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# RECONSTRUCTION OF THE STURM-LIOUVILLE DIFFERENTIAL OPERATORS WITH DISCONTINUITY CONDITIONS AND A CONSTANT DELAY

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In this manuscript, we study second–order differential operators with a constant delay and transmission boundary conditions. We establish properties of the spectral characteristics and investigate the inverse problem of recovering operators from their spectra. Also, we construct the potential function by using the Fourier series and delay point of the Sturm–Liouville differential operator.

**Key words** : Sturm-Liouville differential operators; reconstruction of the potential function; jump conditions; Fourier series.

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## 1. INTRODUCTION

We consider the boundary value problem

$$\ell y := -y''(x) + q(x)y(x-a) = \lambda y(x), \qquad x \in (0,\pi) \tag{1.1}$$

with the boundary conditions

$$y(0) = y^{(j)}(\pi) = 0, \text{ for } j = 0,1 \tag{1.2}$$

and the jump conditions

$$y\left(\frac{\pi}{2}+0\right)=by\left(\frac{\pi}{2}-0\right), \quad y'\left(\frac{\pi}{2}+0\right)=b^{-1}y'\left(\frac{\pi}{2}-0\right)+cy\left(\frac{\pi}{2}-0\right), \tag{1.3}$$

where $q(x) \in L(a,\pi)$ and $q(x)=0$ for $x<a$, $a \in \left(\frac{\pi}{2},\pi\right)$, $b \neq 0$, and $c$ are real. The coefficient $b$ and c are assumed to be known a priori and fixed. For simplicity we use the notation $L_j := L_j(q(x);\, a;\, b,\, c)$ for the problems (1.1)-(1.3).

The method of separation of variables for solving PDEs with a constant delay and discontinuous boundary conditions naturally leds to ODEs with a constant delay and discontinuities inside of the interval which often appear in mathematics. Inverse spectral problem consists in recovering operators from their spectral characteristics. The inverse spectral Sturm-Liouville problem can be regarded as three aspects, e.g., existence, uniqueness and reconstruction of the coefficients given specific properties of eigenvalues and eigenfunctions. This paper deals with second-order differential operators with a constant delay and discontinuous boundary conditions. In this note, by using the fourier series, we study the inverse spectral problem of recovering the potential function and the delay point on the Sturm-Liouville differential operators from their spectral characteristics. Differential equations with delay arise in various problems of mathematics as well as in applications (see the monographs [17-19] and the references therein).

The applications of boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. Inverse problems with a discontinuity condition inside the interval play an important role in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology. As a rule, such problems are related to discontinuous and non-smooth properties of a medium (e.g., see [3-12] and the references therein). Moreover, recently Freiling and Yurko in [13], Pikula, Vladicic, and Markovic, in [15], Vladicic, Pikula in [14], and Chuan-Fu Yang in [16] studied the inverse Sturm–Liuoville with constant delay. Our work generalized the result of [14] under a delay constant and discontinuous conditions inside the interval.

## 2. Asymptotic Form of Solutions and Eigenvalues

Let $\varphi(x,\lambda)$ be the solution of Eq. (1.1) under the initial conditions

$$\varphi(0,\lambda)=0, \quad \varphi'(0,\lambda)=1. \tag{2.1}$$

For each fixed $x$, and $j=0,1$, the functions $\varphi^j(x,\lambda)$ are entire function in $\lambda$ of order $1/2$ (see [13]). The function $\varphi(x,\lambda)$ is the unique solution of the integral equation

$$\varphi(x,\lambda)=\int_0^x \frac{\sin\rho(x-t)}{\rho} q(t)\varphi(t-a,\lambda)dt, \tag{2.2}$$

with $\rho^2 = \lambda$. By solving (2.2) by the method of successive approximations, we get

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \varphi_1(x,\lambda). \tag{2.3}$$

So, by using the jump conditions (1.3) and the initial conditions (2.1), we obtain

$$\varphi_0(x,\lambda) = \begin{cases} \frac{\sin\rho x}{\rho}, & x < \frac{\pi}{2}, \\ a^+\frac{\sin\rho x}{\rho} + a^-\frac{\sin\rho(\pi-x)}{\rho} + \frac{c}{2\rho^2}(\cos\rho(\pi-x) - \cos\rho x), & x > \frac{\pi}{2}, \end{cases} \tag{2.4}$$

$$\varphi_0'(x,\lambda) = \begin{cases} \cos\rho x, & x < \frac{\pi}{2}, \\ a^+\cos\rho x - a^-\cos\rho(\pi-x) + \frac{c}{2\rho}(\sin\rho(\pi-x) + \sin\rho x), & x > \frac{\pi}{2}, \end{cases} \tag{2.5}$$

where

$$a^+ = \frac{b+b^{-1}}{2}, \quad a^- = \frac{b-b^{-1}}{2}, \tag{2.6}$$

and

$$\varphi_1(x,\lambda) = \begin{cases} 0, & x < a, \\ \int_a^x \frac{\sin\rho(x-t)}{\rho} q(t)\varphi_0(t-a,\lambda)dt, & x \ge a, \end{cases} \tag{2.7}$$

$$\varphi_1'(x,\lambda) = \begin{cases} 0, & x < a, \\ \int_a^x \cos\rho(x-t) q(t)\varphi_0(t-a,\lambda)dt, & x \ge a. \end{cases} \tag{2.8}$$

Using the formulas (2.4), (2.5), (2.7) and (2.8), we calculate

$$\varphi_1(x,\lambda) = \frac{1}{\rho^2}\int_a^x q(t)\sin\rho(x-t)\sin\rho(t-a)dt \tag{2.9}$$

and

$$\varphi_1'(x,\lambda) = \frac{1}{\rho}\int_a^x q(t)\cos\rho(x-t)\sin\rho(t-a)dt. \tag{2.10}$$

Denote $\Delta_j(\lambda) := \varphi^{(j)}(\pi,\lambda)$, $j = 0, 1$. The functions $\Delta_j(\lambda)$ are entire functions in $\lambda$ of order $1/2$ and the zeroes of $\Delta_j(\lambda)$ coincide with the eigenvalues $\lambda_n$ and $\mu_n$ of $L_0(q(x); a; b, c)$ and $L_1(q(x); a; b, c)$, respectively. The functions $\Delta_j(\lambda)$ are called the characteristic functions for $L_j(q(x); a; b, c)$. From Eqs. (2.3)-(2.5) and (2.9)-(2.10) we obtain the following asymptotic formula for $|\rho| \to \infty$,

$$\begin{aligned} \Delta_0(\lambda) =& \varphi(\pi,\lambda) \\ =& a^+\frac{\sin\rho\pi}{\rho} + \frac{c}{2\rho^2}(1-\cos\rho\pi) - \frac{1}{2\rho^2}\cos\rho(\pi-a)\int_a^\pi q(t)dt \\ &+ O\left(\frac{\exp(|\tau|(\pi-a))}{\rho^3}\right) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\Delta_1(\lambda) =&\varphi'(\pi,\lambda)\\ =&a^+\cos\rho\pi - a^- + \frac{c}{2\rho}\sin\rho\pi + \frac{1}{2\rho}\sin\rho(\pi-a)\int_a^\pi q(t)dt\\ &+ O\left(\frac{\exp(|\tau|(\pi-a))}{\rho^2}\right). \end{aligned} \tag{2.12}$$

Using (2.11) and (2.12), and applying the well-known method (see [1, Ch. 1]), we obtain the asymptotic formula for the eigenvalues $\lambda_n = \rho_{n0}^2$ and $\mu_n = \rho_{n1}^2$ as $n \to \infty$,

$$\rho_{n0} = n + \frac{1}{2\pi n a^+}\left(\cos na\int_a^\pi q(t)dt + c\,(1-(-1)^n)\right) + o\left(\frac{1}{n}\right), \tag{2.13}$$

$$\rho_{n1} = \rho_{n1}^\circ + O\left(\frac{1}{n}\right), \tag{2.14}$$

where $\rho_{n1}^\circ$ is the root of $\Delta_1^\circ(\lambda) = a^+\cos\rho\pi - a^-$. So, we get $\rho_{n1}^\circ = n - \frac{1}{2} + \frac{1}{\pi}\arccos\frac{a^-}{a^+}$.

*Lemma* 2.1 — The specification of the spectrum $\{\lambda_{nj}\}$, $n_j \geq 1$ and $j = 0, 1$, uniquely determines the characteristic function $\Delta_j(\lambda)$ by the formulaes

$$\Delta_0(\lambda) = a^+\pi\prod_{n=1}^{\infty}\left(\frac{\lambda_n-\lambda}{n^2}\right), \qquad \Delta_1(\lambda) = a^+\prod_{n=1}^{\infty}\left(\frac{\mu_n-\lambda}{{\rho_{n1}^\circ}^2}\right). \tag{2.15}$$

PROOF : By Hadamard's factorization theorem [2, P. 289], $\Delta_0(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta_0(\lambda) = a^+C\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right). \tag{2.16}$$

(the case when $\Delta_0(0) = 0$ requires minor modifications). Consider the function

$$\tilde{\Delta}_0(\lambda) := a^+\frac{\sin\rho\pi}{\rho}$$

By using the Hadamard's factorization we obtain the infinite product

$$\frac{\sin\rho\pi}{\rho} = \pi\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{n^2}\right).$$

so

$$\tilde{\Delta}_0(\lambda) = a^+\pi\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{n^2}\right)$$

then

$$\frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = \frac{C}{\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n}\prod_{n=1}^{\infty}\left(1+\frac{\lambda_n-n^2}{n^2-\lambda}\right).$$

Taking (2.11) and (2.16) into account, we calculate

$$\lim_{\lambda\to-\infty}\frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = 1, \qquad \lim_{\lambda\to-\infty}\left(1+\frac{\lambda_n-n^2}{n^2-\lambda}\right) = 1$$

and hence

$$C = \pi\prod_{n=1}^{\infty}\frac{\lambda_n}{n^2}.$$

Substituting this into (2.16), we arrive at (2.15). □

## 3. Reconstruction of Potential Function

In this section, at the first, we will show that the delay point is uniquely determined by the spectrum.

*Lemma* 3.1 — If $\{\lambda_n\}_{n=1}^{\infty}$ is the spectrum of $L_0(q(x);a;b,c)$, then the delay point $a$ is uniquely determined.

PROOF : There are infinitely many numbers $k\in\mathbb{N}$ with $\sin(ka)\neq 0$. From (2.13)

$$\begin{aligned}\lim_{k\to\infty}\frac{\lambda_{k+2}-(k+2)^2-\lambda_{k-2}+(k-2)^2}{\lambda_{k+1}-(k+1)^2-\lambda_{k-1}+(k-1)^2} &= \lim_{k\to\infty}\frac{\cos a(k+2)-\cos a(k-2)}{\cos a(k+1)-\cos a(k-1)}\\ &= \lim_{k\to\infty}\frac{\sin ka\sin 2a}{\sin ka\sin a}\\ &= 2\cos a.\end{aligned} \tag{3.1}$$

Finally

$$a = \arccos\left(\frac{1}{2}\lim_{k\to\infty}\frac{\lambda_{k+2}-(k+2)^2-\lambda_{k-2}+(k-2)^2}{\lambda_{k+1}-(k+1)^2-\lambda_{k-1}+(k-1)^2}\right). \tag{3.2}$$

*Lemma* 3.2 — If $\{\lambda_n\}_{n=1}^{\infty}$ is the spectrum of $L_0(q(x);a;b,c)$ then $\int_a^\pi q(t)dt$ is uniquely determined.

PROOF : There are infinitely many $k\in\mathbb{N}$ satisfying $\cos ka\neq 0$. From (2.12), we get

$$\begin{aligned}\lim_{k\to\infty}\frac{\pi a^+\left(\lambda_k-\left(k^2+\frac{c}{\pi a^+}(1-(-1)^k)\right)\right)}{\cos ka} &= \lim_{k\to\infty}\frac{\pi a^+\left(\frac{\cos ka}{\pi a^+}\int_a^\pi q(t)dt+o(1)\right)}{\cos ka}\\ &= \int_a^\pi q(t)dt.\end{aligned} \tag{3.3}$$

By applying $q(x) = 0$ for $x \in [0, a)$, we obtain that $\int_0^\pi q(t)dt = \int_a^\pi q(t)dt$. By using Lemma 3.2, we get the first Fourier coefficient of the potential $q$ on $[0,\ \pi]$.

Now, we will obtain the Fourier coefficients of the potential function $q(x)$ and we will prove that these coefficients are uniquely determined from the spectrum of $L_0$ and $L_1$.

Denote by $a_n = \int_0^\pi q(t)\cos 2nt\,dt$ and $b_n = \int_0^\pi q(t)\sin 2nt\,dt$ the Fourier coefficient of the potential $q$. So, we finally come to our main result.

***Theorem* 3.3** — *If $\{\lambda_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$ be the eigenvalues of the boundary value problems $L_0$ and $L_1$, respectively, then the Fourier coefficient $a_n$ and $b_n$, of the potential $q$ uniquely determined for all $n \in \mathbb{N}$.*

PROOF : From the Eq. (2.2), we obtain that the characteristic functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are the following form

$$\Delta_0(\lambda) = a^+ \frac{\sin \rho\pi}{\rho} + \frac{c}{2\rho^2}(1 - \cos \rho\pi) + \frac{1}{\rho^2}\int_a^\pi q(t)\sin \rho(\pi - t)\sin \rho(t - a)dt$$

and

$$\Delta_1(\lambda) = a^+ \cos \rho\pi - a^- + \frac{c}{2\rho}\sin \rho\pi + \frac{1}{\rho}\int_a^\pi q(t)\cos \rho(\pi - t)\sin \rho(t - a)dt.$$

We transform product of trigonometric functions to sum and we get

$$\begin{aligned}\Delta_0(\lambda) =& a^+ \frac{\sin \rho\pi}{\rho} + \frac{c}{2\rho^2}(1 - \cos \rho\pi) - \frac{\cos \rho(\pi - a)}{2\rho^2}\int_a^\pi q(t)dt \\ &+ \frac{\cos \rho(\pi + a)}{2\rho^2}\int_a^\pi q(t)\cos(2\rho t)dt \\ &+ \frac{\sin \rho(\pi + a)}{2\rho^2}\int_a^\pi q(t)\sin(2\rho t)dt\end{aligned}$$

and

$$\begin{aligned}\Delta_1(\lambda) =& a^+ \cos \rho\pi - a^- + \frac{c}{2\rho}\sin \rho\pi + \frac{\sin \rho(\pi - a)}{2\rho}\int_a^\pi q(t)dt \\ &- \frac{\sin \rho(\pi + a)}{2\rho}\int_a^\pi q(t)\cos(2\rho t)dt \\ &+ \frac{\cos \rho(\pi + a)}{2\rho}\int_a^\pi q(t)\sin(2\rho t)dt.\end{aligned}$$

Now by putting $\rho = n$ (for $n \in \mathbb{N}$) and using $a_n = \int_0^\pi q(t)\cos(2nt)dt = \int_a^\pi q(t)\cos(2nt)dt$ and $b_n = \int_0^\pi q(t)\sin(2nt)dt = \int_a^\pi q(t)\sin(2nt)dt$, we get

$$a_n = 2n(nA_n \cos n(\pi + a) - B_n \sin n(\pi + a))$$

and

$$b_n = 2n(nA_n \sin n(\pi + a) + B_n \cos n(\pi + a))$$

where

$$A_n = \Delta_0(n) - a^+ \frac{\sin n\pi}{n} - \frac{c}{2n^2}(1 - \cos n\pi) + \frac{\cos n(\pi - a)}{2n^2} \int_a^\pi q(t)dt$$

and

$$B_n = \Delta_1(n) - a^+ \cos n\pi + a^- - \frac{c}{2n} \sin n\pi - \frac{\sin n(\pi - a)}{2n} \int_a^\pi q(t)dt.$$

By applying Lemma 3.1 and Lemma 3.2, we can compute $A_n$ and $B_n$. We determine the coefficients of Fourier series and compute the potential function $q(x)$.

Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\mu_n\}_{n=1}^\infty$, "be the spectrum of $L_0$ and $L_1$, respectively" are given. So, we apply the following algorithm for reconstruction of the potential function $q(x)$.

*Algorithm* :

1. Using Eqs. (3.2) and (3.3), for recovering the delay point $a$ and Constructing the $a_0 = \int_0^\pi q(t)dt$.
2. Applying Lemma 2.1 for computing $\Delta_0(n)$ and $\Delta_1(n)$.
3. Compute the value of $a_n$ and $b_n$ the coefficient of Fourier series by applying Theorem 3.3.
4. Apply the Fourier series for reconstruction the unknown potential.

3.1 *Examples* : In this section, some examples are presented.

*Example* 3.4 : Suppose that in Eqs. (1.1)-(1.3), $a$, $b$, $c$, and the potential function are

$$q(x) = \begin{cases} 7(x-2)(\pi - x), & x \geq a, \\ \\ 0, & x < a, \end{cases}$$

$a = 2$, $b = 3$, and $c = 3$. Figure 1 shows that the reconstruction of $q(x)$.

*Example* 3.5 : Suppose that in Eqs. (1.1)-(1.3),

$$q(x) = \begin{cases} x, & x \geq a, \\ \\ 0, & x < a, \end{cases}$$

$a = 2$, $b = -3$, and $c = 0$. Figure 2 shows that the reconstruction of $q(x)$.

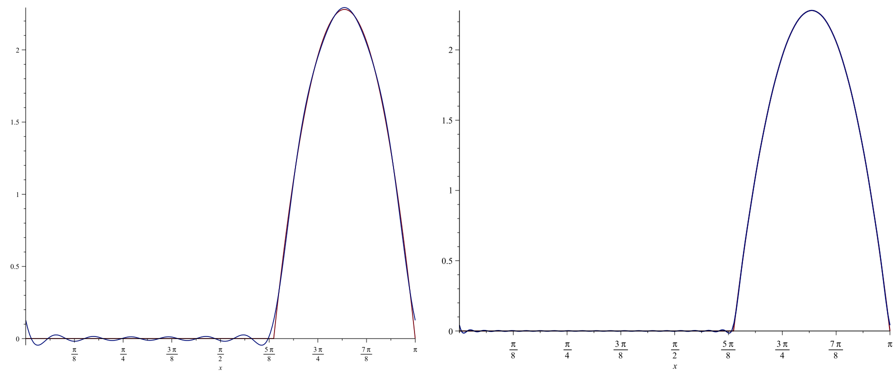

Figure 1: Reconstruction of potential function $q(x) = 5(x-2)(\pi - x)$ for $x \in [2, \pi]$, $b = 3$, and $c = 3$. (left $n = 10$) (right $n = 30$)

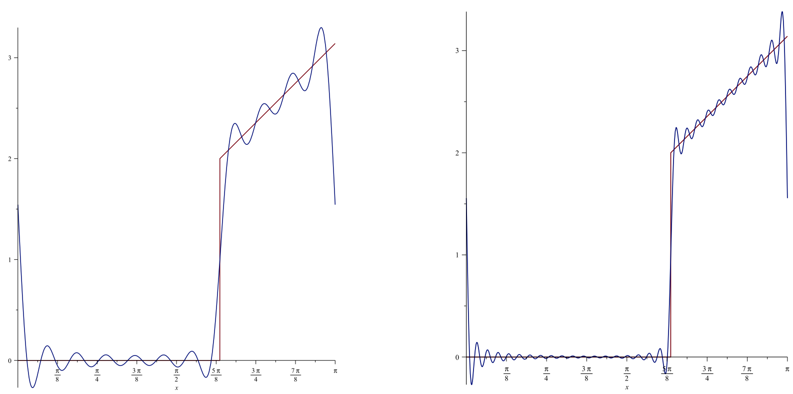

Figure 2: Reconstruction of potential function $q(x) = x$ for $x \in [2, \pi]$, $b = -3$, and $c = 0$. (left $n = 10$) (right $n = 30$)

*Example* 3.6 : Suppose that in Eqs. (1.1)(1.3), $a$, $b$, $c$, and the potential function are

$$q(x) = \begin{cases} 3\cos(x), & x \geq a, \\ \\ 0, & x < a, \end{cases}$$

$a = \frac{\pi}{2}$, $b = 2$, and $c = -3$. Figure 3 shows that the reconstruction of $q(x)$.

## 4. CONCLUSION

In this paper, the inverse Sturm-Liouville differential operator with a constant delay and transmission

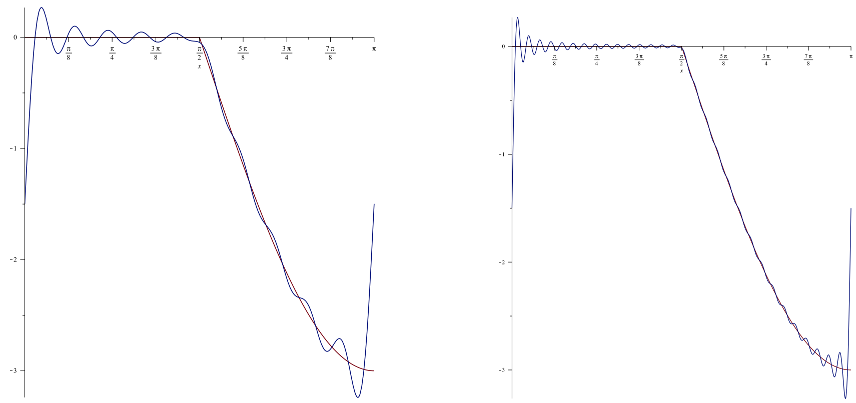

Figure 3: Reconstruction of potential function $q(x) = 3\cos(x)$ for $x \in [\frac{\pi}{2}, \pi]$, $b = 2$ and $c = -3$. (left $n = 10$) (right $n = 30$)

boundary conditions was studied. For this purpose, asymptotic form of solutions, eigenvalues and eigenfunctions of the problem was obtained. So, we recovered the delay point $a$ and constructed $a_0 = \int_0^\pi q(t)dt$. Finally, we reconstructed the potential function $q(x)$ by using the Fourier series.

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