Indian J. Pure Appl. Math., **51**(2): 749-760, June 2020

DOI: 10.1007/s13226-020-0428-2

CERTAIN ESTIMATES OF THE DERIVATIVE OF A MEROMORPHIC FUNCTION ON BOUNDARY OF THE UNIT DISK

Abdullah Mir

Department of Mathematics, University of Kashmir,
Srinagar 190 006, *India*
e-mail: drabmir@yahoo.com

(*Received* 12 *February* 2019; *accepted* 12 *April* 2019)

In this paper we establish a family of comparison inequalities between rational functions with prescribed poles for the sup-norms on the unit circle in the complex plane. Certain estimates for the modulus of the derivative of rational functions as well as some inequalities of Turán's type are also obtained. The obtained results produce many inequalities for polynomials and polar derivatives as special cases.

Key words : Rational functions; Bernstein inequality; poles; zeros.

2010 Mathematics Subject Classification : 30A10, 30C10, 26D10.

1. Introduction

Let $\mathbb{P}_n$ denote the class of all complex polynomials of degree n. A classical majorization result due to Bernstein is that, for two polynomials P and Q with $Q \in \mathbb{P}_n$, deg $P \leq$ deg Q and $Q(z) \neq 0$ for $|z| > 1$, the majorization $|P(z)| \leq |Q(z)|$ on the unit circle $|z| = 1$ implies the majorization of their derivatives $|P'(z)| \leq |Q'(z)|$. In particular, this majorization result allows to establish famous Bernstein inequality [3] for the sup-norms on the unit circle: for $P \in \mathbb{P}_n$, it is true that

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The above inequality (1.1) was proved by Bernstein in 1912. Later in 1985, Malik and Vong [7] used a parameter β and proved the following Bernstein-type inequality from which (1.1) can also be deduced.

***Theorem* A** — *Let P and Q be two polynomials with $Q \in \mathbb{P}_n$ and deg $P \leq$ deg Q. If $Q(z)$ has all its zeros in $|z| \leq 1$ and $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for every $|\beta| \leq 1$,*

$$\left|\frac{zP'(z)}{n} + \frac{\beta P(z)}{2}\right| \leq \left|\frac{zQ'(z)}{n} + \frac{\beta Q(z)}{2}\right| \text{ for } |z| = 1. \tag{1.2}$$

If we take $Q(z) = z^n \max_{|z|=1} |P(z)|$ in Theorem A, we get a generalization of (1.1). That is, if $P(z)$ is a polynomial of degree $\leq n$, then for every $|\beta| \leq 1$,

$$\left|\frac{zP'(z)}{n} + \frac{\beta P(z)}{2}\right| \leq \left|1 + \frac{\beta}{2}\right| \max_{|z|=1} |P(z)| \text{ for } |z| = 1. \tag{1.3}$$

It is easy to see that equality is attained in (1.1) for the polynomial $P(z) = z^n$ and this polynomial has all its zeros at the origin, by restricting the zeros of polynomials, the bound can be sharpened.

In fact, for $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, inequality (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.4}$$

The above inequality was conjectured by Erdös and later proved by Lax [6].

Turán [11] obtained a lower bound for the maximum of $|P'(z)|$ on $|z| = 1$, by proving that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.5}$$

In 1997, Jain [5] had used a parameter β and proved an interesting generalization of (1.5). More precisely, Jain proved that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every β with $|\beta| \leq 1$, we have

$$\max_{|z|=1} \left|zP'(z) + \frac{n\beta}{2} P(z)\right| \geq \frac{n}{2} \left\{1 + Re(\beta)\right\} \max_{|z|=1} |P(z)|. \tag{1.6}$$

For $P \in \mathbb{P}_n$, the polar derivative $D_\alpha P(z)$ of $P(z)$ with respect to the point α is defined as

$$\begin{aligned} D_\alpha P(z) &:= -\left[\frac{P(z)}{(z-\alpha)^n}\right]'(z-\alpha)^{n+1} \\ &= nP(z) + (\alpha - z)P'(z). \end{aligned}$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$. This is the so-called polar derivative of $P(z)$ with respect to α (see [8]). It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} := P'(z), \text{uniformly with respect to } z \text{ for } |z| \leq R, R > 0.$$

Over last 40 years many different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on the multiplicity of zero at $z = 0$, the modulus of largest root of $P(z)$, restrictions on coefficients etc. Many of these generalizations involve the comparison of polar derivative $D_\alpha P(z)$ with various choices of $P(z)$, α and other parameters.

Li, Mohapatra and Rodriguez [13] gave a new perspective to the above inequalities (1.1), (1.4), (1.5) and extended them to rational functions with fixed poles. Essentially, in the inequalities referred to, they replaced the polynomial $P(z)$ by a rational function $r(z)$ with poles $a_1, a_2, ..., a_n$ all lying in $|z| > 1$ and z^n by a Blaschke product $B(z)$. Before proceeding towards their results, let us introduce the set of rational functions involved.

For $a_j \in \mathbb{C}$ with $j = 1, 2, ..., n$, let

$$W(z) := \prod_{j=1}^{n}(z - a_j);\ B(z) := \prod_{j=1}^{n}\left(\frac{1 - \overline{a}_j z}{z - a_j}\right) = \frac{W^*(z)}{W(z)},$$

where

$$W^*(z) = z^n \overline{W\left(\frac{1}{\overline{z}}\right)}$$

and

$$\mathbb{R}_n := \mathbb{R}_n(a_1, a_2, ..., a_n) = \left\{\frac{P(z)}{W(z)} :\ P \in \mathbb{P}_n\right\}.$$

Then $\mathbb{R}_n$ is the set of rational functions with poles $a_1, a_2, ..., a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. For $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$, the conjugate transpose r^* of r is defined by $r^*(z) = B(z)\overline{r(\frac{1}{\overline{z}})}$.

For $r \in \mathbb{R}_n$, Li, Mohapatra and Rodriguez [13] proved the following inequality for rational functions similar to (1.1):

$$|r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)| \text{ for } |z| = 1. \tag{1.7}$$

They improved (1.7) by showing that if $r \in \mathbb{R}_n$ and $|z| = 1$, then

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)|. \tag{1.8}$$

where $r^*(z) = B(z)\overline{r(\frac{1}{\overline{z}})}$.

As an extension of (1.4) and (1.5) to rational functions, Li, Mohapatra and Rodriguez also showed that if $r \in \mathbb{R}_n$ and $r(z) \neq 0$ in $|z| < 1$, then for $|z| = 1$,

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|, \tag{1.9}$$

where as, if $r \in \mathbb{R}_n$ has exactly n zeros in $|z| \leq 1$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|. \tag{1.10}$$

The main aim of this paper is to establish a family of comparison inequalities between rational functions similar to (1.2) and (1.6), essentially by replacing $P(z)$ with a rational function $r(z)$ which has prescribed poles in the region $|z| > 1$ and z^n by a Blaschke product $B(z)$.

2. Main Results

From now on, we shall always assume that all the poles $a_1, a_2, ..., a_n$ of $r(z)$ lie in $|z| > 1$. For the case when all poles are in $|z| < 1$, we can obtain analogous results with suitable modifications. Here, we first prove the following result which in particular provides an extension of inequality (1.6) to rational functions.

***Theorem* 1** — *If $r \in \mathbb{R}_n$ and all the n zeros of r lie in $|z| \leq k$, $k \leq 1$. Then for every β with $|\beta| \leq 1$ and $|z| = 1$,*

$$\left| zr'(z) + \frac{n\beta}{1+k} r(z) \right| \geq \frac{1}{2} \left\{ |B'(z)| + \frac{n}{1+k} \big(1 - k + 2Re(\beta)\big) \right\} |r(z)|. \tag{2.1}$$

Equality in (2.1) holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$ with $B(z) = \left(\frac{1-az}{z-a}\right)^n$ at $z = 1$, $a > 1$ and $\beta = 0$.

Taking $k = 1$ in Theorem 1, we get the following result that should be compared with the polynomial inequality (1.6).

Corollary 1 — If $r \in \mathbb{R}_n$ and all the n zeros of r lie in $|z| \leq 1$. Then for every β with $|\beta| \leq 1$ and $|z| = 1$,

$$\left| zr'(z) + \frac{n\beta}{2} r(z) \right| \geq \frac{1}{2} \Big(|B'(z)| + nRe(\beta) \Big) |r(z)|. \tag{2.2}$$

If we take $\alpha_j = \alpha,\ |\alpha| \geq 1$, for $j = 1, 2, ..., n$, then $W(z) = (z - \alpha)^n$ and

$r(z) = \frac{P(z)}{(z-\alpha)^n}$, and hence

$$\begin{aligned} r'(z) &= \frac{(z-\alpha)^n P'(z) - n(z-\alpha)^{n-1}P(z)}{(z-\alpha)^{2n}} \\ &= -\left\{\frac{nP(z) - (z-\alpha)P'(z)}{(z-\alpha)^{n+1}}\right\} \\ &= \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}. \end{aligned}$$

Also $W^*(z) = (1-\overline{\alpha}z)^n$, which gives $B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n$.

This implies

$$B^{'}(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

Using these observations in (2.2) and assuming $Re(\beta) \geq 0$, we get for $|z| = 1$, that

$$\begin{aligned} \left|zD_\alpha P(z) + \frac{n\beta}{2}(\alpha - z)P(z)\right| &\geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|z-\alpha|} + nRe(\beta)|z-\alpha|\right\}|P(z)| \\ &\geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|\alpha|+1} + nRe(\beta)(|\alpha|-1)\right\}|P(z)| \\ &= \frac{n}{2}(|\alpha|-1)(1+Re(\beta))|P(z)|. \end{aligned}$$

Thus from Corollary 1, we immediately get the following polar derivative analogue of (1.6).

Corollary 2 — If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$. Then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1, |\beta| \leq 1$ $(Re(\beta) \geq 0)$, we have

$$\max_{|z|=1}\left|zD_\alpha P(z) + \frac{n\beta}{2}(\alpha - z)P(z)\right| \geq \frac{n}{2}(|\alpha|-1)\big(1+Re(\beta)\big)\max_{|z|=1}|P(z)|. \tag{2.3}$$

Remark 1 : For $\beta = 0$, the above corollary reduces to a result of Shah [10]. Clearly, Corollary 2 generalizes (1.6) and to obtain (1.6) from above corollary, simply divide both sides of (2.3) by $|\alpha|$ and let $|\alpha| \to \infty$.

Next, we prove the following comparison result that should be compared with the polynomial inequality (1.2) of Malik and Vong [7], which follows as a straightforward consequence of a result due to Li [12].

***Theorem* 2** — *Let $r, s \in \mathbb{R}_n$ and all the zeros of s lie in $|z| \leq 1$ and for $|z| = 1$,*

$$|r(z)| \leq |s(z)|.$$

Then for every β *with* $|\beta| \leq 1$,

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \leq \left|\frac{s'(z)}{B'(z)} + \frac{\beta}{2}\frac{s(z)}{B(z)}\right| \text{ for } |z| = 1. \tag{2.4}$$

Remark 2 : It is pertinent to mention that inequality (2.4) looks to be identical to inequality (3.1) of Lemma 1 (stated in the next section), but we shall see that its importance can not be ignored. Here, we mention some of its consequences. If we take $s(z) = B(z)\max_{|z|=1}|r(z)|$ in Theorem 2, we get a result similar to (1.4), essentially by replacing $P(z)$ with a rational function $r(z)$ having poles in the region $|z| > 1$ and z^n by a Blaschke product $B(z)$.

Corollary 3 — If $r \in \mathbb{R}_n$ and $|z| = 1$, then for every $|\beta| \leq 1$,

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \leq \left|1 + \frac{\beta}{2}\right| \max_{|z|=1}|r(z)|. \tag{2.5}$$

Remark 3 : For $\beta = 0$, inequality (2.5) reduces to (1.7). By applying Theorem 2 to the rational functions $r(z)$ and $B(z)\min_{|z|=1}|r(z)|$, we get the following result.

Corollary 4 — If $r \in \mathbb{R}_n$ has all its zeros in $|z| \leq 1$. Then for every $|\beta| \leq 1$ and $|z| = 1$, we have

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \geq \left|1 + \frac{\beta}{2}\right| \min_{|z|=1}|r(z)|. \tag{2.6}$$

For $\beta = 0$, the above corollary should be compared with the polynomial inequality due to Aziz and Dawood [1].

Remark 4 : Suppose $r \in \mathbb{R}_n$ has all its zeros in $|z| \geq 1$ and $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$, then all the zeros of $r^*(z)$ lie in $|z| \leq 1$. Also $|r(z)| = |r^*(z)|$ for $|z| = 1$. Therefore, by Theorem 2, we have for every $|\beta| \leq 1$ and $|z| = 1$,

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \leq \left|\frac{(r^*(z))'}{B'(z)} + \frac{\beta}{2}\frac{r^*(z)}{B(z)}\right|. \tag{2.7}$$

Taking $\beta = 0$ in (2.7) and scaling both sides by $|B^{'}(z)|$, we get the following inequality.

$$|r'(z)| \leq |(r^*(z))'|. \tag{2.8}$$

By combining (2.8) and (1.8), Li, Mohaparta and Rodriguez [13] were able to prove (1.9).

Finally, we prove the following result which as a special case extends a polynomial inequality of Mohapatra, O'Hara and Rodriguez [9] to rational functions.

***Theorem* 3** — *If $r \in \mathbb{R}_n$ then for every $|\beta| \leq 1$,*

$$(1-|\beta|)\max_{|z|=1}|r(z)| \leq \max_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)}+\frac{\beta}{2}\frac{r(z)}{B(z)}\right|+\left|\frac{(r^*(z))'}{B'(z)}+\frac{\beta}{2}\frac{r^*(z)}{B(z)}\right|\right\} \leq \left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\max_{|z|=1}|r(z)|, \tag{2.9}$$

where $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$.

For $\beta = 0$, the above theorem reduces to the following result (see also [2]), which extends a result of Mohapatra, O'Hara and Rodriguez [9] for polynomials.

Corollary 5 — If $r \in \mathbb{R}_n$, then

$$\max_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)}\right|+\left|\frac{(r^*(z))'}{B'(z)}\right|\right\} = \max_{|z|=1}|r(z)|,$$

where $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$.

A rational function $r \in \mathbb{R}_n$ is said to be self-inversive if $r^*(z) = \zeta r(z)$ for some $|\zeta| = 1$. From Theorem 3, we immediately get the following result for self-inversive rational functions.

Corollary 6 — If $r \in \mathbb{R}_n$ is self-inversive, then for every $|\beta| \leq 1$,

$$\frac{1}{2}(1-|\beta|)\max_{|z|=1}|r(z)| \leq \max_{|z|=1}\left|\frac{r'(z)}{B'(z)}+\frac{\beta}{2}\frac{r(z)}{B(z)}\right| \leq \frac{1}{2}\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\max_{|z|=1}|r(z)|.$$

For $\beta = 0$, the above corollary reduces to the following result, which extends a result of O'Hara and Rodriguez [4] for self-inversive polynomials.

Corollary 7 — If $r \in \mathbb{R}_n$ is self-inversive, then

$$\max_{|z|=1}\left|\frac{r'(z)}{B'(z)}\right| = \frac{1}{2}\max_{|z|=1}|r(z)|.$$

3. LEMMAS

For the proofs of these theorems, we need the following lemmas. The following lemma is due to Li [12].

Lemma 1 — Let $r, s \in \mathbb{R}_n$ and all the zeros of s lie in $|z| \leq 1$ and for $|z| = 1$,

$$|r(z)| \leq |s(z)|.$$

Then for every ρ with $|\rho| \leq \frac{1}{2}$,

$$\left|r'(z) + \rho B'(z) r(z)\right| \leq \left|s'(z) + \rho B'(z) s(z)\right| \text{ for } |z| = 1. \tag{3.1}$$

Lemma 2 — If $r \in \mathbb{R}_n$ and $|z| = 1$, then

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| + \left|\frac{(r^*(z))'}{B'(z)} + \frac{\beta}{2}\frac{r^*(z)}{B(z)}\right| \leq \left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \max_{|z|=1} |r(z)|, \tag{3.2}$$

for every $|\beta| \leq 1$ and $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$.

PROOF OF LEMMA 2 : Let $M := \max_{|z|=1} |r(z)|$. Therefore, for any λ with $|\lambda| > 1$,

$$|r(z)| < |\lambda M B(z)| \text{ for } |z| = 1.$$

By Rouche's theorem, all the zeros of $G(z) = r(z) + \lambda M B(z)$ lie in $|z| < 1$.

If $H(z) = B(z)\overline{G(\frac{1}{\bar{z}})}$, then $|G(z)| = |H(z)|$ for $|z| = 1$. Using (2.4), we have for $|z| = 1$ and $|\beta| \leq 1$,

$$\left|\frac{H'(z)}{B'(z)} + \frac{\beta}{2}\frac{H(z)}{B(z)}\right| \leq \left|\frac{G'(z)}{B'(z)} + \frac{\beta}{2}\frac{G(z)}{B(z)}\right|. \tag{3.3}$$

Now by putting $G(z) = r(z) + \lambda M B(z)$ and $H(z) = r^*(z) + \overline{\lambda} M$ in (3.3), we get for $|z| = 1$ and $|\beta| \leq 1$,

$$\left|\frac{(r^*(z))'}{B'(z)} + \frac{\beta}{2}\frac{r^*(z)}{B(z)} + \frac{\overline{\lambda}\beta M}{2B(z)}\right| \leq \left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)} + \lambda\left(1 + \frac{\beta}{2}\right)M\right|. \tag{3.4}$$

Choosing a suitable argument of λ on the right hand side of (3.4) and using Corollary 3, we get for $|z| = 1$ and $|\beta| \leq 1$,

$$\left|\frac{(r^*(z))'}{B'(z)} + \frac{\beta}{2}\frac{r^*(z)}{B(z)}\right| - |\lambda|\left|\frac{\beta}{2}\right|M \leq |\lambda|\left|1 + \frac{\beta}{2}\right|M - \left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right|. \tag{3.5}$$

Making $|\lambda| \to 1$ in (3.5), we get (3.2) and this completes the proof of the Lemma 2.

4. PROOFS OF THEOREMS

PROOF OF THEOREM 1 : By hypothesis all the n zeros of $r \in \mathbb{R}_n$ lie in $|z| \leq k$, where $k \leq 1$. Let $z_1, z_2, ..., z_n$ be the zeros of $r(z) = \frac{P(z)}{W(z)}$, where $P(z) = c_n \prod_{j=1}^{n}(z - z_j)$; $|z_j| \leq k \leq 1$, $j = 1, 2, ..., n$.

By a direct calculation, we obtain for every β with $|\beta| \leq 1$,

$$\frac{zr'(z)}{r(z)} + \frac{n\beta}{1+k} = \frac{zP'(z)}{P(z)} - \frac{zW'(z)}{W(z)} + \frac{n\beta}{1+k},$$

which gives for $0 \leq \theta < 2\pi$,

$$\begin{aligned} Re\bigg(\frac{zr'(z)}{r(z)}+\frac{n\beta}{1+k}\bigg)\bigg|_{z=e^{i\theta}} &= Re\bigg(\frac{zP'(z)}{P(z)}\bigg)\bigg|_{z=e^{i\theta}} - Re\bigg(\frac{zW'(z)}{W(z)}\bigg)\bigg|_{z=e^{i\theta}} + \frac{n}{1+k}Re(\beta) \\ &= \sum_{j=1}^{n} Re\bigg(\frac{e^{i\theta}}{e^{i\theta}-z_j}\bigg) - Re\bigg(\frac{zW'(z)}{W(z)}\bigg)\bigg|_{z=e^{i\theta}} + \frac{n}{1+k}Re(\beta). \end{aligned} \tag{4.1}$$

It can be easily verified that for $e^{i\theta}$, $0 \leq \theta < 2\pi$ and $|z_j| \leq k \leq 1$,

$$Re\bigg(\frac{e^{i\theta}}{e^{i\theta}-z_j}\bigg) \geq \frac{1}{1+k}. \tag{4.2}$$

Also,

$$B(z) = \prod_{j=1}^{n}\bigg(\frac{1-\overline{a_j}z}{z-a_j}\bigg)$$

$$= \frac{W^*(z)}{W(z)},$$

where $W^*(z) = z^n\overline{W(\frac{1}{\bar{z}})}$.

This gives

$$\frac{zB'(z)}{B(z)} = \frac{z\big(W^*(z)\big)'}{W^*(z)} - \frac{zW'(z)}{W(z)}.$$

Using the fact (e.g, see formula (15) in [13]) that

$$\frac{zB'(z)}{B(z)} = |B'(z)| \text{ for } |z| = 1, \tag{4.3}$$

we get

$$Re\bigg(\frac{z\big(W^*(z)\big)'}{W^*(z)}\bigg) - Re\bigg(\frac{zW'(z)}{W(z)}\bigg) = |B'(z)| \text{ for } |z| = 1. \tag{4.4}$$

Further, we have

$$z\big(W^*(z)\big)' = nz^n\overline{W\left(\frac{1}{\overline{z}}\right)} - z^{n-1}\overline{W'\left(\frac{1}{\overline{z}}\right)},$$

it easily follows for $|z| = 1$, $\big($so that $z\overline{z} = 1\big)$ that

$$\frac{z\big(W^*(z)\big)'}{W^*(z)} = n - \left(\overline{\frac{zW'(z)}{W(z)}}\right),$$

which implies for $|z| = 1$,

$$Re\left(\frac{z\big(W^*(z)\big)'}{W^*(z)}\right) + Re\left(\frac{zW'(z)}{W(z)}\right) = n. \tag{4.5}$$

From (4.4) and (4.5), it follows that

$$Re\left(\frac{zW'(z)}{W(z)}\right)\bigg|_{z=e^{i\theta}} = \frac{n - |B'(e^{i\theta})|}{2}. \tag{4.6}$$

Using (4.2) and (4.6) in (4.1), we get for $0 \leq \theta < 2\pi$ and $|\beta| \leq 1$,

$$\begin{aligned} Re\left(\frac{zr'(z)}{r(z)} + \frac{n\beta}{1+k}\right)\bigg|_{z=e^{i\theta}} &\geq \frac{n}{1+k} - \left(\frac{n - |B'(e^{i\theta})|}{2}\right) + \frac{n}{1+k}Re(\beta) \\ &= \frac{1}{2}\left\{\big|B'(e^{i\theta})\big| + \frac{n}{1+k}\big(1 - k + 2Re(\beta)\big)\right\}, \end{aligned}$$

for points $e^{i\theta},\ 0 \leq \theta < 2\pi$, other than the zeros of $r(z)$. Hence we have

$$\left|\frac{e^{i\theta}r'(e^{i\theta})}{r(e^{i\theta})} + \frac{n\beta}{1+k}\right| \geq \frac{1}{2}\left\{\big|B'(e^{i\theta})\big| + \frac{n}{1+k}\big(1 - k + 2Re(\beta)\big)\right\},$$

which implies

$$\left|e^{i\theta}r'(e^{i\theta}) + \frac{n\beta}{1+k}r(e^{i\theta})\right| \geq \frac{1}{2}\left\{\big|B'(e^{i\theta})\big| + \frac{n}{1+k}\big(1 - k + 2Re(\beta)\big)\right\}|r(e^{i\theta})|, \tag{4.7}$$

for points $e^{i\theta},\ 0 \leq \theta < 2\pi$, other than the zeros of $r(z)$. Since (4.7) is true for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are the zeros of $r(z)$ also, it follows that

$$\left|zr'(z) + \frac{n\beta}{1+k}r(z)\right| \geq \frac{1}{2}\left\{\big|B'(z)\big| + \frac{n}{1+k}\big(1 - k + 2Re(\beta)\big)\right\}|r(z)|,$$

for $|z| = 1$ and for every β with $|\beta| \leq 1$. This completes the proof of Theorem 1.

PROOF OF THEOREM 2 : For arbitrary fixed z_0 with $|z_0| = 1$, the Blaschke product is of the absolute value $|B(z_0)| = 1$. Therefore, the complex number $\rho = \beta/(2B(z_0))$, $|\beta| \leq 1$ is in the allowed range $|\rho| \leq 1/2$. Substituting this value into inequality (3.1), we get

$$\left|r'(z_0) + \frac{\beta r(z_0)B'(z_0)}{2B(z_0)}\right| \leq \left|s'(z_0) + \frac{\beta s(z_0)B'(z_0)}{2B(z_0)}\right|. \tag{4.8}$$

Since $B'(z_0) \neq 0$ (e.g, see formula (14) in [13]), therefore, scaling both sides of (4.8) by $|B'(z_0)|$ and then setting $z = z_0$ yields the desired inequality (2.4). This completes the proof of Theorem 2.

PROOF OF THEOREM 3 : In view of Lemma 2, we have to prove the left hand inequality of (2.8) to prove the theorem. We have $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$. Therefore

$$z(r^*(z))' = zB'(z)\overline{r\Big(\frac{1}{\bar{z}}\Big)} - \frac{B(z)}{z}\overline{r'\Big(\frac{1}{\bar{z}}\Big)},$$

and hence for $|z| = 1$ (so that $z = \frac{1}{\bar{z}}$), we get

$$\frac{z(r^*(z))'}{r^*(z)} = \frac{zB'(z)}{B(z)} - \left(\overline{\frac{zr'(z)}{r(z)}}\right).$$

This implies by using (4.3) for $|z| = 1$, that

$$\frac{z(r^*(z))'}{r^*(z)} = |B'(z)| - \left(\overline{\frac{zr'(z)}{r(z)}}\right),$$

which gives for $|z| = 1$,

$$\begin{aligned}\left|\frac{z(r^*(z))'}{r^*(z)}\right| + \left|\frac{zr'(z)}{r(z)}\right| &\geq \left|\frac{z(r^*(z))'}{r^*(z)} + \frac{zr'(z)}{r(z)}\right| \\ &= |B'(z)|. \end{aligned} \tag{4.9}$$

Using $|r(z)| = |r^*(z)|$ for $|z| = 1$ in (4.9), we get

$$\left|\frac{(r^*(z))'}{B'(z)}\right| + \left|\frac{r'(z)}{B'(z)}\right| \geq |r(z)|. \tag{4.10}$$

Since $|B(z)| = 1$ for $|z| = 1$, we have by using (4.10) and for every $|\beta| \leq 1$,

$$\begin{aligned}&\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| + \left|\frac{(r^*(z))'}{B'(z)} + \frac{\beta}{2}\frac{r^*(z)}{B(z)}\right| \\ &\geq \left|\frac{r'(z)}{B'(z)}\right| + \left|\frac{(r^*(z))'}{B'(z)}\right| - \left|\frac{\beta}{2}\right|\left|\frac{r(z)}{B(z)}\right| - \left|\frac{\beta}{2}\right|\left|\frac{r^*(z)}{B(z)}\right| \\ &\geq |r(z)| - |\beta||r(z)| \\ &= (1 - |\beta|)|r(z)| \text{ for } |z| = 1,\end{aligned}$$

which gives

$$\max_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)}+\frac{\beta}{2}\frac{r(z)}{B(z)}\right|+\left|\frac{(r^*(z))'}{B'(z)}+\frac{\beta}{2}\frac{r^(z)}{B(z)}\right|\right\}\geq(1-|\beta|)\max_{|z|=1}|r(z)|. \tag{4.11}$$

The above inequality (4.11) when combined with Lemma 2 proves Theorem 3 completely.

REFERENCES

1. A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, *J. Approx. Theory*, **54** (1988), 306-313.
2. A. Aziz and B. A. Zargar, Some properties of rational functions with prescribed poles, *Canad. Math. Bull*., **42** (1999), 417-426.
3. S. Bernstein, Sur l'ordre de la meilleure approximation des functions continues par des polynomes de degré donné, *Mem. Acad. R. Belg*., **4** (1912), 1-103.
4. P. J. O'Hara and R. S. Rodriguez, Some properties of self-inversive polynomials, *Proc Amer. Math Soc*., **44** (1974), 331-335.
5. V. K. Jain, Generalization of certain well known inequalities for polynomials, *Glas. Mate*., **32** (1997), 45-51.
6. P. D. Lax, Proof of a conjecture of P. Erdös on the derivative of a polynomial, *Bull. Amer. Math. Soc*., **50** (1944), 509-513.
7. M. A. Malik and M. C. Vong, Inequalities concerning the derivative of polynomials, *Rendiconti Del Circolo Matematico Di Palermo Series II*, **34** (1985), 422-426.
8. M. Marden, Geometry of polynomials, 2nd ed., *Math. surveys*, No. 3, *Amer. Math. Soc*., Providence, RI, (1966).
9. R. N. Mohapatra, P. J. O'Hara, and R. S. Rodriguez, Simple proofs of Bernstein-type inequalities, *Proc. Amer. Math. Soc*., **102** (1988), 629-632.
10. W. M. Shah, A generalization of a theorem of Paul Turán, *J. Ramanujan Math. Soc*., **1** (1996), 67-72.
11. P. Turán, Über die Ableitung von Polynomen, *Compos. Math*., **7** (1939), 89-95.
12. Xin Li, A comparison inequality for rational functions, *Proc. Amer. Math. Soc*., **139** (2011), 1659-1665.
13. Xin Li, R. N. Mohapatra, and R. S. Rodriguez, Bernstein-type inequalities for rational functions with prescribed poles, *J. London Math. Soc*., **51** (1995), 523-531.