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CLASSICAL ORTHOGONAL POLYNOMIALS VIA A SECOND-ORDER LINEAR DIFFERENTIAL OPERATORS

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Let $T_c := D(x-c)\big((x-c)D + 2\mathbb{I}\big)$ be a second-order linear differential operator, where c is an arbitrary complex number, $D := \frac{d}{dx}$ and $\mathbb{I}$ represents the identity on the linear space of polynomials with complex coefficients. The aim of this paper is to describe all of the T_c-classical orthogonal polynomials. Two canonical situations appear: the Laguerre $\{L_n^{(2)}\}_{n\geq 0}$ and the Jacobi $\{P_n^{(\alpha-2,2)}\}_{n\geq 0}$

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1. Introduction

Classical orthogonal polynomials can be characterized by means of several properties [5, 8, 24]. Among them, the so-called Hahn's characterization states that $\{P_n(x)\}_{n\geq 0}$ is a sequence of classical orthogonal polynomials if and only if the sequence of its derivatives $\{P_n'(x)\}_{n\geq 0}$ is also orthogonal. For classical orthogonal polynomials this is simple since for monic Hermite polynomials we have that $\frac{d}{dx}H_n(x) = nH_{n-1}(x)$ is again orthogonal; for monic Laguerre polynomials we have that $\frac{d}{dx}L_n^{(\alpha)}(x) = nL_{n-1}^{(\alpha+1)}(x)$ is also orthogonal; and for monic Jacobi polynomials $\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = nP_{n-1}^{(\alpha+1,\beta+1)}(x)$ is again orthogonal. Even in the case of orthogonality with respect to nonpositive weight functions we have that for Bessel polynomials [13, 14, 25] $\frac{d}{dx}B_n^{(\alpha)}(x) = nB_{n-1}^{(\alpha-2)}(x)$. In what follows we shall refer to this class as D-classical orthogonal polynomials where $D = \frac{d}{dx}$.

The concept of O-classical orthogonal polynomials, where O is an operator on the space of polynomials, has been studied by many authors in the literature (see [1-4, 7, 9, 15-18, 20]).

Let T_c be the following second-order linear differential operator

$$T_c := D(x-c)\big((x-c)D + 2\mathbb{I}\big),$$

where $\mathbb{I}$ denotes the identity operator and D the classical derivative, as already mentioned.

The aim of this contribution is to find the sequences of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ such that the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ where $Q_n(x) := \frac{T_c P_n(x)}{(n+1)(n+2)}$, is also orthogonal. This constitutes the so-called Hahn's characterization of T_c-classical orthogonal polynomials.

Indeed, we prove that such polynomials are a subfamily of the well-known D-classical orthogonal polynomials (i.e. Hermite, Laguerre, Bessel and Jacobi). More precisely, we have as solution only the Laguerre orthogonal polynomials with $\alpha = 2$ when $c = 0$ and the Jacobi polynomials with parameters $(\alpha - 2, 2)$ when $c = 1$.

The techniques that are applied here are similar to those used in [2, 3], where the authors study the Hahn's problem with respect to some differential operators that reduce by exactly one the degree of any polynomial, although the operator T_c preserves the degree of any polynomial. Note that the operator used in reference [3], therein called $\mathscr{O}_{c;1,3,2}$, and the operator defined above are connected by the standard derivative, as follows:

$$\mathscr{O}_{c;1,3,2} = DT_c.$$

Then, by recalling the equalities (2.22) and (2.23) of page 149 of Chihara's book [12], which asserts well-known differential identities fulfilled by the Laguerre and the Jacobi sequences, we can conclude that the work now gives reference [3].

The rest of this paper is organised as follows. In Section 2, we develop the terminology and basic definitions that will be used later on. In Section 3, we exhaustively describe the T_c-classical sequences.

2. Preliminaries and Notations

Let $\mathscr{P}$ be the linear space of polynomials in one variable with complex coefficients. The algebraic dual space of $\mathscr{P}$ will be represented by $\mathscr{P}'$. We denote by $\langle u, p\rangle$ the action of $u \in \mathscr{P}'$ on $p \in \mathscr{P}$ and by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the sequence of moments of u with respect to the polynomial sequence $\{x^n\}_{n\geq 0}$.

Let us define the following operations in $\mathscr{P}'$. For linear functional u, any polynomial q, and any $(a, b, c) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}^2$, let $Du = u',\ qu$, $(x-c)^{-1}u$, $\tau_{-b}u$ and $h_a u$ be the linear functionals defined by duality, [21].

$$\begin{aligned}
&\langle qu, p\rangle := \ \langle u, qp\rangle, \\
&\langle u', p\rangle := -\langle u, p'\rangle, \\
&\langle (x-c)^{-1}u,\ p\rangle := \langle u,\ \theta_c p\rangle, \quad \text{where} \quad \theta_c p(x) = \frac{p(x)-p(c)}{x-c}, \\
&\langle \tau_{-b}u,\ p\rangle := \ \langle u,\ \tau_b p\rangle, \quad \text{where} \quad \tau_b p(x) = p(x-b), \\
&\langle h_a u,\ p\rangle := \langle u,\ h_a p\rangle, \quad \text{where} \quad h_a p(x) = p(ax), \quad \text{for every } p \in \mathscr{P}.
\end{aligned}$$

A linear functional u is called *normalized* if it satisfies $(u)_0 = 1$.

Let $\{P_n\}_{n\geq 0}$ be a infinite sequence of monic polynomials (SMP) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathbb{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Notice that u_0 is said to be the canonical functional associated with the SMP $\{P_n\}_{n\geq 0}$. Recall that any $u \in \mathscr{P}'$ can be represented as $u = \sum_{n=0}^{+\infty} \langle u, P_n\rangle u_n$. So, if $\{u_n^{[1]}\}_{n\geq 0}$ denotes the dual sequence of the SMP $\{P_n^{[1]}\}_{n\geq 0}$ where $P_n^{[1]}(x) := (n+1)^{-1}P'_{n+1}(x)$, $n \geq 0$, then $Du_n^{[1]} = -(n+1)u_{n+1},\ n \geq 0$ [22]. Likewise, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of the shifted SMP $\{\tilde{P}_n\}_{n\geq 0}$, where $\tilde{P}_n(x) := a^{-n}P_n(ax+b)$ with $(a, b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, is given by $\tilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n, n \geq 0$ [22].

Let us recall that a form u (linear functional) is said to be quasi-definite (regular) if there exists a unique sequence of monic polynomials $\{P_n\}_{n\geq 0}$, such that [21, 22]

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m}, \quad n,\ m \geq 0, \quad r_n \neq 0, \quad n \geq 0. \tag{1}$$

The sequence $\{P_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to u. Note that $u = (u)_0 u_0$, with $(u)_0 \neq 0$. For any quasi-definite linear functional u and any polynomial φ such that $\varphi u = 0$, it is then straightforward to prove that $\varphi = 0$, [22].

Lemma 2.1 — [21]. The SMP $\{P_n\}_{n\geq 0}$, with dual sequence $\{u_n\}_{n\geq 0}$, is orthogonal with respect to u_0 if and only if one of the following statements hold:

(i) $u_n = \langle u_0, P_n^2\rangle^{-1} P_n u_0,\ n \geq 0.$

(ii) $\{P_n\}_{n\geq 0}$ satisfies a Three-Term Recurrence Relation

$$\text{(TTRR)}\quad \begin{cases} P_0(x)=1, \quad P_1(x)=x-\beta_0, \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x),\ n\geq 0, \end{cases} \tag{2}$$

where $\beta_n=\langle u_0, xP_n^2\rangle/\langle u_0, P_n^2\rangle \in \mathbb{C}$ and $\gamma_{n+1}=\langle u_0, P_{n+1}^2\rangle/\langle u_0, P_n^2\rangle \in \mathbb{C}\backslash\{0\}$.

A linear functional u is said to be positive-definite if it is quasi-definite i.e. it satisfies (1) and $r_n>0$ for every nonnegative integer n (see [21]). Note that a linear functional u is quasi-definite but it is not necessarily positive-definite.

The orthogonality is preserved by a shifting in the variable. Indeed, for the shifted sequence $\{\tilde{P}_n\}_{n\geq 0}$ where $\tilde{P}_n(x) := a^{-n}P_n(ax+b)$ with $(a,b)\in \mathbb{C}\backslash\{0\}\times\mathbb{C}$, the following Three-Term Recurrence Relation holds (see [12, 21])

$$\text{(TTRR)}\quad \begin{cases} \tilde{P}_0(x)=1, \quad \tilde{P}_1(x)=x-\tilde{\beta}_0, \\ \tilde{P}_{n+2}(x)=(x-\tilde{\beta}_{n+1})\tilde{P}_{n+1}(x)-\tilde{\gamma}_{n+1}\tilde{P}_n(x),\ n\geq 0, \end{cases}$$

where $\tilde{\beta}_n=a^{-1}(\beta_n-b)$ and $\tilde{\gamma}_{n+1}=a^{-2}\gamma_{n+1}$.

Notice that $\{\tilde{P}_n\}_{n\geq 0}$ is orthogonal with respect to the linear functional $\tilde{u}_0=(h_{a^{-1}}\circ\tau_{-b})u_0$.

A linear functional u is said to be D-classical when it is quasi-definite and there exist two polynomials Φ and Ψ, Φ monic, $\deg\Phi=t\leq 2$, and $\deg\Psi=1$, such that u satisfies a Pearson's equation (see [11, 21-23])

$$\text{(PE)}\quad (\Phi u)'+\Psi u=0. \tag{3}$$

In such a case, the corresponding SMOP $\{P_n\}_{n\geq 0}$ is said to be D-classical.

Any shift leaves invariant the D-classical character. Indeed, the shifted linear functional $\tilde{u}=(h_{a^{-1}}\circ\tau_{-b})u$ fulfils [22]

$$(\tilde{\Phi}\tilde{u})'+\tilde{\Psi}\tilde{u}=0,$$

where $\tilde{\Phi}(x)=a^{-t}\Phi(ax+b)$ and $\tilde{\Psi}(x)=a^{1-t}\Psi(ax+b)$.

It is well-known that any D-classical polynomial sequence $\{P_n\}_{n\geq 0}$ can be characterized by taking into account its orthogonality and a First Structure Relation (FSR) or a Second Structure Relation (SSR) as follows, [6, 11-13, 22]:

$$\text{(FSR)}\qquad \Phi(x)P'_{n+1}(x)=r(x;n)P_{n+1}(x)+s_nP_n(x),\ n\geq 0, \tag{4}$$

$$\text{(SSR)}\qquad P_n(x)=P_n^{[1]}(x)+a_nP_{n-1}^{[1]}(x)+b_nP_{n-2}^{[1]}(x),\ n\geq 0, \tag{5}$$

The D-classical orthogonal polynomials are essentially the only polynomial (not just orthogonal polynomial) systems that satisfy a Second-Order Differential Equation (SODE, in short), Bochner [10], (see also [5, 23]), of the form

$$\Phi(x)P''_{n+1}(x) - \Psi(x)P'_{n+1}(x) = \omega_n P_{n+1}(x),\ n \geq 0, \tag{6}$$

with $\deg \phi \leq 2$, $\deg \psi = 1$ and where $(n+1)\big(\frac{1}{2}\phi''(0)n + \psi'(0)\big) = \omega_n \neq 0,\ n \geq 0$. For the four canonical situations (three positive-definite cases, namely *Hermite*, *Laguerre*, and *Jacobi*, and one quasi-definite case, namely *Bessel*), in the next table we summarize the parameters involved in (2)-(6), (for more details, see [5, 11, 21-23]).

Table: Some basic characteristics of classical orthogonal polynomials.

(C$_1$) Hermite: $P_n(x) = H_n(x),\ n \geq 0,$

$\beta_n = 0,\ n \geq 0, \gamma_{n+1} = \frac{n+1}{2},\ n \geq 0,$
$\Phi(x) = 1, \quad \Psi(x) = 2x,$
$r(x;n) = 0, s_n = n+1,\ n \geq 0,$
$a_n = b_n = 0, n \geq 0,$
$\omega_n = -2(n+1), n \geq 0.$

(C$_2$) Laguerre: $P_n(x) = L_n^{(\alpha)}(x),\ n \geq 0, \quad (\alpha \neq -n,\ n \geq 1),$

$\beta_n = 2n+\alpha+1,\ n \geq 0, \gamma_{n+1} = (n+1)(n+\alpha+1),\ n \geq 0,$
$\Phi(x) = x, \quad \Psi(x) = x-\alpha-1,$
$r(x;n) = n+1, s_n = \gamma_{n+1},\ n \geq 0,$
$a_n = n, b_n = 0, n \geq 0,$
$\omega_n = -(n+1), n \geq 0.$

(C$_3$) Bessel: $P_n(x) = B_n^{(\alpha)}(x),\ n \geq 0, \quad (\alpha \neq -\frac{n}{2},\ n \geq 0),$

$\beta_0 = -\frac{1}{\alpha}, \beta_n = \frac{1-\alpha}{(n+\alpha-1)(n+\alpha)},\ n \geq 0, \gamma_n = -\frac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)},\ n \geq 1,$
$\Phi(x) = x^2, \quad \Psi(x) = -2(\alpha x+1),$
$r(x;n) = (n+1)(x - \frac{1}{n+\alpha}), s_n = -(2n+2\alpha+1)\gamma_{n+1},\ n \geq 0,$
$a_n = \frac{n}{(n+\alpha-1)(n+\alpha)}, n \geq 1, a_0 = 0, \quad b_n = \frac{(n-1)n}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)}, n \geq 2, b_0 = b_1 = 0,$
$\omega_n = (n+1)(n+2\alpha), n \geq 0.$

(C$_4$) Jacobi: $P_n(x) = P_n^{(\alpha,\beta)}(x),\ n \geq 0, \quad (\alpha,\ \beta \neq -n,\ \alpha+\beta \neq -n-1,\ n \geq 1),$

$\beta_0 = \frac{\alpha-\beta}{\alpha+\beta+2},\ \beta_n = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \gamma_n = \frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\ n \geq 1,$
$\Phi(x) = x^2-1, \quad \Psi(x) = -(\alpha+\beta+2)x+\alpha-\beta,$
$r(x;n) = (n+1)(x - \frac{\alpha-\beta}{2n+\alpha+\beta+2}), s_n = -(2n+\alpha+\beta+3)\gamma_{n+1},\ n \geq 0,$
$a_n = -\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, n \geq 1, a_0 = 0, b_n = -\frac{4(n-1)n(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, n \geq 2, b_0 = b_1 = 0,$
$\omega_n = (n+1)(n+\alpha+\beta+2), n \geq 0.$

Notice that a linear functional D-classical is not necessarily positive-definite.

Now, recall that the latter operator is given by

$$\begin{aligned} T_c : \mathscr{P} &\longrightarrow \mathscr{P} \\ f &\longmapsto (x-c)^2 f'' + 4(x-c)f' + 2f, \end{aligned}$$

It is clear that T_c preserves the degree of any polynomial. We have

$$T_c(x-c)^n = (n+1)(n+2)(x-c)^n,\ n \geq 0. \tag{7}$$

By transposition of the operator T_c, we get

$${}^tT_c = (x-c)^2 D^2. \tag{8}$$

For any SMP $\{P_n\}_{n\geq 0}$, we define

$$Q_n(x) := \frac{T_c\, P_n(x)}{(n+1)(n+2)}, \quad n \geq 0. \tag{9}$$

Clearly, $\{Q_n\}_{n\geq 0}$ is a SMP and $\deg Q_n = n$. If $\{v_n\}_{n\geq 0}$ denotes the dual sequence of $\{Q_n\}_{n\geq 0}$, then we have

$$(x-c)^2 v_n'' = (n+1)(n+2)u_n, \quad n \geq 0. \tag{10}$$

Note that for $c = 0$ and using the representation of Laguerre polynomials in terms of hypergeometric series [19]:

$$L_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} \frac{\Gamma(n+\alpha+1)}{\Gamma(\nu+\alpha+1)}\, x^\nu, \tag{11}$$

with $\alpha = 2$, we obtain

$$\Big(x^2D^2 + 4xD + 2\mathbb{I}\Big) L_n^{(2)}(x) = (n+1)(n+2) L_n^{(0)}(x), \quad n \geq 0.$$

Equivalently

$$L_n^{(0)}(x) = \frac{T\, L_n^{(2)}(x)}{(n+1)(n+2)},\ n \geq 0, \tag{12}$$

where $T := x^2D^2 + 4xD + 2\mathbb{I}$. This implies that $Q_n(x) = L_n^{(0)}(x),\ n \geq 0$, which is an example of solution of our problem. The goal of this manuscript is to describe all of the sequences of T_c-classical orthogonal polynomials in the Hahn's sense, where $T_c,\ c \in \mathbb{C}$ is the above operator.

3. The T_c-Classical Orthogonal Polynomials

Definition 3.1 — Let u_0 be a quasi-definite linear functional and let $\{P_n\}_{n\geq 0}$ be the corresponding SMOP. We call $\{P_n\}_{n\geq 0}$ is T_c-classical if $\{T_cP_n\}_{n\geq 0}$ is also orthogonal. In this case, u_0 is also said to be an T_c-classical linear functional.

Any shift leaves invariant the T_c-classical character.

Lemma 3.1 — When $\{P_n\}_{n\geq 0}$ is T_c-classical, then for any $(a,b)\in\mathbb{C}\backslash\{0\}\times\mathbb{C}$ the shifted polynomial sequence $\{\tilde{P}_n\}_{n\geq 0}$ given by $\tilde{P}_n(x)=a^{-n}P_n(ax+b)$, $n\geq 0$, is $T_{\tilde{c}}$-classical, where $\tilde{c}=a^{-1}(c-b)$.

Proof : Assume that $\{P_n\}_{n\geq 0}$ is T_c-classical. By Definition 3.1, the SMP $\{Q_n(\,;c)\}_{n\geq 0}$ given by (9) that is,

$$(n+1)(n+2)Q_n(x)=(x-c)^2P_n''(x)+4(x-c)P_n'(x)+2P_n(x),\ n\geq 0 \tag{13}$$

is orthogonal.

For any fixed $(a,b)\in\mathbb{C}\backslash\{0\}\times\mathbb{C}$, let $\{\tilde{P}_n\}_{n\geq 0}$ and $\{\tilde{Q}_n\}_{n\geq 0}$ be the shifted SMOP given by $\tilde{P}_n(x)=a^{-n}P_n(ax+b)$ and $\tilde{Q}_n(x)=a^{-n}Q_n(ax+b)$. By replacing x by $ax+b$ in (13), we get

$$(n+1)(n+2)\tilde{Q}_n(x)=(x-\tilde{c})^2\tilde{P}_n''(x)+4(x-\tilde{c})\tilde{P}_n'(x)+2\tilde{P}_n(x),\ n\geq 0$$

where $\tilde{c}=a^{-1}(c-b)$.
Equivalently $\tilde{Q}_n(x)=\frac{T_{\tilde{c}}\tilde{P}_n(x)}{(n+1)(n+2)}$, $n\geq 0$. Hence, $\{\tilde{P}_n\}_{n\geq 0}$ is $T_{\tilde{c}}$-classical.

Our next goal is to describe all of the the T_c-classical polynomial sequences. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are SMOP satisfying

$$\begin{cases} P_0(x)=1,\ P_1(x)=x-\beta_0, \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x),\ \gamma_{n+1}\neq 0,\ n\geq 0, \end{cases} \tag{14}$$

$$\begin{cases} Q_0(x)=1,\ Q_1(x)=x-\xi_0, \\ Q_{n+2}(x)=(x-\xi_{n+1})Q_{n+1}(x)-\lambda_{n+1}Q_n(x),\ \lambda_{n+1}\neq 0,\ n\geq 0. \end{cases} \tag{15}$$

The dual sequences of $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ will be denoted by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, respectively. By Lemma 2.1(i), we get

$$u_n=\frac{P_n}{\langle u_0,P_n^2\rangle}u_0,\ n\geq 0 \quad ; \quad v_n=\frac{Q_n}{\langle v_0,Q_n^2\rangle}v_0,\ n\geq 0. \tag{16}$$

In the following result, we prove that the sequence $\{Q_n\}_{n\geq 0}$ is D-classical.

Lemma 3.2 — The following properties hold.

(i) (SSR) $Q_n(x) = Q_n^{[1]}(x) + c_n Q_{n-1}^{[1]}(x) + d_n Q_{n-2}^{[1]}(x),\ n \geq 0$, where

$$c_n = \frac{n}{2}\big(\beta_n - \xi_n\big),\ n \geq 0, \quad d_n = \frac{n-1}{2}\big(\frac{n}{n+2}\gamma_n - \lambda_n\big),\ n \geq 1,\ d_0 = 0.$$

(ii) (PE) $(\Phi v_0)' + \Psi v_0 = 0$, where

$$\begin{aligned} \kappa\Phi(x) &= d_2(\lambda_1\lambda_2)^{-1}Q_2(x) + c_1\lambda_1^{-1}Q_1(x) + 1, \\ \Psi(x) &= (\kappa\lambda_1)^{-1}Q_1(x).\ (\kappa \text{ is a normalization factor}) \end{aligned}$$

PROOF : Let us introduce the sequence of monic polynomials $\{Z_n\}_{n\geq 0}$ given by

$$(n+2)Z_{n+1}(x) := (x-c)^2 P_n'(x) + 2(x-c)P_n(x),\ n \geq 0. \tag{17}$$

By taking derivatives in both hand sides of (13) and taking into account (17), we get

$$Z_n^{[1]}(x) = Q_n(x),\ n \geq 0. \tag{18}$$

Notice that $Z_n(x)$ is a monic primitive of $Q_n(x)$.
From (14) and (17), we obtain

$$(n+4)Z_{n+3}(x) = (n+3)(x-\beta_{n+1})Z_{n+2}(x) - (n+2)\gamma_{n+1}Z_{n+1}(x) + (x-c)^2P_{n+1}(x),\ n \geq 0,$$

$$3Z_2(x) = 2(x-\beta_0)Z_1(x) + (x-c)^2P_0(x).$$

By differentiating in both sides of the previous identities and inserting (18), we get

$$\begin{aligned} 2(n+3)Z_{n+2}(x) &= (n+3)(n+4)Q_{n+2}(x) - (n+2)[(n+3)(x-\beta_{n+1})Q_{n+1}(x) \\ &\quad -(n+1)\gamma_{n+1}Q_n(x)],\ n \geq 0, \\ 2Z_1(x) &= 3Q_1(x) - (x-\beta_0)Q_0(x). \end{aligned}$$

Then, by (15), it follows that

$$Z_{n+2}(x) = Q_{n+2}(x) + e_{n+1}Q_{n+1}(x) + f_nQ_n(x),\ n \geq 0, \tag{19}$$

$$Z_1(x) = Q_1(x) + e_0Q_0(x). \tag{20}$$

where $e_n = \frac{n+1}{2}(\beta_n - \xi_n)$ and $f_n = \frac{n+2}{2}(\frac{n+1}{n+3}\gamma_{n+1} - \lambda_{n+1})$.
By differentiating both hand sides of (19) and using (18), (i) holds.

Let $\{v_n^{[1]}\}_{n\geq 0}$ be the dual sequence of $\{Q_n^{[1]}\}_{n\geq 0}$. From (i), we have $\langle v_0^{[1]}, Q_n\rangle = 0,\ n \geq 3$, $\langle v_0^{[1]}, Q_2\rangle = d_2$, $\langle v_0^{[1]}, Q_1\rangle = c_1$, and $\langle v_0^{[1]}, Q_0\rangle = 1$. So, $v_0^{[1]} = d_2 v_2 + c_1 v_1 + v_0$, and by (16), we get $v_0^{[1]} = \kappa\Phi(x)v_0$, where $\kappa\Phi(x) = d_2\lambda_1^{-1}\lambda_2^{-1}Q_2(x) + c_1\lambda_1^{-1}Q_1(x) + 1$ and κ is a normalization factor. Because $(v_0^{[1]})' = -v_1 = -\lambda_1^{-1}Q_1 v_0$, then $(\Phi v_0)' + \Psi v_0 = 0$, where $\Psi(x) = (\kappa\lambda_1)^{-1}Q_1(x)$. Hence, (ii) holds. □

Lemma 3.3 — There are two non-zero polynomials F and G, with $\deg F \leq 2$ and $\deg G \leq 1$, such that

(i) $(x-c)^2 v_0 = F(x)u_0$.

(ii) $F(x)Q_n''(x) + G(x)Q_n'(x) + 2Q_n(x) = \rho_n P_n(x),\ n \geq 0$, where

$$\begin{aligned} F(x) &= \frac{1}{2}\Big(\rho_2 P_2(x) - G(x)Q_2'(x) - 2Q_2(x)\Big), \\ G(x) &= \rho_1 P_1(x) - 2Q_1(x), \\ \rho_n &= (n+1)(n+2)\frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_n^2\rangle} = n(n-1)\frac{F''(0)}{2} + nG'(0) + 2. \end{aligned}$$

(iii) The following relations hold

(a) $A(x)F(x) = 2\Phi^2(x)$,

(b) $\Phi(x)G(x) = -2\big(\Phi'(x) + \Psi(x)\big)F(x)$,

where,

$$A(x) = \big(2\Phi'(x) + \Psi(x)\big)\big(\Phi'(x) + \Psi(x)\big) - \big(\Phi''(x) + \Psi'(x)\big)\Phi(x).$$

PROOF : From (10) and (16), we obtain

$$(x-c)^2 Q_n(x)v_0'' + 2(x-c)^2 Q_n'(x)v_0' + (x-c)^2 Q_n''(x)v_0 = \rho_n P_n(x)u_0,\ n \geq 0, \tag{21}$$

where $\rho_n = (n+1)(n+2)\frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_n^2\rangle}$.

From (21) with $n = 0$, we obtain

$$(x-c)^2 v_0'' = 2u_0. \tag{22}$$

Using (21) and (23), it follows that

$$(x-c)^2 Q_n''(x)v_0 + 2(x-c)^2 Q_n'(x)v_0' = (\rho_n P_n(x) - 2Q_n(x))u_0. \tag{23}$$

For $n=1$, (23) becomes

$$2(x-c)^2 v_0' = G(x)u_0, \tag{24}$$

where $G(x)=\rho_1 P_1(x)-2Q_1(x)$.

By inserting (24) in (23), we obtain

$$(x-c)^2 Q''(x)_n v_0 = (\rho_n P_n(x) - 2Q_n(x) - G(x)Q'(x)_n)u_0. \tag{25}$$

Hence, taking $n=2$ in (25), (i) holds.

Meanwhile, by substituting $(x-c)^2 v_0 = Fu_0$ in (25) and taking into account the quasi-definiteness of u_0, we deduce (ii).

By using Lemma 3.2(ii), we can write

$$\Phi v_0' = -(\Phi' + \Psi)v_0, \quad \Phi^2 v_0'' = Av_0, \tag{26}$$

where $A=(2\Phi'+\Psi)(\Phi'+\Psi)-(\Phi''+\Psi')\Phi$.

In contrast, if we multiply (22) by Φ^2 and we take into account (26), (i) and also the quasi-definiteness of u_0, we get (iii) (a).

In the same way, multiplying both sides of (24) by Φ and using (26) (i), and then taking into account the quasi-definiteness of u_0, we deduce (iii) (b). □

Now, we will describe all of the T_c-classical polynomial sequences.

***Theorem* 3.1** — *The T_c-classical polynomial sequences are, up to a suitable affine transformation in the variable, one of the following D-classical polynomial sequences*

(i) $P_n(x)=L_n^{(2)}(x)$ *and* $Q_n(x)=L_n^{(0)}(x),\ n\geq 0$, *with* $c=0$.

(ii) $P_n(x)=P_n^{(\alpha-2,2)}(x)$ *and* $Q_n(x)=P_n^{(\alpha,0)}(x),\ n\geq 0$, *with* $c=1$ *and where* $\alpha\neq -n+2,\ n\geq 1$.

PROOF : From Lemma 3.2, $\{Q_n\}_{n\geq 0}$ is D-classical. According to Lemma 3.1, we will analyze the following situations:

(S$_1$). $\{Q_n\}_{n\geq 0}$ is the Hermite SMOP. From Table **(C$_1$)**, $A(x)=2(2x^2-1)$. Then, from Lemma 3.3(iii) (a), we get $2(2x^2-1)F(x)=2$. This yields a contradiction.

(S$_2$). $\{Q_n\}_{n\geq 0}$ is the Laguerre SMOP. From Table **(C$_2$)**, $A(x) = x^2-2\alpha x+\alpha(\alpha-1)$. Therefore, from Lemma 3.3, (iii), we have

$$\big(x^2 - 2\alpha x + \alpha(\alpha-1)\big)F(x) = 2x^2, \tag{27}$$

$$-2(x-\alpha)F(x) = xG(x). \tag{28}$$

From (27), we can deduce, $F(x) = 2$ and $\alpha = 0$. Hence, we get $A(x) = x^2$ and $G(x) = -4$. According to Lemma 3.3 (ii), we have $2Q_n''(x) - 4Q_n'(x) + 2Q_n(x) = \rho_n P_n(x),\ n \geq 0$. Thus, $\rho_n = 2,\ n \geq 0$. Therefore,

$$Q_n''(x) - 2Q_n'(x) + Q_n(x) = P_n(x), \quad n \geq 0.$$

This gives, for $n = 1$, $\beta_0 = 3$. Since $2 = \rho_1 = 6\langle v_0, Q_1^2\rangle\langle u_0, P_1^2\rangle^{-1} = 6\lambda_1\gamma_1^{-1}$, we get $\gamma_1 = 3$. In contrast, we have by Lemma 3.2(i), $\beta_n = 2n+3,\ n \geq 1$ and $\gamma_n = n(n+2),\ n \geq 2$. Hence, we obtain $\beta_n = 2n+3$ and $\gamma_{n+1} = (n+1)(n+3)$, $n \geq 0$. Then, $P_n(x) = L_n^{(2)}(x)$, $n \geq 0$, with $Q_n(x) = L_n^{(0)}(x)$, $n \geq 0$, and $(x-c)^2 v_0 = 2u_0$. Making $n = 1$ in (13), we get $c = 0$. Consequently, the following relations hold:

$$L_n^{(0)}(x) = \frac{TL_n^{(2)}(x)}{(n+1)(n+2)},\ n \geq 0, \tag{29}$$

$$L_n^{(0)\prime\prime}(x) - 2L_n^{(0)\prime}(x) + L_n^{(0)}(x) = L_n^{(2)}(x),\ n \geq 0. \tag{30}$$

(S$_3$). $\{Q_n\}_{n\geq 0}$ is the Bessel SMOP with parameter $\alpha \neq -n/2,\ n \geq 0$. In this case, we get $A(x) = 2(1-\alpha)(3-2\alpha)x^2+4(2\alpha-3)x+4$. Using Lemma 3.3(iii) (a), we obtain $A(x)F(x) = 2x^4$. This requires that $\deg A = 2$ and $\deg F = 2$ because $\deg A \leq 2$, $\deg F \leq 2$, and $\deg A+\deg F = 4$. However, from the previous equation, we must have $A(0) = 0$, that contradicts the fact that $A(0) = 4$.

(S$_4$). $\{Q_n\}_{n\geq 0}$ is the Jacobi SMOP with parameters α and β satisfying $\alpha,\ \beta \neq -n,\ \alpha+\beta \neq -n-1$, $n \geq 1$. Then, we have

$$A(x) = (\alpha+\beta-1)\big((\alpha+\beta)x^2 + 2(\beta-\alpha)x\big) + (\alpha-\beta)^2 - (\alpha+\beta). \tag{31}$$

By using Lemma 3.3(iii) (a), we obtain $A(x)F(x) = 2(x^2-1)^2$. The fact that $\deg A+\deg F = 4$, $\deg F \leq 2$ and $\deg A \leq 2$, yields $\deg A = 2$ and $\deg F = 2$. Meanwhile, the previous equation becomes $F(x)$ divides $(x^2-1)^2$, hence there are three situations to be considered. Either $F(x) = \mu(x+1)^2,\ F(x) = \mu(x-1)^2$, or $F(x) = \mu(x-1)(x+1)$, where μ is a non-zero real number.

(S$_{4,1}$). $F(x) = \mu(x+1)^2,\ \mu \neq 0$. According to Lemma 3.3(iii), we easily obtain

$$\mu A(x) = 2(x-1)^2, \tag{32}$$

$$(x-1)G(x) = 2\mu\big((\alpha+\beta)x + \beta - \alpha\big). \tag{33}$$

From (31) and (32), we get

$$\begin{cases} (\alpha+\beta-1)(\alpha+\beta)\mu = 2, \\ (\alpha+\beta-1)(\alpha-\beta)\mu = 2, \\ \big((\alpha-\beta)^2 - (\alpha+\beta)\big)\mu = 2. \end{cases}$$

Clearly, $\beta = 0$ and then $\mu = \frac{2}{\alpha(\alpha-1)}$ with $\alpha(\alpha-1) \neq 0$. Hence, (32) and (33) gives

$$A(x) = \alpha(\alpha-1)(x-1)^2, \tag{34}$$

$$G(x) = \frac{4}{\alpha-1}(x+1). \tag{35}$$

Then, by virtue of Lemma 3.3(ii), we can deduce

$$\frac{2}{\alpha(\alpha-1)}(x+1)^2 Q_n''(x) + \frac{4}{\alpha-1}(x+1)Q_n'(x) + 2Q_n(x) = \rho_n P_n(x),\ n \geq 0.$$

Thus $\rho_n = \frac{2n(n-1)}{\alpha(\alpha-1)} + \frac{4n}{\alpha-1} + 2,\ n \geq 0$.

For $n = 1$, we obtain $\beta_0 = \frac{\alpha-4}{\alpha+2}$. Since $2\frac{\alpha+1}{\alpha-1} = \rho_1 = 6\langle v_0, Q_1^2\rangle\langle u_0, P_1^2\rangle^{-1} = 6\lambda_1\gamma_1^{-1}$, we get $\gamma_1 = \frac{12(\alpha-1)}{(\alpha+2)^2(\alpha+3)}$.
From Table

(C$_4$) and by Lemma 3.2(i), we finally get

$$\beta_n = \frac{\alpha(\alpha-4)}{(2n+\alpha)(2n+\alpha+2)},\ n \geq 0,$$

$$\gamma_{n+1} = \frac{4(n+1)(n+3)(n+\alpha+1)(n+\alpha-1)}{(2n+\alpha+1)(2n+\alpha+2)^2(2n+\alpha+3)},\ n \geq 0.$$

Thus, we conclude that $P_n(x) = P_n^{(\alpha-2,2)}(x)$, and $Q_n(x) = P_n^{(\alpha,0)}(x),\ \ n \geq 0$ with $\alpha \neq -n+2,\ n \geq 1$. In addition, by (13) with $n = 1$, (14) and (15), we get $c = \frac{1}{2}(3\xi_0 - \beta_0) = 1$. By Lemma 3.3(i), $(x-1)^2 v_0 = \frac{2}{\alpha(\alpha-1)}(x+1)^2 u_0$. Consequently,

$$P_n^{(\alpha,0)}(x) = \frac{T_1 P_{n+1}^{(\alpha-2,2)}(x)}{(n+1)(n+2)}, \tag{36}$$

$$(x+1)^2 P_n^{(\alpha,0)''}(x) + 2\alpha(x+1)P_n^{(\alpha,0)'}(x) + \alpha(\alpha-1)P_n^{(\alpha,0)}(x) = \chi_{n,\alpha} P_n^{(\alpha-2,2)}(x),\ n \geq 0, \tag{37}$$

where $\chi_{n,\alpha} = (n+\alpha)(n+\alpha-1),\ n \geq 0$.

($S_{4,2}$). $F(x) = \mu(x-1)^2$, $\mu \neq 0$. By a similar computation as in **($S_{4,1}$)**, we get $P_n(x) = P_n^{(2,\beta-2)}(x),\ n \geq 0$, where $\beta \neq -n+2,\ \ n \geq 1$, and also $Q_n(x) = P_n^{(0,\beta)}(x),\ n \geq 0$, and $c = -1$. Through a suitable shift, we get **($S_{4,1}$)**. Indeed, from Lemma 3.1 and taking into account $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ (see [12]), the SMOP $\{\tilde{P}_n\}_{n\geq 0}$ where $\tilde{P}_n(x) = (-1)^n P_n(-x) = P_n^{(\beta-2,2)}(x)$, is T_1-classical.

($S_{4,3}$). $F(x) = \mu(x-1)(x+1)$, $\mu \neq 0$. According to Lemma 3.3(iii), we obtain

$$\mu A(x) = 2(x^2-1), \tag{38}$$

$$G(x) = 2\mu\big((\alpha+\beta)x + \beta - \alpha\big). \tag{39}$$

From (31) and (38), we obtain the following system

$$\begin{cases} (\alpha+\beta-1)(\alpha+\beta)\mu = 2, \\ (\alpha+\beta-1)(\alpha-\beta)\mu = 0, \\ \big((\alpha-\beta)^2 - (\alpha+\beta)\big)\mu = -2. \end{cases}$$

Then $\alpha = \beta = \mu = 1$. Hence, (38) and (39) becomes $F(x) = x^2 - 1$, $G(x) = 4x$ and $A(x) = 2(x^2-1)$. According to Lemma 3.3(ii), we get

$$(x^2-1)Q_n''(x) + 4xQ_n'(x) + 2Q_n(x) = \rho_n P_n(x),\ n \geq 0, \tag{40}$$

where $\rho_n = (n+1)(n+2),\ n \geq 0$.

For $n = 1$ in (40), we obtain $\beta_0 = 0$. Given $6 = \rho_1 = 6\lambda_1\gamma_1^{-1}$, we have $\gamma_1 = \lambda_1$. From Table **(C_4)** and by virtue of Lemma 3.2(i), we finally get $\beta_n = \xi_n = 0, n \geq 0$, and $\gamma_n = \lambda_n = \frac{n(n+2)}{(2n+1)(2n+3)},\ n \geq 1$. This implies that, $P_n(x) = Q_n(x) = P_n^{(1,1)}(x), n \geq 0$, and then $u_0 = v_0$. In addition, by (13) with $n = 1$, (14) and (15), we get $c = 0$. Using Lemma 3.3(i), we finally get $u_0 = 0$, which contradicts the quasi-definiteness of u_0. □

Remark 3.1 : By applying the standard derivative in (29) (resp. (36)) and using (2.22) and (2.23) of page 149 of Chihara's book [12], we get (38) and (46) of the reference [3], by the fact that $\mathscr{O}_{c;1,3,2} = DT_c$.

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REFERENCES

1. F. Abdelkarim and P. Maroni, The D_w-classical orthogonal polynomials, *Result. Math.*, **32** (1997), 1-28.
2. B. Aloui, F. Marcellán and R. Sfaxi, *Classical orthogonal polynomials with respect to a lowering operator generalizing the Laguerre operator*, Integral Transforms Spec. Funct., **24**(8) (2013), 636-648.
3. B. Aloui, Hahn's problem with respect to a third-order differential operator, *Transylv. J. Math. Mech.*, **6**(2) (2014), 85-100.
4. B. Aloui, Characterization of Laguerre polynomials as orthogonal polynomials connected by the Laguerre degree raising operator, *Ramanujan J.*, **45**(2) (2018), 475-481.
5. W. Al-Salam, *Characterization theorems for orthogonal polynomials*, In: Orthogonal Polynomials: Theory and Practice, P. Nevai Editor, *NATO ASI Series C*, Vol. 294. Kluwer Acad. Publ., Dordrecht, (1990), 1-24.
6. W. Al-Salam and T. S. Chihara, Another characterization of the classical orthogonal polynomials, *SIAM J. Math. Anal.*, **3** (1972), 65-70.
7. I. Area, A. Godoy, A. Ronveaux and A. Zarzo, Classical symmetric orthogonal polynomials of a discrete variable, *Integral Transforms Spec. Funct.*, **15** (2004), 1-12.
8. N. M. Atakishiyev, M. Rahman, and S. K. Suslov, On classical orthogonal polynomials, *Constructive Approx.*, **11** (1995), 181-223.
9. Y. B. Cheikh and M. Gaied, Characterization of the Dunkl-classical symmetric orthogonal polynomials, *Appl. Math. Comput.*, **187** (2007), 105-114.
10. S. Bochner, Über sturm-Liouvillesche polynomsysteme, *Math Z.*, **29** (1929), 730-736.
11. A. Branquinho, J. Petronilho, and F. Marcellán, Classical orthogonal polynomials, a functional approach, *Acta Appl. Math.*, **34** (1994), 283-303.
12. T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.
13. E. Godoy, I. Area, A. Ronveaux, and A. Zarzo, Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case, *J. Comput. Appl. Math.*, **84** (1997), 257-275.
14. E. Grosswald, *Bessel Polynomials*, (Lecture Notes in Mathematics 698) Berlin: Springer, 1978.
15. V. W. Hahn, Über die Jacobischen polynome und zwei verwandte polynomklassen, *Math. Z.*, **39** (1935), 634-638.
16. V. W. Hahn, Über Orthogonalpolynome, die q-Differenzengleichungen genügen, *Math. Nach.*, **2** (1949), 4-34.

17. L. Khériji and P. Maroni, The H_q-classical orthogonal polynomialsm *Acta. Appl. Math.*, **71** (2002), 49-115.

18. K. H. Kwon and G. J. Yoon, Generalized Hahn's theorem, *J. Comput. Appl. Math.*, **116** (2000), 243-262.

19. N. N. Lebedev, *Special functions and their applications*, (Revised English Edition, Translated and Edited by Richard Silverman 1965), Dover Publications New York, 1972.

20. A. F. Loureiro and P. Maroni, Quadratic decomposition of Appell sequences, *Expo. Math.*, **26** (2008), 177-186.

21. P. Maroni, *Une théorie algébrique des polynômes orthogonaux Applications aux polynômes orthogonaux semi-classiques*, In *Orthogonal Polynomials and their Applications*, C. Brezinski *et al.*, Editors, IMACS *Ann. Comput. Appl. Math.*, **9** (1991), 95-130.

22. P. Maroni, Variations around classical orthogonal polynomials. Connected problems, *J. Comput. Appl. Math.*, **48** (1993), 133-155.

23. P. Maroni, *Fonctions Eulériennes, Polynômes Orthogonaux Classiques*. Techniques de l'Ingénieur, Traité Généralités (Sciences Fondamentales) **A 154** Paris. (1994), 1-30.

24. A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical orthogonal polynomials of a discrete variable*, Springer Series in Computational Physics, Springer, Berlin, 1991.

25. A. Zarzo, I. Area, E. Godoy, and A. Ronveaux, Inversion problems for classical continuous and discrete orthogonal polynomials, *J. Phys. A: Math. Gen.*, **30** (1997), L35-L40.