Indian J. Pure Appl. Math., **51**(2): 777-788, June 2020

DOI: 10.1007/s13226-020-0430-8

MAXIMIZATION AND MINIMIZATION PROBLEMS RELATED TO AN EQUATION WITH THE *p*-LAPLACIAN

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(*Received* 22 *April* 2018; *after final revision* 31 *March* 2019; *accepted* 18 *April* 2019)

In this paper, we investigate two optimization problems related to a quasilinear elliptic equation with p-Laplacian, logistic-type growth rate function such that the admissible set is a class of rearrangements of a fixed function. Under some suitable assumptions, we prove existence and representation of the maximizers and existence, uniqueness and representation of the minimizer. Also, when the domain of the equation is a ball, we show that the maximizer is unique and symmetric.

Key words : Existence; maximization; minimization; rearrangement; symmetric; uniqueness.

2010 Mathematics Subject Classification : 35J20, 49J20.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ and $1 < p < N$. We consider the following weighted eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ_p denotes the p-Laplacian operator, defined by $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, $m \in L^r(\Omega)$, with $r > \frac{N}{p}$ and $m \geq 0$ on Ω. Suppose that λ_1 be the first eigenvalue of (1). We now consider the following equation involving the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda m(x)u^{p-1} - u^{\alpha-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $m \in L^r(\Omega)$, $r > \frac{N}{p}$, $0 < t_0 \leq m(x)$ in Ω, where t_0 is a positive constant, and $1 < p < \alpha$. Let $\lambda > \lambda_1$ be a real number and $r > \frac{N(\alpha-1)}{p(\alpha-p)}$. Drabek and Hernandez in [14] have proved that the problem (2) has exactly one positive solution u such that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Let m_0 be a given function in $L^\infty(\Omega)$ such that $L \leq m_0 \leq H$ in Ω, where L and H are positive constants, and $\mathcal{R} := \mathcal{R}(m_0)$ be a rearrangement class of m_0. For each $m \in \mathcal{R}$ we denote the first eigenvalue of the problem (1) by $\lambda_1(m)$. Cuccu, Emamizadeh and Porru in [7] have proved that there exists $\hat{m}$ in $\overline{\mathcal{R}}$, the weak closure of $\mathcal{R}$, such that $\lambda_1(\hat{m}) = \sup_{m\in\mathcal{R}} \lambda_1(m)$.

Let $\lambda > \lambda_1(\hat{m})$ be a fixed real number. For each $m \in \overline{\mathcal{R}}$ we denote the unique solution of (2) by u_m. The problems (1) and (2) are interesting in combustion, mathematical biology and chemical reactions. Our interests in this paper are in the maximization and minimization of the quantity $\int_\Omega u_m^\alpha \mathrm{d}x$, as m varies in $\overline{\mathcal{R}}$. Under some suitable assumptions, we prove existence and representation of the maximizers and existence, uniqueness and representation of the minimizer. Also, when the domain of the equation is a ball, we show that the maximizer is unique and symmetric. In the our recent paper, [3], we investigated similar results for the p-Laplacian equation on a multiply connected domain. See the next section for precise definition of rearrangement of functions. Rearrangement optimization problems have been investigated in recent years by many authors, see [7-13, 16-18, 20].

2. Preliminaries

In this section we collect some well known results.

Let E be a (Lebesgue) measurable set in $\mathbb{R}^N$. Real measurable functions f and g on E are *rearrangement* of each other whenever

$$\mathcal{L}_N(\{x \in E : f(x) \geq \gamma\}) = \mathcal{L}_N(\{x \in E : g(x) \geq \gamma\}), \ \ \forall \gamma \in \mathbb{R},$$

where $\mathcal{L}_N$ denotes the N-dimensional Lebesgue measure. It is well known that if $f \in L^s(E)$, $1 \leq s \leq \infty$, and g be a rearrangement of f, then $g \in L^s(E)$ and in fact $\|f\|_s = \|g\|_s$, where $\|.\|_s$ denotes the standard norm on $L^s(E)$. We denote the rearrangement class of f by $\mathcal{R}(f)$ which comprises all functions which are rearrangements of f. The readers can see [4-6] for more results about rearrangements of functions.

We now collect some useful lemmas.

Lemma 2.1 — [5]. Let $s > 1$, $\frac{1}{s} + \frac{1}{s'} = 1$ and $f_0 \in L^s(E)$. Then

(i) $\overline{\mathcal{R}(f_0)}$, the weak closure of $\mathcal{R}(f_0)$ in $L^s(E)$, is compact with respect to $L^{s'}$-topology, weak topology, on $L^s(E)$.

(ii) $\overline{\mathcal{R}(f_0)}$ is convex.

Lemma 2.2 — Let $g : E \to \mathbb{R}$ and $w : E \to \mathbb{R}$ be two measurable functions. If every level set of w has measure zero then there exists an increasing function ξ such that $\xi(w) \in \mathcal{R}(g)$. Furthermore, there exists a decreasing function η such that $\eta(w) \in \mathcal{R}(g)$.

PROOF : The first assertion follows from Lemma 2.9 of [5]. The second assertion follows by applying the first one to $-w$.

Lemma 2.3 — Let $s > 1$, $\frac{1}{s} + \frac{1}{s'} = 1$, $f_0 \in L^s(E)$ and $g \in L^{s'}(E)$.

(i) If there is an increasing function ξ such that $\xi(g) \in \mathcal{R}(f_0)$, then

$$\int_E fg\mathrm{d}x \le \int_E \xi(g)g\mathrm{d}x, \ \ \forall f \in \overline{\mathcal{R}(f_0)},$$

and $\xi(g)$ is the unique maximizer relative to $\overline{\mathcal{R}(f_0)}$.

(ii) If there is a decreasing function η such that $\eta(g) \in \mathcal{R}(f_0)$, then

$$\int_E fg\mathrm{d}x \ge \int_E \eta(g)g\mathrm{d}x, \ \ \forall f \in \overline{\mathcal{R}(f_0)},$$

and $\eta(g)$ is the unique minimizer relative to $\overline{\mathcal{R}(f_0)}$.

PROOF : The first assertion follows from Lemma 2.4 of [5]. The second assertion can be proved by putting $-g$ in place of g and $\xi(t) = \eta(-t)$. □

Lemma 2.4 — [4]. Let $1 \le s \le \infty$ and s' be the conjugate exponent of s. Let $g \in L^s(E)$ and $\Psi : L^s(E) \to \mathbb{R}$ be convex.

(i) Suppose that Ψ is sequentially continuous in the $L^{s'}$-topology on $L^s(E)$. Then Ψ attains a maximum value relative to $\mathcal{R}(g)$.

(ii) Suppose that Ψ is strictly convex, that g^* is a maximizer for Ψ relative to $\mathcal{R}(g)$ and that w is a member of sub-gradient of Ψ at g^*. Then $g^* = \xi(w)$ almost everywhere in E for some increasing function ξ.

3. MAXIMIZATION AND MINIMIZATION

Let $p < \alpha < \frac{Np}{N-p}$ and $\lambda > \lambda_1(\hat{m})$. It is well known that, if $m \in \overline{\mathcal{R}}$ then u_m is the (unique positive) solution of the boundary value problem (2) whenever

$$\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla v\mathrm{d}x = \lambda \int_\Omega m(x)u_m^{p-1}v\mathrm{d}x - \int_\Omega u_m^{\alpha-1}v\mathrm{d}x,$$

for all $v \in W_0^{1,p}(\Omega)$. By setting $v = u_m$ in above equation, we infer that

$$\int_\Omega |\nabla u_m|^p \mathrm{d}x = \lambda \int_\Omega m(x) u_m^p \mathrm{d}x - \int_\Omega u_m^\alpha \mathrm{d}x. \tag{3}$$

We now define the functional $\Psi : \overline{\mathcal{R}} \times W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\Psi(m,v) := \frac{\lambda}{p} \int_\Omega m(x)|v|^p \mathrm{d}x - \frac{1}{\alpha} \int_\Omega |v|^\alpha \mathrm{d}x - \frac{1}{p} \int_\Omega |\nabla v|^p \mathrm{d}x.$$

Thus

$$\sup_{v \in W_0^{1,p}(\Omega)} \Psi(m,v) = \Psi(m,u_m) = \left(\frac{1}{p} - \frac{1}{\alpha}\right) \int_\Omega u_m^\alpha \mathrm{d}x, \tag{4}$$

where the second equality follows from (3). We define the functional $\Phi : \overline{\mathcal{R}} \to \mathbb{R}$ by

$$\Phi(m) := \int_\Omega u_m^\alpha \mathrm{d}x.$$

Our interest is in the following optimization problems

$$\max_{m \in \mathcal{R}} \Phi(m), \tag{5}$$

and

$$\min_{m \in \overline{\mathcal{R}}} \Phi(m). \tag{6}$$

We now prove some useful results.

Remark 3.1 : Let c_1 and c_2 be positive constants and $1 < p < \alpha$. It's easy to show that the function $f(t) = c_1 t^p - c_2 t^\alpha$; $t \in [0,\infty)$ has an absolute maximum value;

$$f(t) \leq f(t_{max}), \ \ \forall\, t \in [0,\infty),$$

where $t_{max} = (\frac{pc_1}{\alpha c_2})^{\frac{1}{\alpha - p}}$.

Lemma 3.1 — For $m \in \overline{\mathcal{R}}$ the functional $m \mapsto \Phi(m)$ is continuous with respect to weak* topology in $L^\infty(\Omega)$.

PROOF : Let $m \in \overline{\mathcal{R}}$ and $\{m_i\}$ be a sequence that converges weak* to m in $L^\infty(\Omega)$; this is denoted by $m_i \overset{*}{\rightharpoonup} m$. From (4) we have

$$\begin{aligned}
&\left(\frac{1}{p}-\frac{1}{\alpha}\right)\Phi(m)+\frac{\lambda}{p}\int_\Omega (m_i-m)u_m^p \mathrm{d}x \\
&= \Psi(m_i,u_m) \\
&\leq \left(\frac{1}{p}-\frac{1}{\alpha}\right)\Phi(m_i) \\
&= \Psi(m,u_{m_i})+\frac{\lambda}{p}\int_\Omega (m_i-m)u_{m_i}^p \mathrm{d}x \\
&\leq \left(\frac{1}{p}-\frac{1}{\alpha}\right)\Phi(m)+\frac{\lambda}{p}\int_\Omega (m_i-m)u_{m_i}^p \mathrm{d}x. \qquad (7)
\end{aligned}$$

Since $m_i \overset{*}{\rightharpoonup} m$, we deduce that

$$\lim_{i\to\infty}\int_\Omega (m_i-m)u_m^p \mathrm{d}x = 0. \qquad (8)$$

Now, we prove that

$$\lim_{i\to\infty}\int_\Omega (m_i-m)u_{m_i}^p \mathrm{d}x = 0. \qquad (9)$$

From (3) we have

$$\|\nabla u_{m_i}\|_p^p = \lambda \int_\Omega m_i u_{m_i}^p \mathrm{d}x - \int_\Omega u_{m_i}^\alpha \mathrm{d}x.$$

Since $\{m_i\}$ is bounded and $p<\alpha$ there exist two positive constants c_1 and c_2 such that

$$\|\nabla u_{m_i}\|_p^p \leq c_1\|u_{m_i}\|_p^p - c_2\|u_{m_i}\|_p^\alpha. \qquad (10)$$

Thus, from (10) and Remark 3.1 we deduce that $\{\|\nabla u_{m_i}\|_p\}$ is bounded. Therefore, a subsequence of $\{u_{m_i}\}$, denoted again by $\{u_{m_i}\}$, convergence weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ to some function w in $L^s(\Omega)$, for all s that $p \leq s < \frac{Np}{N-p}$, see [1]. Since

$$\int_\Omega (m_i-m)u_{m_i}^p \mathrm{d}x = \int_\Omega (m_i-m)w^p \mathrm{d}x + \int_\Omega (m_i-m)(u_{m_i}^p - w^p)\mathrm{d}x,$$

and

$$\left|\int_\Omega (m_i-m)(u_{m_i}^p - w^p)\mathrm{d}x\right| \leq 2H\|u_{m_i}^p - w^p\|_1,$$

we infer (9) when $i\to\infty$. Therefore, by (7), (8) and (9) we obtain that

$$\lim_{i\to\infty}\Phi(m_i) = \Phi(m). \qquad (11)$$

This yields the weak* continuity. □

Remark 3.2 : According to the proof of the Lemma 3.1, we claim that w is equal to u_m a.e. in Ω. We know that

$$\left(\frac{1}{p}-\frac{1}{\alpha}\right)\Phi(m_i)=\frac{\lambda}{p}\int_\Omega m_i u_{m_i}^p \mathrm{d}x-\frac{1}{\alpha}\int_\Omega u_{m_i}^\alpha \mathrm{d}x-\frac{1}{p}\int_\Omega |\nabla u_{m_i}|^p \mathrm{d}x.$$

Also

$$\lim_{i\to\infty}\int_\Omega m_i u_{m_i}^p \mathrm{d}x=\int_\Omega m w^p \mathrm{d}x,$$

$$\lim_{i\to\infty}\int_\Omega u_{m_i}^\alpha \mathrm{d}x=\int_\Omega w^\alpha \mathrm{d}x,$$

and

$$\liminf_{i\to\infty}\int_\Omega |\nabla u_{m_i}|^p \mathrm{d}x\geq\int_\Omega |\nabla w|^p \mathrm{d}x.$$

Thus by above results and (11) we deduce

$$\left(\frac{1}{p}-\frac{1}{\alpha}\right)\Phi(m)\leq\Psi(m,w)\leq(\frac{1}{p}-\frac{1}{\alpha})\Phi(m).$$

By the uniqueness of the maximizer of $\Psi(m,.)$ we obtain $w=u_m$ a.e. in Ω.

Lemma 3.2 — The functional $\Phi:\overline{\mathcal{R}}\to\mathbb{R}$ is strictly convex.

PROOF : We know

$$\Phi(m)=\left(\frac{1}{p}-\frac{1}{\alpha}\right)^{-1}\sup_{v\in W_0^{1,p}(\Omega)}\Psi(m,v).$$

Since $\Psi(.,v)$ is affine, thus Φ is convex. Now we prove strict convexity. Assume for $t\in(0,1)$ and $m_1,m_2\in\overline{\mathcal{R}}$ we have

$$\Phi(tm_1+(1-t)m_2)=t\Phi(m_1)+(1-t)\Phi(m_2).$$

Set $m=tm_1+(1-t)m_2$. We have

$$t\Psi(m_1,u_m)+(1-t)\Psi(m_2,u_m)=t\Psi(m_1,u_{m_1})+(1-t)\Psi(m_2,u_{m_2}),$$

hence

$$t(\Psi(m_1,u_m)-\Psi(m_1,u_{m_1}))=(1-t)(\Psi(m_2,u_{m_2})-\Psi(m_2,u_m)).$$

Since $t \in (0,1)$, by uniqueness of the maximizers of $\Psi(m_1,.)$ and $\Psi(m_2,.)$, we infer that $u_m = u_{m_1} = u_{m_2}$ a.e. in Ω. From this fact and the boundary value problem (2) we obtain

$$-\Delta_p u_m = \lambda m_1(x) u_m^{p-1} - u_m^{\alpha-1} \text{ a.e. in } \Omega,$$
$$-\Delta_p u_m = \lambda m_2(x) u_m^{p-1} - u_m^{\alpha-1} \text{ a.e. in } \Omega.$$

Therefore $m_1 = m_2$ a.e. in Ω, so Φ is strictly convex.

Lemma 3.3 — For $m \in \overline{\mathcal{R}}$, $\Phi(m)$ is Gâteaux differentiable with derivative $\frac{\lambda\alpha}{\alpha-p} u_m^p$.

PROOF : Let $\{t_j\}$ be a sequence of positive numbers that tends to zero. Let $m \in \overline{\mathcal{R}}$, $\nu \in L^\infty(\Omega)$ and $m_j := m + t_j(\nu - m)$, $j \in \mathbb{N}$. So, $m_j \to m$ in $L^\infty(\Omega)$ as $j \to \infty$. From (7) we have

$$\begin{aligned}\left(\frac{1}{p} - \frac{1}{\alpha}\right)\Phi(m) + \frac{\lambda}{p}\int_\Omega (m_j - m)u_m^p \mathrm{d}x &\le \left(\frac{1}{p} - \frac{1}{\alpha}\right)\Phi(m_j) \\ &\le \left(\frac{1}{p} - \frac{1}{\alpha}\right)\Phi(m) + \frac{\lambda}{p}\int_\Omega (m_j - m)u_{m_j}^p \mathrm{d}x.\end{aligned}$$

As a consequence of Remark 3.2, $u_{m_j} \to u_m$ in $L^p(\Omega)$. This coupled with above inequalities, implies that

$$\lim_{j\to\infty} \frac{\Phi(m + t_j(\nu - m)) - \Phi(m)}{t_j} = \int_\Omega (\nu - m)\frac{\lambda\alpha}{\alpha - p} u_m^p \mathrm{d}x.$$

Therefore, the proof of the lemma follows. □

Now, we are ready to prove that the maximization problem (5) is solvable.

***Theorem* 3.4** — *The maximization problem (5) is solvable; that is, there exists $m^* \in \mathcal{R}$ such that*

$$\Phi(m^*) = \max_{m\in\mathcal{R}} \Phi(m).$$

Moreover, there exists an increasing function ξ such that $m^* = \xi(u_{m*})$ almost everywhere in Ω.

PROOF : From Lemma 3.1, Lemma 3.2 and Lemma 2.4(i) we infer that there exists $m^* \in \mathcal{R}$ such that $\Phi(m) \le \Phi(m^*)$, for all $m \in \mathcal{R}$. From Lemma 3.3, $\Phi(m)$ is Gâteaux differentiable with derivative $\frac{\lambda\alpha}{\alpha-p} u_m^p$. Since Φ is strictly convex, by Lemma 2.4(ii), there is an increasing function ξ such that $m^* = \xi(u_{m^*})$. □

We now prove the minimization problem (6) has a unique solution.

***Theorem* 3.5** — *(a) The minimization problem (6) has a unique solution $m_* \in \overline{\mathcal{R}}$.*

(b) If for every $\gamma \in [L, H]$, $\mathcal{L}_N(\{x \in \Omega : m_*(x) = \gamma\}) = 0$, then $m_* = \eta(u_{m_*})$ for some decreasing function η.

PROOF : (a) We know Φ is weakly* continous in $L^\infty(\Omega)$, Lemma 3.1, and $\mathcal{R}$ is weakly compact, Lemma 2.1(i). Thus, there exists $m_* \in \overline{\mathcal{R}}$ such that

$$\Phi(m_*) = \min_{m \in \overline{\mathcal{R}}} \Phi(m).$$

Since Φ is strictly convex, we infer that m_* is unique.

(b) By the assumption, m_* has no significant flat section. This coupled with $-\Delta_p u_{m_*} = \lambda m_*(x) u_{m_*}^{p-1} - u_{m_*}^{\alpha-1}$, in Ω, implies that every level set of u_{m_*} in Ω has measure zero. Because, if for some constant c the set $B = \{x \in \Omega : u_{m_*}(x) = c\}$ be such that $\mathcal{L}_N(B) > 0$, then

$$0 = \lambda m_*(x) c^{p-1} - c^{\alpha-1}, \text{ in } B.$$

Therefore m_* is constant in B, which is a contradiction. By applying Lemma 2.2 we derive that there exists a decreasing function η such that $\eta(u_{m_*}^p) \in \mathcal{R}$. Now, from Lemma 2.3(ii) we have

$$\int_\Omega m u_{m_*}^p \mathrm{d}x \geq \int_\Omega \eta(u_{m_*}^p) u_{m_*}^p \mathrm{d}x, \quad \forall m \in \overline{\mathcal{R}}. \tag{12}$$

Let $0 < t < 1$ and $m \in \overline{\mathcal{R}}$. We define $m_t = tm + (1-t)m_*$. By Lemma 2.1(ii), $\overline{\mathcal{R}}$ is convex, so $m_t \in \overline{\mathcal{R}}$ for all $0 < t < 1$. From Lemma 3.3, for sufficiently small t we have

$$\Phi(m_*) \leq \Phi(m_t) = \Phi(m_*) + t \frac{\lambda \alpha}{\alpha - p} \int_\Omega (m - m_*) u_{m_*}^p \mathrm{d}x + o(t).$$

Thus, when $t \to 0^+$ we deduce

$$\int_\Omega m u_{m_*}^p \mathrm{d}x \geq \int_\Omega m_* u_{m_*}^p \mathrm{d}x, \quad \forall m \in \overline{\mathcal{R}}. \tag{13}$$

Therefore, by (12), (13) and Lemma 2.3(ii) we derive $m_* = \eta(u_{m_*}^p) \in \mathcal{R}$. □

4. SYMMETRIC DOMAIN

In this section we assume Ω is a ball centered at the origin. If $f : \Omega \to [0, \infty)$ be a measurable function then we denote the Schwarz decreasing rearrangement of f by $f^\#$ that is defined by

$$f^\#(x) = \sup\{t > 0 : \mu_f(t) \geq |x|^N \omega_N\},$$

where

$$\mu_f(t) = \mathcal{L}_N(\{x \in \Omega : f(x) \geq t\}),$$

and ω_N is the volume of the unit ball in $\mathbb{R}^N$. The function $f^\#$ is radially symmetric, decreasing in the variable $|x|$, and it satisfies $\mu_f(t) = \mu_{f^\#}(t)$, for all t. Now we recall two well-known results.

Lemma 4.1 — [19]. If $f, g : \Omega \to [0, \infty)$ are in $L^2(\Omega)$, then

$$\int_\Omega fg\mathrm{d}x \leq \int_\Omega f^\# g^\# \mathrm{d}x.$$

Lemma 4.2 — [19]. Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative function. Then $u^\# \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^p \mathrm{d}x \geq \int_\Omega |\nabla u^\#|^p \mathrm{d}x. \tag{14}$$

Furthermore, if we have equality in (14) and $\mathcal{L}_N(\{x \in \Omega : \nabla u^\# = 0\}) = 0$ then $u = u^\#$ a.e. in Ω.

Now, we are ready to state our results.

***Theorem* 4.3** —

(i) For every $m \in \mathcal{R}$ we have $\Phi(m) \leq \Phi(m^\#)$.

(ii) Assume

$$\mathcal{L}_N(\{x \in \Omega : m_0(x) = c\}) = 0, \ \ \forall c \in [L, H]. \tag{15}$$

If $\Phi(m) = \Phi(m^\#)$, then $m = m^\#$ a.e. in Ω.

PROOF : Let $\beta = \frac{1}{p} - \frac{1}{\alpha}$. From Lemma 4.1 and Lemma 4.2 we have

$$\begin{aligned}
\beta\Phi(m) &= \Psi(m, u_m) \\
&= \frac{\lambda}{p}\int_\Omega m(u_m)^p \mathrm{d}x - \frac{1}{\alpha}\int_\Omega (u_m)^\alpha \mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u_m|^p \mathrm{d}x \\
&\leq \frac{\lambda}{p}\int_\Omega m^\#(u_m^\#)^p \mathrm{d}x - \frac{1}{\alpha}\int_\Omega (u_m^\#)^\alpha \mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u_m^\#|^p \mathrm{d}x \\
&= \Psi(m^\#, u_m^\#) \\
&\leq \Psi(m^\#, u_{m^\#}) \\
&= \beta\Phi(m^\#).
\end{aligned} \tag{16}$$

So, (i) is proved. Now, assume $\Phi(m) = \Phi(m^\#)$. Thus by (16) we infer that $\Psi(m^\#, u_m^\#) = \Psi(m^\#, u_{m^\#})$. Hence by uniqueness we deduce that $u_m^\# = u_{m^\#}$ a.e. in Ω. Also, from (16) we have

$\Psi(m, u_m) = \Psi(m^{\#}, u_m^{\#})$. Therefore

$$\lambda \left(\int_\Omega m(u_m)^p \mathrm{d}x - \int_\Omega m^{\#}(u_m^{\#})^p \mathrm{d}x \right) = \int_\Omega |\nabla u_m|^p \mathrm{d}x - \int_\Omega |\nabla u_m^{\#}|^p \mathrm{d}x. \tag{17}$$

The equality (17) together with Lemma 4.1 and Lemma 4.2 yield

$$\int_\Omega |\nabla u_m|^p \mathrm{d}x = \int_\Omega |\nabla u_m^{\#}|^p \mathrm{d}x. \tag{18}$$

Since

$$-\Delta_p u_{m^{\#}} = \lambda m^{\#}(x)(u_{m^{\#}})^{p-1} - (u_{m^{\#}})^{\alpha-1} \text{ in } \Omega,$$

by assumption (15) we infer that

$$\mathcal{L}_N(\{x \in \Omega : \nabla u_{m^{\#}} = 0\}) = 0.$$

Thus, by (18) and Lemma 4.2 we deduce that $u_m^{\#} = u_{m^{\#}} = u_m$ a.e. in Ω. Hence

$$-\Delta_p u_m = \lambda m(x)(u_m)^{p-1} - (u_m)^{\alpha-1} \text{ in } \Omega,$$

and

$$-\Delta_p u_m = \lambda m^{\#}(x)(u_m)^{p-1} - (u_m)^{\alpha-1} \text{ in } \Omega.$$

Therefore $m = m^{\#}$ a.e. in Ω. □

Corollary 4.4 — If

$$\mathcal{L}_N(\{x \in \Omega : m_0(x) = c\}) = 0, \ \forall c \in [L, H],$$

then $m_0^{\#}$ is the unique solution of the maximization problem (5).

PROOF : Let $g \in \mathcal{R}$ be a solution of (5). By Theorem 4.3 for every $m \in \mathcal{R}$ we have

$$\Phi(m) \leq \Phi(g) \leq \Phi(g^{\#}) \leq \Phi(g).$$

Therefore $g = g^{\#} = m_0^{\#}$ a.e. in Ω.

ACKNOWLEDGEMENT

We are grateful to the Research Council of Shahid Chamran University of Ahvaz for financial support (SCU.MM98.441).

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