\mathcal{U})$ is the left regular representation of $\mathcal{U}$ on $l^2(\mathcal{U})$. For any $f \in H$, we have

$$\theta_A f = \sum_{U\in\mathcal{U}} \chi_U \otimes AU^* f = \sum_{U\in\mathcal{U}} \lambda_U\, \chi_I \otimes AU^* f = \sum_{U\in\mathcal{U}} \Lambda_U(\chi_I \otimes AU^* f).$$

Similar as Proposition 2.5, we can easily get $P_A = \theta_A S_A^{-1}\theta_A^*$, $P_B = \theta_B S_B^{-1}\theta_B^*$, where P_A, P_B are respectively the orthogonal projections from $l^2(\mathcal{U}) \otimes K$ onto $\mathrm{ran}\theta_A$ and $\mathrm{ran}\theta_B$. Then we get $P_A P_B$ is invertible from $\mathrm{ran}\theta_B$ onto $\mathrm{ran}\theta_A$ and $P_A^\perp P_B^\perp$ is invertible from $(\mathrm{ran}\theta_B)^\perp$ onto $(\mathrm{ran}\theta_A)^\perp$.

Let $\widetilde{H} = H \oplus (\mathrm{ran}\theta_B)^\perp$ and $\sigma_U = U \oplus \Lambda_U$ for each $U \in \mathcal{U}$. Obviously, H is σ-invariant. Let $C = A \oplus Q_I P_B^\perp$, where $Q_U : l^2(\mathcal{U}) \otimes K \to K$ is a complete wandering operator for $\Lambda(\mathcal{U})$ on $l^2(\mathcal{U}) \otimes K$ by [17, Proposition 11] for any $U \in \mathcal{U}$. Then

$$\sigma_U C^* = UA^* \oplus \Lambda_U P_B^\perp Q_I^* = UA^* \oplus P_B^\perp \Lambda_U Q_I^*,$$

as $\mathcal{U}$ is a unitary group.

Therefore, for any $x \in H, y \in (\mathrm{ran}\theta_B)^\perp$, we have

$$\begin{aligned}\theta_C(x \oplus y) &= \sum_{U \in \mathcal{U}} \chi_U \otimes C\sigma_U^*(x \oplus y) \\ &= \sum_{U \in \mathcal{U}} \chi_U \otimes (AU^* x + Q_I \Lambda_U^* P_B^\perp y) \\ &= \theta_A x + P_B^\perp y.\end{aligned}$$

Let $Q = \theta_A \theta_B^*$. Then $Q^2 = Q$ by the dualities of A and B, which means

$$l^2(\mathcal{U}) \otimes K = \mathrm{ran}\theta_A \dotplus (\mathrm{ran}\theta_B)^\perp = \mathrm{ran}\theta_B \dotplus (\mathrm{ran}\theta_A)^\perp.$$

Thus θ_C is invertible from $\widetilde{H}$ onto $l^2(\mathcal{U}) \otimes K$, which implies C is a g-Riesz basis generator for $\sigma(\mathcal{U}) := \{\sigma_U : U \in \mathcal{U}\}$ on $\widetilde{H}$, and $A = CP$, where P is the orthogonal projection from $\widetilde{H}$ onto H.

Let $D = B \oplus Q_I P_A^\perp \tau^*$ and $\rho = P_A^\perp P_B^\perp$, where $\tau : (\mathrm{ran}\theta_A)^\perp \to (\mathrm{ran}\theta_B)^\perp$ such that $\tau P_A^\perp P_B^\perp = P_B^\perp$. Let $\Lambda_1(U) = P_A^\perp \Lambda_U P_A^\perp$, $\Lambda_2(U) = P_B^\perp \Lambda_U P_B^\perp$. Then for every $u \in (\mathrm{ran}\theta_B)^\perp$, we have

$$\begin{aligned}\rho\Lambda_2(U)u &= \rho P_B^\perp \Lambda_U P_B^\perp u = P_A^\perp P_B^\perp P_B^\perp \Lambda_U P_B^\perp u \\ &= \Lambda_U P_A^\perp P_B^\perp u = P_A^\perp \Lambda_U P_A^\perp \rho u \\ &= \Lambda_1(U)\rho u.\end{aligned}$$

Then $\tau\Lambda_1(U)v = \Lambda_2(U)\tau v$ for any $v \in (\mathrm{ran}\theta_A)^\perp$. Thus we get

$$\sigma_U D^* = UB^* \oplus \Lambda_U \tau P_A^\perp Q_I^* = UB^* \oplus \tau P_A^\perp \Lambda_U Q_I^*.$$

Obviously, $Q_I \Lambda(\mathcal{U}) P_A^\perp \tau^*$ is a g-frame for $(\mathrm{ran}\theta_B)^\perp$. Therefore, $Q_I P_A^\perp \tau^*$ is a g-frame generator for $\Lambda(\mathcal{U})$ on $(\mathrm{ran}\theta_B)^\perp$.

For any $x \in H, y \in (\text{ran}\theta_B)^{\perp}$, we get

$$\begin{aligned}\theta_D(x \oplus y) &= \sum_{U \in \mathcal{U}} \chi_U \otimes D\sigma_U^*(x \oplus y)\\ &= \sum_{U \in \mathcal{U}} \chi_U \otimes (BU^*x + Q_I P_A^{\perp} \tau^* \Lambda_U^* y)\\ &= \theta_B x + P_A^{\perp} \tau^* y.\end{aligned}$$

Similarly, θ_D is invertible from $\widetilde{H}$ onto $l^2(\mathcal{U}) \otimes K$ by $l^2(\mathcal{U}) \otimes K = \text{ran}\theta_B \dotplus (\text{ran}\theta_A)^{\perp}$, which implies D is a *g*-Riesz basis generator for $\sigma(\mathcal{U})$ on $\widetilde{H}$, and $B = DP$.

Finally, we will show the dualities of C and D. In fact, for every $x, x_1 \in H$ and $y, y_1 \in (\text{ran}\theta_B)^{\perp}$,

$$\begin{aligned}&\sum_{U \in \mathcal{U}} \langle \sigma_U C^* D \sigma_U^* x \oplus y, x_1 \oplus y_1 \rangle\\ &= \sum_{U \in \mathcal{U}} \langle (UA^* \oplus P_B^{\perp} \Lambda_U Q_I^*)(BU^* \oplus Q_I \Lambda_U^* P_A^{\perp} \tau^*)(x \oplus y), x_1 \oplus y_1 \rangle\\ &= \sum_{U \in \mathcal{U}} \langle (BU^* \oplus Q_I \Lambda_U^* P_A^{\perp} \tau^*)(x \oplus y), (AU^* \oplus Q_I \Lambda_U^* P_B^{\perp})(x_1 \oplus y_1) \rangle\\ &= \sum_{U \in \mathcal{U}} \langle BU^*x + Q_I \Lambda_U^* P_A^{\perp} \tau^* y, AU^* x_1 + Q_I \Lambda_U^* P_B^{\perp} y_1 \rangle\\ &= \sum_{U \in \mathcal{U}} \langle BU^* x, AU^* x_1 \rangle + \sum_{U \in \mathcal{U}} \langle BU^* x, Q_I \Lambda_U^* P_B^{\perp} y_1 \rangle\\ &\quad + \sum_{U \in \mathcal{U}} \langle Q_I \Lambda_U^* P_A^{\perp} \tau^* y, AU^* x_1 \rangle + \sum_{U \in \mathcal{U}} \langle Q_I \Lambda_U^* P_A^{\perp} \tau^* y, Q_I \Lambda_U^* P_B^{\perp} y_1 \rangle.\\ &= \langle \theta_B x, \theta_A x_1 \rangle + \langle \theta_B x, P_B^{\perp} y_1 \rangle + \langle P_A^{\perp} \tau^* y, \theta_A x_1 \rangle + \langle P_A^{\perp} \tau^* y, P_B^{\perp} y_1 \rangle\\ &= \langle x, x_1 \rangle + \langle y, y_1 \rangle = \langle x \oplus y, x_1 \oplus y_1 \rangle.\end{aligned}$$

Then $C, D \in B(\widetilde{H}, K)$ are dual *g*-frame generators for $\sigma(\mathcal{U})$ on $\widetilde{H}$.

The converse is obvious since $U = \sigma_U P$ and H is σ-invariant.

If a complete wandering operator for $\mathcal{U} = \mathcal{U}_0$ exists, the dilation of *g*-frame dual generators for a subspace is not true in general. In the following we will show the dilation of dual *g*-frame generators for a subspace. If $T \in B(H, K)$ is a complete wandering operator for $\mathcal{U}$, for any *g*-Bessel generator $\Gamma \in B(H, K)$, we define $\widetilde{\theta}_{\Gamma}$ as

$$\widetilde{\theta}_{\Gamma} f = \sum_{U \in \mathcal{U}} UT^* \Gamma U^* f \ , \ \forall \ f \in H.$$

Proposition 3.2 — Let $\mathcal{U} = \mathcal{U}_0$ be a trivial abstract wavelet system on H, $T \in B(H, K)$ be a complete wandering operator for $\mathcal{U}$ and $M \subset H$ such that $\mathcal{U}M \subset M$. If $A \in B(H, K)$ is a *g*-frame generator for $\mathcal{U}$ on M, and $B \in B(H,K)$ is a dual g-frame generator of A on M. Then the followings are equivalent:

(1) There exists a pair of dual g-Riesz generators $C, D \in B(H,K)$ for $\mathcal{U}$ on H such that $A = CP, B = DP$, where P is the orthogonal projection from H onto M.

(2) $P^{\perp} \sim Q_B^{\perp}$, where Q_B is the orthogonal projection from H onto $\mathrm{ran}\widetilde{\theta}_B$.

PROOF : For any g-Bessel generator $\Gamma \in B(H,K)$ for $\mathcal{U}$ on H, by the definition of $\widetilde{\theta}_\Gamma$, we have $\widetilde{\theta}_\Gamma \in \mathcal{U}'$. Hence, there exists a partial isometry V_Γ with the initial space $(\ker\widetilde{\theta}_\Gamma)^{\perp}$ and the final space $\mathrm{ran}\widetilde{\theta}_\Gamma$ such that $\widetilde{\theta}_\Gamma = V_\Gamma S_\Gamma^{\frac{1}{2}}$ by the polar decomposition of $\widetilde{\theta}_\Gamma$. Then we get $V_\Gamma \in \mathcal{U}'$ since $\mathcal{U}'$ is a von Neumann algebra, and $V_\Gamma^* V_\Gamma, V_\Gamma V_\Gamma^*$ are the orthogonal projection from H onto $\mathrm{ran}V_\Gamma^*$ and $\mathrm{ran}V_\Gamma$ respectively.

For g-frame generators A and B, we can get two partial isometric operators $V_A, V_B \in \mathcal{U}'$ such that

$$(\ker V_A)^{\perp} = (\ker\widetilde{\theta}_A)^{\perp} = M = (\ker V_B)^{\perp} = (\ker\widetilde{\theta}_B)^{\perp},$$

and

$$\mathrm{ran}V_A = \mathrm{ran}\widetilde{\theta}_A, \mathrm{ran}V_B = \mathrm{ran}\widetilde{\theta}_B.$$

Therefore,

$$P = V_A^* V_A = V_B^* V_B, Q_A = V_A V_A^*, Q_B = V_B V_B^*,$$

where P, Q_A, Q_B are the orthogonal projections from H onto M, $\mathrm{ran}\widetilde{\theta}_A$, $\mathrm{ran}\widetilde{\theta}_B$ respectively. Consequencely, $P, Q_A, Q_B \in \mathcal{U}'$.

If $P^{\perp} \sim Q_B^{\perp}$, we will construct the dual g-Riesz basis generators next. In fact, there exists a partial isometry $F \in \mathcal{U}'$ with the initial space $(\mathrm{ran}\widetilde{\theta}_B)^{\perp}$ and the final space $M^{\perp}$ such that $P^{\perp} = FF^*, Q_B^{\perp} = F^*F$. Let $C = A + TF^* \in B(H,K)$. We claim C is a g-Riesz basis generator for $\mathcal{U}$ on H.

We can easily obtain $\widetilde{\theta}_C = \widetilde{\theta}_A + F^*$. For any $f \in H$, since $\widetilde{\theta}_B^* f \in (\ker\widetilde{\theta}_A)^{\perp} = M$ and $\widetilde{\theta}_A^*\widetilde{\theta}_B = \widetilde{\theta}_B^*\widetilde{\theta}_A = I_M = P$, we have $\widetilde{\theta}_B^*\widetilde{\theta}_A(\widetilde{\theta}_B^* f) = \widetilde{\theta}_B^* f$. Thus $\widetilde{\theta}_B^*(f - \widetilde{\theta}_A\theta_B^* f) = 0$, which means

$$f - \widetilde{\theta}_A\widetilde{\theta}_B^* f \in \ker\widetilde{\theta}_B^* = (\mathrm{ran}\widetilde{\theta}_B)^{\perp} = \mathrm{ran}F^*.$$

Hence, there is a $g \in M^{\perp}$ such that $F^*g = f - \widetilde{\theta}_A\widetilde{\theta}_B^* f$.

Let $h = g + \widetilde{\theta}_B^* f$. We get $\widetilde{\theta}_A h = \widetilde{\theta}_A\widetilde{\theta}_B^* f, F^*h = F^*g$. Then

$$\widetilde{\theta}_C h = \widetilde{\theta}_A h + F^*h = \widetilde{\theta}_A\widetilde{\theta}_B^* f + F^*g = f,$$

which implies $\widetilde{\theta}_C$ is surjective.

Let $\widetilde{\theta}_C f = \widetilde{\theta}_A f + F^* f = 0$ for $f \in H$. Then

$$\widetilde{\theta}_A P f = -F^* P^{\perp} f \in (\mathrm{ran}\widetilde{\theta}_B)^{\perp} = \ker \widetilde{\theta}_B^*.$$

Therefore, $\widetilde{\theta}_B^* \widetilde{\theta}_A P f = 0$, which means $Pf = 0$. Thus we obtain $F^* P^{\perp} f = 0$. Since F^* is an isometry on $M^{\perp}$, we have $P^{\perp} f = 0$. Then $f = 0$. Hence, $\widetilde{\theta}_C$ is injective.

Consequently, $C\mathcal{U}$ is a g-Riesz basis on H. Let $D = CS_C^{-1} \in B(H,K)$, where S_C is the frame operator of $C\mathcal{U}$. Since $S_C^{-1} \in \mathcal{U}'$, we get D is the (unique) canonical dual g-frame generator of C for $\mathcal{U}$ on H.

Let $D = E + Z \in B(H,K)$, where $E \in B(M,K)$, $Z \in B(M^{\perp},K)$. We need to show $B = E = DP$.

In fact, for any $f \in M$, $g \in H$, we get

$$\begin{aligned}
\langle f, g\rangle &= \sum_{U\in\mathcal{U}} \langle UD^*CU^* f, g\rangle = \sum_{U\in\mathcal{U}} \langle CU^*Pf, DU^*Pg\rangle \\
&= \sum_{U\in\mathcal{U}} \langle CPU^* f, DPU^* g\rangle = \sum_{U\in\mathcal{U}} \langle AU^* f, EU^* g\rangle \\
&= \sum_{U\in\mathcal{U}} \langle UE^*AU^* f, g\rangle.
\end{aligned}$$

Thus E, A are dual g-frame generators for $\mathcal{U}$ on M. On the other hand, as $ZU^*f = DP^{\perp}U^*Pf = DU^*P^{\perp}Pf = 0$ for any $f \in M$, for every $h \in H$, we have

$$\begin{aligned}
\langle f, h\rangle &= \sum_{U\in\mathcal{U}} \langle UC^*DU^* f, h\rangle = \sum_{\langle U\in\mathcal{U}} \langle U(A+TF^*)^*(E+Z)U^* f, h\rangle \\
&= \sum_{U\in\mathcal{U}} \langle EU^* f, AU^* h\rangle + \sum_{U\in\mathcal{U}} \langle EU^* f, TU^*F^* h\rangle \\
&\quad + \sum_{U\in\mathcal{U}} \langle ZU^* f, AU^* h\rangle + \sum_{U\in\mathcal{U}} \langle ZU^* f, TU^*F^* h\rangle \\
&= \sum_{U\in\mathcal{U}} \langle UA^*EU^* f, h\rangle + \sum_{U\in\mathcal{U}} \langle UT^*EU^* f, F^* h\rangle \\
&= \langle f + F\widetilde{\theta}_E f, h\rangle.
\end{aligned}$$

Thus $F\widetilde{\theta}_E f = 0, \widetilde{\theta}_E f \in \ker F = \mathrm{ran}\widetilde{\theta}_B$. So there exists $g \in M$ such that $\widetilde{\theta}_E f = \widetilde{\theta}_B g$, then

$$f = \widetilde{\theta}_A^* \widetilde{\theta}_E f = \widetilde{\theta}_A^* \widetilde{\theta}_B g = g.$$

which means $B = E$.

For the converse, suppose that there exist dual g-Riesz generators $C, D \in B(H, K)$ for $\mathcal{U}$ on H such that $A = CP, B = DP$, where P is the orthogonal projection from H onto M. Let $F = CP^{\perp}$. Then $\widetilde{\theta}_F = \widetilde{\theta}_C - \widetilde{\theta}_A$ since $P \in \mathcal{U}'$. So F is a g-frame generator for $\mathcal{U}$ on $M^{\perp}$. Then for $f \in M^{\perp}$, we get

$$\widetilde{\theta}_B^* \widetilde{\theta}_F f = \widetilde{\theta}_B^* (\widetilde{\theta}_C - \widetilde{\theta}_A) = P \widetilde{\theta}_D^* \widetilde{\theta}_C f - \widetilde{\theta}_B^* \widetilde{\theta}_A f = 0.$$

Thus $\text{ran}\widetilde{\theta}_F \subset \ker\widetilde{\theta}_B^* = (\text{ran}\widetilde{\theta}_B)^{\perp}$. Since $\widetilde{\theta}_C$ is invertible, for every $f \in (\text{ran}\widetilde{\theta}_B)^{\perp} \ominus \text{ran}\widetilde{\theta}_F$, there exists a $g \in H$ such that

$$\widetilde{\theta}_C g = \widetilde{\theta}_A g + \widetilde{\theta}_F g = f \in (\text{ran}\widetilde{\theta}_B)^{\perp}.$$

It follows that

$$\widetilde{\theta}_A g = f - \widetilde{\theta}_F g \in (\text{ran}\widetilde{\theta}_B)^{\perp}.$$

By the dualities of A and B, we have $H = \text{ran}\widetilde{\theta}_A \dotplus (\text{ran}\widetilde{\theta}_B)^{\perp}$, which implies $\widetilde{\theta}_A g = 0$, $g \in \ker\widetilde{\theta}_A = M^{\perp}$. Hence, $f = \widetilde{\theta}_F g \in \text{ran}\widetilde{\theta}_F$. Then $f = 0$, which shows $\widetilde{\theta}_F : M^{\perp} \to (\text{ran}\widetilde{\theta}_B)^{\perp}$ is invertible. By the polar decomposition of $\widetilde{\theta}_F$, we have $\widetilde{\theta}_F = V_F S_F^{\frac{1}{2}}$. Then $V_F \in \mathcal{U}'$, and $P^{\perp} = V_F^* V_F$, $Q_B^{\perp} = V_F V_F^*$, which implies $P^{\perp} \sim Q_B^{\perp}$.

In general, for a general g-frame, or a general vector-valued frame, there exists another case of dilation with respect to a projection(idempotent operator) from a larger space onto H, which is equivalent to the existence of a dual Parseval g-frame. Next, we will show the equivalence of this case for the g-frame generators.

Proposition 3.3 — Let $\mathcal{U} = \mathcal{U}_0$ be a trivial abstract wavelet system on H. If $A \in B(H, K)$ is a complete g-frame generator for $\mathcal{U}$, then the followings are equivalent:

(1) There is a complete Parseval g-frame generator $B \in B(H, K)$ for $\mathcal{U}$, which is a dual g-frame generator of A.

(2) $P_C \leq P_A^{\perp}$ and $1 \leq a_A$, where P_A, P_C are the orthogonal projections from $l^2(\mathcal{U})$ onto $\text{ran}\theta_A$, $\text{ran}\theta_C$ respectively, $C \in B(H, K)$ is a Parseval g-frame generator for $\mathcal{U}$ on $M := \overline{\text{ran}}(I - S_A^{-1})$ and a_A is the lower bound of A.

(3) There is a Hilbert space $\widetilde{H} \supset H$, and a unitary operator group $\sigma(\mathcal{U})$ on $\widetilde{H}$ such that H is σ-invariant, $Q\widetilde{H} = H$, $A = TQ^*$, $U = \sigma_U|_H$ and $UQ = Q\sigma_U$ for any $U \in \mathcal{U}$, where Q is a projection from $\widetilde{H}$ onto H.

PROOF : $(1) \Leftrightarrow (2)$.

Similar as the proof of Proposition 3.1, let $\Lambda(\mathcal{U}) := \{\Lambda_U := \lambda_U \otimes I_K \in B(l^2(\mathcal{U}) \otimes K), U \in \mathcal{U}\}$, where $\lambda(\mathcal{U})$ is the left regular representation of $\mathcal{U}$ on $l^2(\mathcal{U})$. By [17, Proposition 11], $Q_U : l^2(\mathcal{U}) \otimes K \to K$ is a complete wandering operator for $\Lambda(\mathcal{U})$ on $l^2(\mathcal{U}) \otimes K$ for $U \in \mathcal{U}$.

For any $f \in H$ and *g*-Bessel generator $\Gamma \in B(H, K)$ for $\mathcal{U}$, by the definition of θ_Γ, we have

$$\theta_\Gamma f = \sum_{U \in \mathcal{U}} \chi_U \otimes \Gamma U^* f = \sum_{U \in \mathcal{U}} \Lambda_U(\chi_I \otimes \Gamma U^* f).$$

Suppose $B \in B(H, K)$ is a complete Parseval *g*-frame generator and is a dual *g*-frame generator of A.

For any $f \in H$, we have $||f||^2 = ||\theta_B^* \theta_A f||^2 \leq ||\theta_A f||^2$ by the dualities. So $a_A \geq 1$.

Let $D = B - AS_A^{-1} \in B(H, K)$, where S_A is the *g*-frame operator for A. Then D is obvious a *g*-Bessel generator for $\mathcal{U}$ on H since $S_A^{-1} \in \mathcal{U}'$. Moreover, we get $\theta_D^* \theta_D = I - S_A^{-1}$. Thus, $M = \overline{\text{ran}}(I - S_A^{-1})$ is a invariant subspace for $\mathcal{U}$. We next show the existence of a Parseval *g*-frame generator for $\mathcal{U}$ on M.

In fact, by the polar decomposition of θ_D, that is, $\theta_D = TS_D^{\frac{1}{2}}$, where $T \in B(M, \overline{\text{ran}}\theta_D)$ is an isometry. Since $\Lambda_U \theta_D = \theta_D U$ for every $U \in \mathcal{U}$, we have

$$\Lambda_U T S_D^{\frac{1}{2}} = T S_D^{\frac{1}{2}} U = TUS_D^{\frac{1}{2}}.$$

Then, for every $f \in M$, we have $\Lambda_U T f = TUf$. Let $C = Q_I T \in B(H, K)$. Thus

$$CU^* f = Q_I T U^* f = Q_I \Lambda_U^* T f,$$

as $\mathcal{U}$ is a group. Hence, $C \in B(H, K)$ is a Parseval *g*-frame generator for $\mathcal{U}$ on M, $\theta_C = T$ and $\text{ran}\theta_C = \overline{\text{ran}}\theta_D$.

Because $\theta_D^* \theta_A = (\theta_B^* - S_A^{-1}\theta_A^*)\theta_A = 0$, we have $\text{ran}\theta_C \perp \text{ran}\theta_A$. So $\text{ran}\theta_C \subset (\text{ran}\theta_A)^\perp$. Hence, $P_C \leq P_A^\perp$.

For the converse, suppose $P_C \leq P_A^\perp$ and $1 \leq a_A$. Then $0 \leq I - S_A^{-1}$. Let $\tau = (I - S_A^{-1})^{\frac{1}{2}}$. Then $\tau \in \mathcal{U}'$ and $M = \overline{\text{ran}}(I - S_A^{-1}) = \overline{\text{ran}}\tau$. We will construct the dual Parseval *g*-frame generator of A.

We claim there exists a Parseval *g*-frame generator $E \in B(M, K)$ for $\mathcal{U}$ on M. In fact, let $E = AS_A^{-\frac{1}{2}}P$, where P is the orthogonal projection from H onto M. Because M is $\mathcal{U}$-invariant, we obtain E is a Parseval *g*-frame generator for $\mathcal{U}$ on M.

Since $P_C \leq P_A^\perp$ and $\Lambda(\mathcal{U})'$ is a von Neumann algebra, there exists a subprojection $Q \leq P_A^\perp$ such that $P_C \sim Q$. So there exists a partial isometry $\Delta \in \Lambda(\mathcal{U})'$ such that $P_C = \Delta\Delta^*$, $Q = \Delta^*\Delta$. Let

$\Psi = Q_I\Delta^*\theta_C$. Then

$$\Psi U^* = Q_I\Delta^*\theta_C U^* = Q_I\Delta^*\Lambda_U^*\theta_C = Q_I\Lambda_U^*\Delta^*\theta_C.$$

Hence, Ψ is also a Parseval g-frame generator for $\mathcal{U}$ on M and

$$\mathrm{ran}\theta_\Psi = \mathrm{ran}Q \subset (\mathrm{ran}\theta_A)^\perp.$$

Let $\Phi = \Psi\tau \in B(M, K)$. Thus Φ is a g-Bessel generator for $\mathcal{U}$ on M since $\tau \in \mathcal{U}'$. For every $f \in M$, we obtain $\theta_\Phi f = \theta_\Psi \tau f$. So $\theta_A^*\theta_\Phi f = \theta_A^*\theta_\Psi\tau f = 0$.

Let $B = \Phi + AS_A^{-1} \in B(H, K)$. Then B is a g-Bessel generator for $\mathcal{U}$ on M. Thus

$$\theta_B^*\theta_B = (\theta_\Phi + \theta_A S_A^{-1})^*(\theta_\Phi + \theta_A S_A^{-1}) = \tau^2 + S_A^{-1} = I,$$

which shows B is a complete Parseval g-frame generator for $\mathcal{U}$ on H. On the other hand,

$$\theta_B^*\theta_A = (\theta_\Phi + \theta_A S_A^{-1})^*\theta_A = I.$$

Therefore, $B \in B(H, K)$ is a dual g-frame generator of A.

$(1) \Leftrightarrow (3)$.

Suppose $B \in B(H, K)$ is a complete Parseval g-frame generator, a dual g-frame generator of A. By Proposition 3.1 or [15, Theorem 3.10], there exists a Hilbert $\widetilde{H} = H \oplus (\mathrm{ran}\theta_B)^\perp$, a unitary operator group $\sigma(\mathcal{U}) := \{\sigma_U = U \oplus \Lambda_U,\ U \in \mathcal{U}\}$ on $\widetilde{H}$, a complete wandering operator $T = B \oplus Q_I P_B^\perp \in B(\widetilde{H}, K)$ for $\sigma(\mathcal{U})$ on $\widetilde{H}$ such that $U = \sigma_U\,|_H$, and H is σ-invariant.

Let $Q = \theta_A^*\theta_T \in B(\widetilde{H}, H)$. For every $f \in \widetilde{H}$,

$$Qf = \sum_{U\in\mathcal{U}} UA^*T\,\sigma_U^* f.$$

Since $\sigma_V\,T^*k \in \widetilde{H}$ for $V \in \mathcal{U}$, for any $k \in K$, we have

$$Q\,\sigma_V\,T^*k = \sum_{U\in\mathcal{U}} UA^*T\,\sigma_U^*\,\sigma_V\,T^*k = VA^*k.$$

Specially,

$$QT^*k = A^*k,\ VQT^*k = VA^*k = Q\,\sigma_V\,T^*k.$$

Then $VQ = Q\,\sigma_V$ since $\mathcal{U}$ is a group. We also get $Q\widetilde{H} \subset H$ by the definition of Q. We will show Q is a projection with $\mathrm{ran}Q = H$. In fact, for any $f \in H$,

$$Qf = \sum_{U\in\mathcal{U}} UA^*T\,\sigma_U^* Pf = \sum_{U\in\mathcal{U}} UA^*BUPf = f.$$

Hence, $H \subset Q\widetilde{H}$. We obtain $Q^2 = Q$, $Q\widetilde{H} = H$.

For the converse, let $B = TQP \in B(H, K)$, where P from $\widetilde{H}$ onto H is the orthogonal projection. For each $U \in \mathcal{U}$, we obtain

$$BU^* = TQPU^* = TQ\,\sigma_U^*\,P = TU^*QP = T\,\sigma_U^*\,PQP = T\,\sigma_U^*\,P.$$

Therefore, $B \in B(H, K)$ is a complete Parseval *g*-frame generator for $\mathcal{U}$ on H. For every $f, g \in H$,

$$\begin{aligned}\sum_{U\in\mathcal{U}} \langle UA^*BU^*f, g\rangle &= \sum_{U\in\mathcal{U}} \langle UA^*TQPU^*f, g\rangle \\ &= \sum_{U\in\mathcal{U}} \langle UQT^*TQPU^*f, g\rangle \\ &= \sum_{U\in\mathcal{U}} \langle Q\,\sigma_U\,T^*T\,\sigma_U^*\,QPf, g\rangle \\ &= \langle QQPf, g\rangle = \langle f, g\rangle.\end{aligned}$$

Then $B \in B(H, K)$ is a dual *g*-frame generator of A. □

REFERENCES

1. M. Abdollahpour, Dilation of dual *g*-frames to dual *g*-Riesz bases, *Banach J. Math. Anal*., **9**(1)(2015), 54-66.
2. P. Casazza, D. Han, and D. Larson, Frames for Banach spaces, Contemp, *Math., Amer. Math. Soc*., **247** (1999), 149-182.
3. O. Christensen, An Introduction of Frames and Riesz Bases, Birkh äuser, Boston (2003).
4. J. Conway, *A course in functional analysis*, 2nd edn. Springer, New York, 1990.
5. X. Dai and D. Larson, Wandering vectors for unitary systems and orthogonal wavelets, *Mem. Am. Math. Soc*., **134**(640) (1988).
6. J. Gabardo and D. Han, Frame representations for group-like unitary systems, *J. Operat. Theor*., **49**(2) (2003), 223-244.
7. D. Han., Approximations for Gabor and wavelet frames, *Trans. Am. Math. Soc*., **355**(8) (2003), 3329-3342.
8. D. Han, Dilations and Completions for Gabor Systems, *J. Fourier Anal. Appl*., **15**(2) (2009), 201-217.
9. D. Han, Frame representations and Parseval duals with applications to Gabor frames, *Trans. Am. Math. Soc*., 360(6)(2008), 3307-3326.

10. D. Han. The existence of tight Gabor duals for Gabor frames and subspace Gabor frames, *J. Funct. Anal*., **256**(1) (2009), 129-148.

11. D. Han, K. Kornelson, D. Larson, and E. Weber, Frames for undergraduates, *Amer. Math. Soc*., Providence, 2007.

12. D. Han and D. Larson, Frames, bases and group representations, *Mem. Amer. Math. Soc*., **147**(697) (2000), 1-94.

13. D. Han and D. Larson, Wandering vector multipliers for unitary groups, *Trans. Am. Math. Soc*., **353**(8) (2001), 3347-3370.

14. D. Han, D. Larson, B. Liu, and R. Liu, Operator-valued measures, dilations and the theory of frames, *Mem. Amer. Math. Soc*., **229**(1075) (2014), 1-84.

15. D. Han, P. Li, B. Meng, and W. Tang, Operator valued frames and structured quantum channels, *Sci. China-Math*., **54**(11) (2011), 2361-2372.

16. V. Kaftal, D. Larson, and S. Zhang, Operator-valued frames, *Trans. Am. Math. Soc*., **361**(12) (2009), 6349-6385.

17. B. Meng. Operator-valued frame generators for group-like unitary systems, *Oper. Matrices*, **7**(2) (2013), 441-464.

18. A. Najati, M. Faroughi, and A. Rahimi, G-frame and stability of G-frames in Hilbert spaces, *Methods of Functional Analysis and Topology*, **14**(3) (2008), 271-286.

19. W. Sun, G-frames and *g*-Riesz bases, *J. Math. Anal. Appl*., **322**(1) (2006), 437-452.