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FUSION FRAMES FOR OPERATORS IN HILBERT C^*-MODULES

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In this paper we introduce K-fusion frames on a Hilbert C^*-module H, where K is an adjointable operator on H. We obtain several characterizations of K-fusion frames. In addition, we extend the concept of duality to K-fusion frames and study some of its properties.

Key words : Frame; fusion frame; operator; duality; Hilbert C^*-module.

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1. INTRODUCTION

The concept of frame in Hilbert spaces has been introduced by Duffin and Schaeffer [3] in 1952 to study some deep problems in nonharmonic Fourier series. After it has been reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [2], frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. Traditionally, frames have been used in signal processing, image processing, data compression and sampling theory. In 2000, Frank-Larson [6] introduced the notion of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces, and Jing [10] continued to consider them. It is well known that Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Recentely, Khosravi and Khosravi [11] introduced the fusion frame theory in Hilbert C^*-modules.

Atomic systems for subspaces were first introduced by Feichtinger and Werther [5] based on examples arising in sampling theory. In [7], Gavruta introduced atomic systems for operators in Hilbert spaces, and frames for operators allowing the reconstruction of elements from the range of a linear and bounded operator.

Recently, Najati [13] generalized the notion of frames for operators for Hilbert spaces to Hilbert C^*-modules and studied som of their properties. In this paper we define the concept of fusion frames for operators in Hilbert C^*-modules.

The paper is organized in the following manner. In Section 2, we recall the definitions and basic properties. Section 3 is devoted to introduce fusion frames for operators in Hilbert C^*-modules. In Section 4, we study the concept of duality of fusion frames for operators in Hilbert C^*-modules based on the notion of duality of fusion frames introduced in [8, 9].

2. PRELIMINARIES

In the following we briefly recall some definitions and basic properties of operators and fusion frames in Hilbert C^*-modules. Throughout this paper, the symbols $\mathbb{I}, \mathbb{C}$ and $\mathcal{A}$ refer, respectively to a finite or countable index set, the field of complex numbers, and a unital C^*-algebra with identity $1_{\mathcal{A}}$.

Definition 2.1 — (see [12]). A pre-Hilbert C^*-module over $\mathcal{A}$ or, simply, a pre Hilbert $\mathcal{A}$-module, is a left $\mathcal{A}$-module H with a sesquilinear form $\langle .,. \rangle : H \times H \longrightarrow \mathcal{A}$, called an $\mathcal{A}$-valued inner product, that possesses the following properties:

(1) $\langle f, f \rangle \geq 0$, for any $f \in H$,

(2) $\langle f, f \rangle = 0$ if and only if $f = 0$,

(3) $\langle f, g \rangle = \langle g, f \rangle^*$, for any $f, g \in H$,

(4) $\langle \lambda f, h \rangle = \lambda \langle f, g \rangle$, for any $\lambda \in \mathbb{C}$,

(5) $\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$ for any $a, b \in \mathcal{A}$ and $f, g, h \in H$.

The map $f \longmapsto \|f\| = \|\langle f, f \rangle\|_{\mathcal{A}}^{1/2}$, defines a norm on H. If a pre-Hilbert C^*-module H is complete with respect to this norm, then $(H, \mathcal{A}, \langle .,. \rangle)$ is called a Hilbert C^*-module over $\mathcal{A}$ or, simply Hilbert $\mathcal{A}$-module. We write H or $(H, \langle .,. \rangle)$ instead of $(H, \mathcal{A}, \langle .,. \rangle)$ when the $\mathcal{A}$-valued inner product and the C^*-algebra are well-known. The Hilbert $\mathcal{A}$-module H is called to be a full Hilbert $\mathcal{A}$ -module when the linear span of $\{\langle f, g \rangle : f, g \in H\}$ is dense in $\mathcal{A}$.

The C^*-algebra $\mathcal{A}$ itself can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle = ab^*$. The standard Hilbert $\mathcal{A}$-module $\ell_2(H, I)$ is defined by :

$\ell_2(H, I) := \{\{f_i\}_{i\in I} \subseteq H : \sum_{i\in\mathbb{I}}\langle f_i, f_i\rangle$ converges in $\|.\|, on\mathcal{A}\}$, with

$$\langle\{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle.$$

Let $(H, \langle ., .\rangle_1)$ and $(K, \langle ., .\rangle_2)$ be two Hilbert $\mathcal{A}$-modules. A map $T : H \longrightarrow K$ (not neccessarily linear or bounded) is said to be adjointable (with respect to the $\mathcal{A}$-linear products $(H, \langle ., .\rangle_1)$ and $(K, \langle ., .\rangle_2)$, if there exists a map $T^* : K \longrightarrow H$ satisfying

$$\langle Tf, g\rangle_2 = \langle f, T^*g\rangle_1,$$

whenever $f \in H$, and $g \in K$.The map T^* is called the adjoint of T.

We also reserve the notation $End^*_{\mathcal{A}}(H, K)$ for the set of all adjointable operators from H to K and $End^*_{\mathcal{A}}(H, H)$ is abbreviated to $End^*_{\mathcal{A}}(H)$.

Definition 2.2 — (see [6]). A sequence $\{f_i\}_{i\in\mathbb{I}}$ of elements in a Hilbert $\mathcal{A}$-module H is said to be a frame for H if there exist two constants $A, B > 0$ such that

$$A\langle f, f\rangle \leq \sum_{i\in\mathbb{I}}\langle f, f_i\rangle\langle f_i, f\rangle \leqslant B\langle f, f\rangle, \tag{2.1}$$

holds for every $f \in H$. The number A and B are called frame bounds.

Definition 2.3 — (see [13]). Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$. A family $\{f_i\}_{i\in\mathbb{I}} \subseteq H$ is called a K-frame for H if there exist constants $A_1, B_1 > 0$ such that

$$A_1\langle K^*f, K^*f\rangle \leq \sum_{i\in\mathbb{I}}\langle f, f_i\rangle\langle f_i, f\rangle \leqslant B_1\langle f, f\rangle. \quad \forall f \in H \tag{2.2}$$

The numbers $A_1$and B_1 are called K-frame bounds.

(Throughout this paper, the sum like those in the middles of (2.1) and (2.2) are assumed to be convergent in the norm sense).

Definition 2.4 — (see [11]). Let $\mathcal{A}$ be a unital C^*-algebra,H be a Hilbert $\mathcal{A}$-module and let $\{\nu_i : i \in I\}$ be a family of weights in $\mathcal{A}$, i.e ν_i is a positive, invertible element from the center of $\mathcal{A}$, and let $\{M_i : i \in I\}$ be a family of closed orthogonally complemented submodules of H, then $\{(M_i, \nu_i) : i \in I\}$ is a fusion frame, if there exist real constants $0 < C \leq D < \infty$ such that

$$C\langle f, f\rangle \leq \sum_{i\in I}\nu_i^2\langle\pi_{M_i}(f), \pi_{M_i}(f)\rangle \leq D\langle f, f\rangle \quad ,\forall f \in H. \tag{2.3}$$

We call C and D the lower and upper bounds of fusion frame. If $C = D = \lambda$, the family $\{(M_i, \nu_i) : i \in I\}$ is called λ-tight fusion frame. If in (2.3), we only require to have the upper bound, then $\{(M_i, \nu_i) : i \in I\}$ is a Bessel fusion frame with Bessel bound D, and if in (2.3), the sum converges in norm, it is called standard fusion frame.

Definition 2.5 — Let $\{(M_i, \nu_i) : i \in I\}$ be a standard fusion frame, for H. The operator $T_{M,\nu} : \ell_2(H, I) \longrightarrow H$ defined by $T_{M,\nu}(f) = \sum_{i\in I} \nu_i \pi_{M_i}(f_i)$ for all $f = \{f_i\}_{i\in I}$ in $\ell_2(H, I)$ is called the synthesis operator. The fusion frame operator $S_{M,\nu} : H \longrightarrow H$ is defined by $S_{M,\nu}(f) = \sum_{i\in I} \nu_i^2 \pi_{M_i}(f)$ for all $f \in H$, which is positive, invertible and self-adjoint operator, and for all $f \in H$, we have

$$f = \sum_{i\in I} \nu_i^2 S_{M,\nu}^{-1}(\pi_{M_i}(f)) = \sum_{i\in I} \nu_i^2 \pi_{M_i}(S_{M,\nu}^{-1}(f)).$$

Lemma 2.6 — (see [4]). Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ be Hilbert $\mathcal{A}$-modules. Also let $T' \in End_{\mathcal{A}}^*(\mathcal{G}, \mathcal{F})$ and $T \in End_{\mathcal{A}}^*(\mathcal{E}, \mathcal{F})$ with $\overline{Ran}(T^*)$ orthogonally complemented. The following statements are equivalent

(1) $T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$,

(2) There exists $\mu > 0$ such that $\|T'^* z\| \leq \mu \|T^* z\|$ for all $z \in F$,

(3) There exists $D \in End_{\mathcal{A}}^*(\mathcal{G}, \mathcal{E})$ such that $T' = TD$ i.e $TX = T'$ has a solution ,

(4) $Ran(T') \subseteq Ran(T)$.

Lemma 2.7 — (see [1]). Let H and K be two Hilbert $\mathcal{A}$-modules and $T : H \longrightarrow K$ be an adjointable map . Then

(1) If T is injective and T has closed range, then the adjointable map T^*T is invertible and $\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2$,

(2) If T is surjective, then the adjointable map TT^* is invertible and $\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2$.

3. Fusion Frames for Operators in Hilbert C^*-modules

In this section we introduce fusion frames for operators in Hilbert C^*-modules.

Definition 3.1 — Let $K \in End_{\mathcal{A}}^*(H)$, $\{M_i : i \in I\}$ be a family of closed orthogonally complemented submodules of H and let $\{\nu_i : i \in I\}$ be a family of weights in $\mathcal{A}$, then $\{(M_i, \nu_i) : i \in I\}$ is said to be a K-fusion frame for H, if there exist two constants $C, D > 0$ such that

$$C\langle K^*f, K^*f\rangle \leq \sum_{i\in I} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle \leq D\langle f, f\rangle, \qquad \forall f \in H.$$

The numbers C and D are called K-fusion frame bounds. Particularly, if

$$C\langle K^*f, K^*f\rangle = \sum_{i\in \mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle, \qquad \forall f \in H,$$

then we call $\{(M_i, \nu_i) : i \in I\}$ is a C-tight K-fusion frame for H.

Remark 3.2 : (a) If $K \in End_{\mathcal{A}}^*(H)$ is a surjective operator, then every K-fusion frame $\{(M_i, \nu_i) : i \in I\}$ for H, is a fusion frame. Indeed, if we let C and D be the K-fusion frame bounds, then for any $f \in H$, we have

$$C\|(KK^*)^{-1}\|^{-1}\langle f, f\rangle \leq C\langle K^*f, K^*f\rangle \leq \sum_{i\in \mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle \leq D\langle f, f\rangle.$$

(b) Let $\{(M_i, \nu_i) : i \in I\}$ be a fusion frame, then $\{(M_i, \nu_i) : i \in I\}$ is a 1-tight $S_{M,\nu}$-fusion frame, where $S_{M,\nu}$ is fusion frame operator. To see this, observe that $\langle S_{M,\nu}f, S_{M,\nu}f\rangle = \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle$. Thus $\{(M_i, v_i) : i\}$ is a l-tight $S_{M,v}$-fusion frame.

Thus $\{(M_i, \nu_i) : i \in I\}$ is a 1-tight $S_{M,\nu}$-fusion frame.

(c) Every fusion frame is an id-fusion frame. □

The following proposition gives a necessary and sufficient conditions for tight fusion frames to be tight K-fusion frames in Hilbert C^*-modules.

Proposition 3.3 — Let $\{(M_i, \nu_i) : i \in I\}$ be an A -tight fusion frame for H and $K \in End_{\mathcal{A}}^*(H)$. The family $\{(M_i, \nu_i) : i \in I\}$ is a tight K-fusion frame for H, if and only if there exists a number $M > 0$ such that $KK^* = Mid_H$.

PROOF : We assume first that $\{(M_i, \nu_i) : i \in I\}$ is a B-tight K-fusion frame for H, then for any $f \in H$, we have

$$B\langle K^*f, K^*f\rangle = \sum_{i\in \mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle = A\langle f, f\rangle.$$

Thus, $\langle KK^*f, f\rangle = \dfrac{A}{B}\langle f, f\rangle$ and the polarization formula shows that $\langle KK^*f, g\rangle = \dfrac{A}{B}\langle f, g\rangle$ for any $f, g \in H$. Consequently, $KK^* = \dfrac{A}{B} id_H$.

For the other implication, let $KK^* = Mid_H$ for some positive number M, then for any $f \in H$, we have

$$\sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle = \frac{A}{M}\langle Mf, f\rangle = \frac{A}{M}\langle KK^*f, f\rangle = \frac{A}{M}\langle K^*f, K^*f\rangle.$$

This shows that $\{(M_i, \nu_i) : i \in I\}$ is a $\dfrac{A}{M}$-tight K-fusion frame for H. □

Let $K \in End_{\mathcal{A}}^*(H)$, by $F_K(H)$ and $F_K^T(H)$ we denote the sets of all K-fusion frames and all tight K-fusion frames of H respectively. Our next result presents a relation between two sets of tight K-fusion frames and the involved operators.

Proposition 3.4 — Let H be a finitely or countably generated Hilbert $\mathcal{A}$-module and $K_1, K_2 \in End_{\mathcal{A}}^*(H)$. Then $F_{K_1}^T(H) \subset F_{K_2}^T(H)$ if and only if there exists a constant $A > 0$ such that $AK_2K_2^* = K_1K_1^*$.

PROOF : Let $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ be a tight K_1-fusion frame for H, thus there exists $A_1 > 0$ such that

$$A_1\langle K_1^*f, K_1^*f\rangle = \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle, f \in H.$$

Since $F_{K_1}^T(H) \subseteq F_{K_2}^T(H)$, there exists $A_2 > 0$ such that

$$A_2\langle K_2^*f, K_2^*f\rangle = \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle, f \in H.$$

Altogether we obtain $A_2\langle K_2^*f, K_2^*f\rangle = A_1\langle K_1^*f, K_1^*f\rangle$, equivalently $\dfrac{A_2}{A_1}K_2K_2^* = K_1K_1^*$.

Suppose now that, $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ is a tight K_1-fusion frame. Then there exists $A_1 > 0$ such that

$$A_1\langle K_1^*f, K_1^*f\rangle = \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle, f \in H.$$

Since $AK_2K_2^* = K_1K_1^*$, we have $AA_1K_2K_2^* = A_1K_1K_1^*$. Hence

$$AA_1\langle K_2^*f, K_2^*f\rangle = \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle, f \in H.$$

Therefore, $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ is a AA_1-tight K_2-fusion frame of H and consequently, $F_{K_1}^T(H) \subseteq F_{K_2}^T(H)$. □

We also have the relation between two sets of K-fusion frames and the ranges of the involved operators.

Proposition 3.5 — Let $K_1, K_2 \in End^*_{\mathcal{A}}(H)$. If $Ran(K_2) \subseteq Ran(K_1)$ and $\overline{Ran}(K_1^*)$ is orthogonally complemented, then $F_{K_1}(H) \subseteq F_{K_2}(H)$.

PROOF : By Lemma 2.6, we know that there exists $\mu > 0$ such that $K_2 {K_2}^* \leq \mu K_1 {K_1}^*$. Let $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ be a K_1-fusion frame for H, with bounds C and D, then

$$\frac{C}{\mu}\langle {K_2}^* f, {K_2}^* f\rangle \leq C\langle {K_1}^* f, {K_1}^* f\rangle \leq \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle \leq D\langle f, f\rangle.$$

This shows that $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ is a K_2-fusion frame for H with bounds $\frac{C}{\mu}$ and D, therefore we have $F_{K_1}(H) \subseteq F_{K_2}(H)$. □

The converse of the above proposition remains true if we replace $F_{K_1}(H)$ by $F^T_{K_1}(H)$.

Proposition 3.6 — Let $K_1, K_2 \in End^*_{\mathcal{A}}(H)$. If $F^T_{K_1}(H) \subseteq F_{K_2}(H)$ and $\overline{Ran}(K_1^*)$ is orthogonally complemented then $Ran(K_2) \subseteq Ran(K_1)$.

PROOF : Let $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ be a A-tight K_1-fusion frame for H, thus

$$A\langle {K_1}^* f, {K_1}^* f\rangle = \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle, \qquad f \in H.$$

Since $F^T_{K_1}(H) \subseteq F_{K_2}(H)$, there exist $C, D > 0$ such that

$$C\langle {K_2}^* f, {K_2}^* f\rangle \leq \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle \leq D\langle f, f\rangle, \quad f \in H.$$

Therefore, $C\|{K_2}^* f\|^2 \leq A\|{K_1}^* f\|^2$, for all $f \in H$. By Lemma 2.6, we have $Ran(K_2) \subseteq Ran(K_1)$. □

Proposition 3.7 — Let $K \in End^*_{\mathcal{A}}(H)$. Suppose that the operator $T : H \to \ell^2(H, \mathbb{I})$ is given by $Tf = \{\nu_i \pi_{M_i}(f)\}_{i\in I}$ and $\overline{Ran}(T)$ is orthogonally complemented, then $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ is a K-fusion frame for H if and only if there exist constants $C, D > 0$ such that

$$C\|K^* f\|^2 \leq \|\sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle\| \leq D\|f\|^2, f \in H. \qquad (3.1)$$

PROOF : Evidently, every K-fusion frame of H satisfies (3.1).

For the converse, we suppose that a sequence $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ fulfills (3.1). For all $f \in H$, the left-hand inequality of (3.1) gives $\|K^*f\|^2 \leq \frac{1}{C}\|Tf\|^2$, then Lemma 2.6 implies that there exists a constant $\mu > 0$ such that $KK^* \leq \mu TT^*$, and hence

$$\frac{1}{\mu}\langle K^*f, K^*f\rangle \leq \langle Tf, Tf\rangle = \sum_{i\in\mathbb{I}} \nu_i^2 \langle \pi_{M_i}(f), \pi_{M_i}(f)\rangle, f \in H.$$

To complete the proof, it remains to show that $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ is a fusion Bessel sequence.

For any $\{f_i\}_{i\in\mathbb{I}} \in \ell^2(H, \mathbb{I})$ and any $\mathbb{J} \subseteq \mathbb{I}$, the right-hand inequality of (3.1) leads to

$$\begin{aligned}
\|\sum_{i\in\mathbb{J}} \nu_i\pi_{M_i}(f_i)\|^2 &= sup_{f\in H,\|f\|=1}\|\langle\sum_{i\in\mathbb{J}} \nu_i\pi_{M_i}(f_i), f\rangle\|^2\\
&= sup_{f\in H,\|f\|=1}\|\sum_{i\in\mathbb{J}}\langle f_i, \nu_i\pi_{M_i}(f)\rangle\|^2\\
&\leq sup_{f\in H,\|f\|=1}\|\sum_{i\in\mathbb{J}}\langle f_i, f_i\rangle\|\|\sum_{i\in\mathbb{J}}\nu_i^2\langle\pi_{M_i}f, \pi_{M_i}f\rangle\|\\
&\leq D\|\sum_{i\in\mathbb{J}}\langle f_i, f_i\rangle\|.
\end{aligned}$$

Thus the series $\sum_{i\in\mathbb{I}} \nu_i\pi_{M_i}(f_i)$ converges in H unconditionally. Since

$$\langle Tf, \{f_i\}_{i\in\mathbb{I}}\rangle = \sum_{i\in\mathbb{I}}\langle\nu_i\pi_{M_i}(f), f_i\rangle = \langle f, \sum_{i\in\mathbb{I}}\nu_i\pi_{M_i}(f_i)\rangle,$$

T is adjointable. Now for any $f \in H$ we have

$$\langle Tf, Tf\rangle = \sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}(f), \pi_{M_i}(f)\rangle \leq \|T\|^2\langle f, f\rangle.\square$$

Using Theorem 3.7 we can easily prove the following result, which offers a condition for getting a K-fusion frame from a fusion frame.

Proposition 3.8 — Let $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ be a fusion frame for H with bounds C, D and the synthesis operator $T_{M,\nu}$. Let $K \in End^*_{\mathcal{A}}(H)$ and $\overline{Ran}(T^*_{M,\nu})$ be orthogonally complemented, then $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ is a K-fusion frame for H.

PROOF : Let $S_{M,\nu}$ be the fusion frame operator of $\{(M_i, \nu_i) : i \in \mathbb{I}\}$, then for every $f \in H$ the reconstruction formula gives, $Kf = \sum_{i\in\mathbb{I}}\nu_i^2\pi_{M_i}S^{-1}_{M,\nu}Kf$ and so

$$\|K^*f\|^2 = sup_{\|h\|=1}\|\langle K^*f, h\rangle\|^2 = sup_{\|h\|=1}\|\sum_{i\in\mathbb{I}}\langle\nu_i\pi_{M_i}f, \nu_i\pi_{M_i}S^{-1}Kh\rangle\|^2$$

$\leq sup_{\|h\|=1}\|\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle\|\|\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}S^{-1}Kh,\pi_{M_i}S^{-1}Kh\rangle\|$

$\leq DC^{-2}\|K\|^2\|\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle\|.\square$

Proposition 3.9 — Let $K\in End_{\mathcal{A}}^*(H)$, and for each $i\in\mathbb{I}$,$\{f_{ik}\}_{k\in\mathbb{J}_i}$ is a frame for M_i where M_i is closed orthogonally complemented submodules of H with bounds A_i and B_i and let

$$0<A=inf_{i\in\mathbb{I}}A_i\leq sup_{i\in\mathbb{I}}B_i=B<\infty$$

Then the following conditions are equivalent

(1) $\{(M_i,\nu_i):i\in\mathbb{I}\}$ is a K-fusion frame for H where $\{\nu_i:i\in\mathbb{I}\}$ is a weight in $\mathcal{A}$,

(2) $\{\nu_i f_{ik}\}_{i\in\mathbb{I},k\in\mathbb{J}_i}$ is a K-frame for H.

PROOF : $(1)\Rightarrow(2)$ For any $f\in H$, we have

$$\begin{aligned}A\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle&\leq\sum_{i\in\mathbb{I}}A_i\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle\\&\leq\sum_{i\in\mathbb{I}}\sum_{k\in\mathbb{J}_i}\langle\pi_{M_i}(\nu_i f),f_{ik}\rangle\langle f_{ik},\pi_{M_i}(\nu_i f)\rangle\\&\leq B\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle.\end{aligned}$$

Hence $\{\nu_i f_{ik}\}_{i\in\mathbb{I},k\in\mathbb{J}_i}$ is a K-frame for H.

$(2)\Rightarrow(1)$

$$\begin{aligned}A\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle&\leq\sum_{i\in\mathbb{I}}\sum_{k\in\mathbb{J}_i}\langle f,\nu_i\pi_{M_i}f_{ik}\rangle\langle\nu_i\pi_{M_i}f_{ik},f\rangle\\&\leq B\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle,\end{aligned}$$

and on the other hand since $\{\nu_i f_{ik}\}_{i\in\mathbb{I},k\in\mathbb{J}_i}$ is a K-frame for H, there exist $A_1,B_1>0$ such that

$$A_1\langle K^*f,K^*f\rangle\leq\sum_{i\in\mathbb{I}}\sum_{k\in\mathbb{J}_i}\langle f,\nu_i\pi_{M_i}f_{ik}\rangle\langle\nu_i\pi_{M_i}f_{ik},f\rangle\leq B_1\langle f,f\rangle,$$

Therefore

$$\frac{A_1}{B}\langle K^*f,K^*f\rangle\leq\sum_{i\in\mathbb{I}}\nu_i^2\langle\pi_{M_i}f,\pi_{M_i}f\rangle\leq\frac{B_1}{A}\langle f,f\rangle.\square$$

Lemma 3.10 — Let $\{(M_i,\nu_i):i\in\mathbb{I}\}$ be a K-fusion frame for H, then

(1) $\{(M_i, \nu_i) : i \in \mathbb{I}\}$ is a fusion Bessel sequence,

(2) $\{(M_i, |\alpha|\nu_i) : i \in \mathbb{I}\}$ is a K-fusion frame for each $\alpha \in \mathbb{R} - \{0\}$.

PROOF : It is clear.

4. DUALITY OF FUSION FRAMES FOR OPERATORS IN HILBERT C^*-MODULES

In this section, we define a concept of dual by means of a bounded operator and associated synthesis operators to investigate the relation between a fusion frame for operator and a fusion Bessel sequence.

Definition 4.1 — Let $\{f_i\}_{i\in\mathbb{I}}$ be a K-frame and $\{g_i\}_{i\in\mathbb{I}}$ be a Bessel sequence of H. Then $\{g_i\}_{i\in\mathbb{I}}$ is called a dual K-frame of $\{f_i\}_{i\in\mathbb{I}}$ if

$$Kf = \sum_{i\in\mathbb{I}} \langle f, f_i\rangle g_i, \quad \forall f \in H. \tag{4.1}$$

Inspired by the concept of Q-dual fusion frames in [8], we next introduce what we call V-dual K-fusion frames in Hilbert C^*-modules.

Definition 4.2 — Let $\mathcal{M} = \{(M_i, \nu_i) : i \in \mathbb{I}\}$ be a K-fusion frame for H and $\mathcal{W} = \{(W_i, \omega_i) : i \in \mathbb{I}\}$ be a fusion Bessel sequence for H with synthesis operators $T_{\mathcal{M}}$ and $T_{\mathcal{W}}$, respectively. Then $\mathcal{W}$ is called a V-dual K -fusion frame of $\mathcal{M}$, if there exists a bounded operator $V : \ell^2(H, \mathbb{I}) \to \ell^2(H, \mathbb{I})$ such that $T_{\mathcal{W}} V T^*_{\mathcal{M}} = K$.

Proposition 4.3 — Let $\mathcal{M} = \{(M_i, \nu_i) : i \in \mathbb{I}\}$ and $\mathcal{W} = \{(W_i, \omega_i) : i \in \mathbb{I}\}$ be two K-fusion frames for H. Let $\{f_{ik}\}_{k\in\mathbb{J}_i}$ be a frame for M_i and $\{\tilde{f_{ik}}\}_{k\in\mathbb{J}_i}$ be a frame for W_i, with frame bounds A_i, B_i and $\tilde{A}_i, \tilde{B}_i$, respectively. Suppose that

$$0 < A = inf_{i\in\mathbb{I}} A_i \leq B = sup_{i\in\mathbb{I}} B_i,$$

$$0 < \tilde{A} = inf_{i\in\mathbb{I}} \tilde{A}_i \leq \tilde{B} = sup_{i\in\mathbb{I}} \tilde{B}_i,$$

then the following conditions are equivalent

(1) $\{\omega_i \tilde{f_{ik}}\}_{i\in\mathbb{I},k\in\mathbb{J}_i}$ is a dual K-frame of $\{\nu_i f_{ik}\}_{i\in\mathbb{I},k\in\mathbb{J}_i}$,

(2) $\{(W_i, \omega_i) : i \in \mathbb{I}\}$ is a V- dual K-fusion frame of $\{(M_i, \nu_i) : i \in \mathbb{I}\}$, such that $V : \ell^2(H, \mathbb{I}) \to \ell^2(H, \mathbb{I})$ is defined by $V(\{h_i\}_{i\in\mathbb{I}}) = \{\sum_{k\in\mathbb{J}_i} \langle g_i, f_{ik}\rangle \tilde{f_{ik}}\}_{i\in\mathbb{I}}$ where

$$g_i = \begin{cases} 0 & \text{if } h_i \notin M_i \\ h_i & \text{if } h_i \in M_i \end{cases}.$$

PROOF : Let us denote by T_i and $\tilde{T_i}$ the synthesis operators associated to the frames $\{f_{ik}\}_{k\in\mathbb{J}_i}$ and $\{\tilde{f_{ik}}\}_{k\in\mathbb{J}_i}$ respectively. Recall that T_i is an operator from $\ell^2(\mathcal{A})$ onto M_i is defined by

$$T_i\{c_k\}_{k\in\mathbb{J}_i} = \sum_{k\in\mathbb{J}_i} c_k f_{ik}, \forall\{c_k\}_{k\in\mathbb{J}_i} \in \ell^2(\mathcal{A}, \mathbb{I}),$$

where

$$\ell^2(\mathcal{A}, \mathbb{I}) = \{\{a_i\}_{i\in\mathbb{I}} \subseteq \mathcal{A} : \sum_{i\in\mathbb{I}} a_i a_i^* \text{ converges in} \|.\|\}.$$

Now define the operator $V : \ell^2(H, \mathbb{I}) \to \ell^2(H, \mathbb{I})$ by $V(\{h_i\}_{i\in\mathbb{I}}) = \{\sum_{k\in\mathbb{J}_i} \langle g_i, f_{ik}\rangle \tilde{f_{ik}}\}_{i\in\mathbb{I}}$ where

$$g_i = \begin{cases} 0 & \text{if } h_i \notin M_i \\ h_i & \text{if } h_i \in M_i \end{cases}.$$

Since

$$\sum_{i\in\mathbb{I}} \langle \tilde{T_i} T_i^*(g_i), \tilde{T_i} T_i^*(g_i)\rangle \leq \sum_{i\in\mathbb{I}} \|\tilde{T_i}\|^2 \|T_i^*\|^2 \langle g_i, g_i\rangle,$$

it follows that

$$\begin{aligned} \|V(\{h_i\}_{i\in\mathbb{I}})\|^2 &= \|\{\sum_{k\in\mathbb{J}_i} \langle g_i, f_{ik}\rangle \tilde{f_{ik}}\}_{i\in\mathbb{I}}\|^2 \\ &= \|\sum_{i\in\mathbb{I}} \langle \tilde{T_i} T_i^*(g_i), \tilde{T_i} T_i^*(g_i)\rangle\| \\ &\leq \tilde{B} B \|\{h_i\}\|^2 < \infty. \end{aligned}$$

Therefore, V is well-defined and also bounded. Let $T_{\mathcal{M}}$ and $T_{\mathcal{W}}$ be the synthesis operators of $\mathcal{M} = \{(M_i, \nu_i) : i \in \mathbb{I}\}$ and $\mathcal{W} = \{(W_i, \omega_i) : i \in \mathbb{I}\}$ respectively, then

$$T_{\mathcal{W}} V T_{\mathcal{M}}^* f = \sum_{i\in\mathbb{I}} \sum_{k\in\mathbb{J}_i} \langle f, \nu_i \pi_{M_i} f_{ik}\rangle (\omega_i \pi_{W_i} \tilde{f_{ik}}).$$

Hence the last term is equal to Kf for all $f \in H$ if and only if $\{\omega_i \tilde{f_{ik}}\}_{i\in\mathbb{I},k\in\mathbb{J}_i}$ is a dual K-frame of $\{\nu_i f_{ik}\}_{i\in\mathbb{I},k\in\mathbb{J}_i}$. □

Remark 4.4 : (a) Proposition 4.3 can be seen as a generalization of Theorem 4.3 in [9], where dual frames are related with dual fusion frame systemes.

(b) If we let $V = id_{\ell^2(H,\mathbb{I})}$ in Definition 4.2 then

$$Kf = \sum_{i\in\mathbb{I}} \omega_i \nu_i \pi_{W_i} \pi_{M_i} f, \tag{4.2}$$

for all $f \in H$. In this case we call $\{(W_i, \omega_i) : i \in \mathbb{I}\}$ an id-dual K-fusion frame of $\{(M_i, \nu_i) : i \in \mathbb{I}\}$. □

Proposition 4.5 — Let $\mathcal{M} = \{(M_i, \nu_i) : i \in \mathbb{I}\}$ be a K-fusion frame for H and $\mathcal{W} = \{(W_i, \omega_i) : i \in \mathbb{I}\}$ be an id-dual K-fusion frame of $\mathcal{M} = \{(M_i, \nu_i) : i \in \mathbb{I}\}$ with synthesis operator $T_{\mathcal{W}}$. Suppose that $\overline{Ran}(T^*_{\mathcal{W}})$ is orthogonally complemented, then $\mathcal{W} = \{(W_i, \omega_i) : i \in \mathbb{I}\}$ is also a K-fusion frame for H. Moreover $\mathcal{M} = \{(M_i, \nu_i) : i \in \mathbb{I}\}$ and $\mathcal{W} = \{(W_i, \omega_i) : i \in \mathbb{I}\}$ in (4.2) can be interchanged if and only if $K = K^*$.

PROOF : Denote by D the fusion Bessel bound of $\mathcal{M} = \{(M_i, \nu_i) : i \in \mathbb{I}\}$, then for any $f \in H$ we have

$$\begin{aligned}
\|K^* f\|^2 &= sup_{h\in H,\|h\|=1} \|\langle K^* f, h\rangle\|^2 \\
&= sup_{h\in H,\|h\|=1} \|\sum_{i\in\mathbb{I}} \langle \omega_i \pi_{W_i}(f), \nu_i \pi_{M_i}(h)\rangle\|^2 \\
&\leq sup_{h\in H,\|h\|=1} \|\sum_{i\in\mathbb{I}} \langle \omega_i \pi_{W_i}(f), \omega_i \pi_{W_i}(f)\rangle\| \|\sum_{i\in\mathbb{I}} \langle \nu_i \pi_{M_i}(h), \nu_i \pi_{M_i}(h)\rangle\| \\
&\leq D \|\sum_{i\in\mathbb{I}} \omega_i^2 \langle \pi_{W_i}(f), \pi_{W_i}(f)\rangle\|,
\end{aligned}$$

by Proposition 3.7, $\mathcal{W} = \{(W_i, \omega_i) : i \in \mathbb{I}\}$ is a K-fusion frame for H. The " Moreover" part follows immediately from the observation that for all $f, g \in H$, we have

$$\langle K^* f, g\rangle = \langle \sum_{i\in\mathbb{I}} \omega_i \nu_i \pi_{M_i} \pi_{W_i}(f), g\rangle. \square$$

5. CONCLUSIONS

We introduced K-fusion frames on a Hilbert C^*-module H, characterized tight K-fusion frames for H,and studied some aspects of duality in this setting.

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