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EXISTENCE OF POSITIVE SOLUTIONS FOR AN ELLIPTIC SYSTEM INVOLVING NONLOCAL OPERATOR

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Using the method of sub-super solutions and comparison principle, we study the existence of positive solutions for a class of quasilinear elliptic systems with sign-changing weights involving nonlocal operator.

Key words : Kirchhoff-type systems; sub-super solutions; comparison principle; sign-changing weights.

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1. Introduction

In this paper, we consider the existence of positive solutions for a class of quasilinear elliptic systems of the form

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = a(x)\left(\lambda u^\alpha v^\gamma + \mu f(u)\right) & \text{in } \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\Delta_q v = b(x)\left(\lambda u^\delta v^\beta + \mu g(v)\right) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < p, q < N$, $\alpha, \beta, \gamma, \delta$ are constants and λ, μ are positive parameters.

Problems involving the p-Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [7]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids. The structure of positive solutions for quasilinear reaction-diffusion systems (nonlinear Newtonian filtration systems) and semilinear reaction-diffusion systems (Newtonian filtration systems) is a front topic in the study of static electric fields in dielectric media, in which the potential is described by the boundary value problem of a static non-Newtonian filtration system, called the Poisson Boltzmann problem. This kind of problems also appears in the study of the non-Newtonian or Newtonian turbulent filtration in porous media and so on, which have extensive engineering background. In recent years, problems involving Kirchhoff type operators have been studied in many papers; we refer to [2, 4, 5, 6,13, 14, 15], in which the authors have used variational and topological methods to get the existence of solutions for (1.1). The main tool used in this study is the method of sub- and supersolutions. Our result in this note improves the previous one [8] in which $M_1(t) = M_2(t) \equiv 1$. We emphasize that it is really necessary to impose the boundedness of the Kirchhoff functions M_i, $i = 1, 2$. To our best knowledge, this is a new research topic for nonlocal problems; see [2, 3, 9].

In this paper, we denote by $W_0^{1,r}(\Omega)$ $(1 \leq r < \infty)$ the completion of $C_0^\infty(\Omega)$, with respect to the norm

$$\|u\|_r = \left(\int_\Omega |\nabla u|^r \, dx \right)^{\frac{1}{r}}.$$

Let us consider the following eigenvalue problem for the r-Laplace operator $-\Delta_r u$, see [11]:

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Let $\phi_{1,r} \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (1.2) such that $\phi_{1,r} > 0$ in Ω and $\|\phi_{1,r}\|_\infty = 1$. It can be shown that $\frac{\partial \phi_{1,r}}{\partial \nu} < 0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants m, η, σ_r such that

$$\begin{cases} |\nabla \phi_{1,r}|^r - \lambda_{1,r} \phi_{1,r}^r \geq m & \text{on } \overline{\Omega}_\eta, \\ \phi_{1,r} \geq \sigma_r & \text{on } \Omega \setminus \overline{\Omega}_\eta, \end{cases}$$

where $\overline{\Omega}_\eta := \{x \in \Omega : \ d(x, \partial\Omega) \leq \eta\}$.

We also consider the unique solution $e_r \in W_0^{1,r}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_r e_r = 1 & \text{in } \Omega, \\ e_r = 0 & \text{on } \partial\Omega, \end{cases}$$

to discuss our result. It is known that $e_r > 0$ in Ω and $\frac{\partial e_r}{\partial \nu} < 0$ on $\partial\Omega$.

Here we assume that the weight functions $a(x)$ and $b(x)$ take negative values in a subset of $\overline{\Omega}_\eta$, but require $a(x)$ and $b(x)$ be strictly positive in $\Omega \setminus \overline{\Omega}_\eta$. To be precise we assume that there exist positive constants a_0, a_1, b_0 and b_1 such that $a(x) \geq -a_0$, $b(x) \geq b_0$ on $\overline{\Omega}_\eta$ and $a(x) \geq a_1$, $b(x) \geq b_1$ on $\Omega \setminus \overline{\Omega}_\eta$.

2. Existence of Positive Solutions

We will study the existence of positive solutions by using the method of sub- and supersolutions. A pair of functions (ψ_1, ψ_2) is said to be a subsolution of problem (1.1) if it is in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx \leq \int_\Omega a(x)\left(\lambda\psi_1^\alpha\psi_2^\gamma + \mu f(\psi_1)\right)w\,dx, \quad \forall w \in W,$$

and

$$M_2\left(\int_\Omega |\nabla\psi_2|^p\,dx\right)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\,dx \leq \int_\Omega b(x)\big(\lambda\psi_1^\delta\psi_2^\beta + \mu g(\psi_2)\big)w\,dx, \quad \forall w \in W,$$

where $W := \{w \in C_0^\infty(\Omega) :\ w \geq 0 \text{ in } \Omega\}$. A pair of functions $(z_1, z_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is said to be a supersolution if

$$M_2\left(\int_\Omega |\nabla z_1|^p\,dx\right)\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx \geq \int_\Omega a(x)\left(\lambda z_1^\alpha z_2^\gamma + \mu f(z_1)\right)w\,dx, \quad \forall w \in W,$$

and

$$M_2\left(\int_\Omega |\nabla z_2|^p\,dx\right)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx \geq \int_\Omega b(x)\big(\lambda z_1^\delta z_2^\beta + \mu g(z_2)\big)w\,dx, \quad \forall w \in W.$$

Lemma 2.1 — (see [10]). Suppose that there exist sub- and supersolutions (ψ_1, ψ_2) and (z_1, z_2), respectively, of problem (1.1) such that $(\psi_1, \psi_2) \leq (z_1, z_2)$. Then problem (1.1) has a solution (u, v) such that $\psi_1 \leq u \leq z_1$ and $\psi_2 \leq v \leq z_2$.

Let us make the following hypotheses on problem (1.1):

$(H1)$ $\alpha, \beta \geq 0$, $\gamma, \delta > 0$ and $\theta = (p-1-\alpha)(q-1-\beta) - \gamma\delta > 0$.

$(H2)$ There exist positive constants a_0, a_1, b_0 and b_1 such that $a(x) \geq -a_0$, $b(x) \geq b_0$ on $\overline{\Omega}_\eta$ and $a(x) \geq a_1$, $b(x) \geq b_1$ on $\Omega \setminus \overline{\Omega}_\eta$.

$(H3)$ $M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$, $i = 1, 2$, are two continuous and increasing functions and $0 < m_i \leq M_i(t)$ for all $t \in \mathbb{R}_0^+$.

$(H4)$ Suppose that there exists $\epsilon > 0$ such that

$$\min\left\{\frac{m}{2a_0\epsilon^{d_1-1}}, \frac{m}{2b_0\epsilon^{d_2-1}}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\} \leq \max\left\{\frac{\lambda_{1,p}}{2c_1\epsilon^{d_1-1}}, \frac{\lambda_{1,q}}{2c_2\epsilon^{d_2-1}}\right\},$$

where c_1, c_2, d_1 and d_2 are some positive constants that will be determined in the proof of the main result (Theorem 2.1), and

$$\min\left\{\frac{m\epsilon}{2a_0 f\left(\epsilon^{\frac{1}{p-1}}\right)}, \frac{m\epsilon}{2b_0 g\left(\epsilon^{\frac{1}{q-1}}\right)}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\} \geq \max \left\{\frac{\lambda_{1,p}\epsilon}{2a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)}, \frac{\lambda_{1,q}\epsilon}{2b_1 g\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)}\right\}.$$

$(H5)$ $f, g \in C^1([0,+\infty),[0,+\infty))$ are increasing and homomorphism such that $\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} g(t) = +\infty$.

$(H6)$ $\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = \lim_{t\to+\infty} \frac{g(t)}{t^{q-1}} = 0$.

The following result plays an important role in our arguments, we refer the interested readers to [1, 9, 17] for details.

Lemma 2.2 — Assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function satisfying

$$M(t) \geq M_0 > 0 \text{ for all } t \in \mathbb{R}^+.$$

If the functions $u, v \in W_0^{1,p}(\Omega)$ satisfies

$$M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \leq M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \quad (2.1)$$

for all $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$, then $u \leq v$ in Ω.

PROOF : Our proof is based on the arguments presented in [18, 19]. Define the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ by the formula

$$\Phi(u) := \frac{1}{p}M\left(\int_\Omega |\nabla u|^p\,dx\right), \quad u \in W_0^{1,p}(\Omega).$$

It is obvious that the functional Φ is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in W_0^{1,p}(\Omega)$ is the functional $\Phi' \in W_0^{-1,p}(\Omega)$, given by

$$\Phi'(u)(\varphi) = M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx, \quad \varphi \in W_0^{1,p}(\Omega).$$

Clearly, Φ' is continuous and bounded, since the function M is continuous. We will show that Φ' is strictly monotone in $W_0^{1,p}(\Omega)$. Indeed, for any $u, v \in W_0^{1,p}(\Omega)$, $u \neq v$, without loss of generality, we may assume that

$$\int_\Omega |\nabla u|^p \, dx \geq \int_\Omega |\nabla v|^p \, dx$$

(otherwise, changing the role of u and v in the following proof). Therefore, we have

$$M\left(\int_\Omega |\nabla u|^p \, dx\right) \geq M\left(\int_\Omega |\nabla v|^p \, dx\right), \tag{2.2}$$

since M is a monotone function. Using Cauchy's inequality, we have

$$\nabla u \cdot \nabla v \leq |\nabla u||\nabla v| \leq \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2). \tag{2.3}$$

Using (2.15), we get

$$\int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx \geq \frac{1}{2}\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2) \, dx, \tag{2.4}$$

and

$$\int_\Omega |\nabla v|^p \, dx - \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u \, dx \geq \frac{1}{2}\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2 - |\nabla u|^2) \, dx. \tag{2.5}$$

If $|\nabla u| \geq |\nabla v|$, using (2.14)-(2.17) we have

$$\begin{aligned}
I_1 &:= \Phi'(u)(u) - \Phi'(u)(v) - \Phi'(v)(u) + \Phi'(v)(v) \\
&= M\left(\int_\Omega |\nabla u|^p \, dx\right)\left(\int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx\right) \\
&\quad - M\left(\int_\Omega |\nabla v|^p \, dx\right)\left(\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u \, dx - \int_\Omega |\nabla v|^p \, dx\right) \\
&\geq \frac{1}{2}M\left(\int_\Omega |\nabla u|^p \, dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2) \, dx \\
&\quad - \frac{1}{2}M\left(\int_\Omega |\nabla v|^p \, dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - |\nabla v|^2) \, dx \\
&= \frac{1}{2}M\left(\int_\Omega |\nabla v|^p \, dx\right)\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2) \, dx \\
&\geq \frac{M_0}{2}\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2) \, dx.
\end{aligned} \tag{2.6}$$

If $|\nabla v| \geq |\nabla u|$, changing the role of u and v in (2.14)-(2.17) we have

$$\begin{aligned}
I_2 &:= \Phi'(v)(v) - \Phi'(v)(u) - \Phi'(u)(v) + \Phi'(u)(u) \\
&= M\left(\int_\Omega |\nabla v|^p\,dx\right)\left(\int_\Omega |\nabla v|^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u\,dx\right) \\
&\quad - M\left(\int_\Omega |\nabla u|^p\,dx\right)\left(\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx - \int_\Omega |\nabla u|^p\,dx\right) \\
&\geq \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}(|\nabla v|^2 - |\nabla u|^2)\,dx \\
&\quad - \frac{1}{2}M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}(|\nabla v|^2 - |\nabla u|^2)\,dx \\
&= \frac{1}{2}M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega (|\nabla v|^{p-2} - |\nabla u|^{p-2})(|\nabla v|^2 - |\nabla u|^2)\,dx \\
&\geq \frac{M_0}{2}\int_\Omega (|\nabla v|^{p-2} - |\nabla u|^{p-2})(|\nabla v|^2 - |\nabla u|^2)\,dx.
\end{aligned} \tag{2.7}$$

From (2.18) and (2.19) we have

$$\big(\Phi'(u) - \Phi'(v)\big)(u-v) = I_1 = I_2 \geq 0, \quad \forall u, v \in W_0^{1,p}(\Omega).$$

Moreover, if $u \neq v$ and $\big(\Phi'(u) - \Phi'(v)\big)(u-v) = 0$, then we have

$$\int_\Omega (|\nabla u|^{p-2} - |\nabla v|^{p-2})(|\nabla u|^2 - |\nabla v|^2)\,dx = 0,$$

so $|\nabla u| = |\nabla v|$ in Ω. Thus, we deduce that

$$\begin{aligned}
\big(\Phi'(u) - \Phi'(v)\big)(u-v) &= \Phi'(u)(u-v) - \Phi'(v)(u-v) \\
&= M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}|\nabla u - \nabla v|^2\,dx \\
&= 0,
\end{aligned} \tag{2.8}$$

i.e., $u - v$ is a constant. In view of $u = v = 0$ on $\partial\Omega$ we have $u \equiv v$, which is contrary with $u \neq v$. Therefore $\big(\Phi'(u) - \Phi'(v)\big)(u-v) > 0$ and Φ' is strictly monotone in $W_0^{1,p}(\Omega)$.

Let u, v be two functions such that (2.13) is verified. Taking $\varphi = (u-v)^+$, the positive part of $u - v$, as a test function of (2.13), we have

$$\begin{aligned}
(\Phi'(u) - \Phi'(v))(\varphi) &= M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \\
&\quad - M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \\
&\leq 0.
\end{aligned} \tag{2.9}$$

Relations (2.9) and (2.20) mean that $u \leq v$. □

From Lemma 2.1 we can establish the basic principle of the sub- and supersolutions method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = h(x,u,v) & \text{in } \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\Delta_q v = k(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $h, k : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

$(HK1)$ $h(x,s,t)$ and $k(x,s,t)$ are Carathéodory functions and they are bounded if s, t belong to bounded sets.

$(HK2)$ There exists a function $g : \mathbb{R} \to \mathbb{R}$ being continuous, nondecreasing, with $g(0) = 0$, $0 \leq g(s) \leq C(1 + |s|^{\min\{p,q\}-1})$ for some $C > 0$, and applications $s \mapsto h(x,s,t) + g(s)$ and $t \mapsto k(x,s,t) + g(t)$ are nondecreasing, for a.e. $x \in \Omega$.

If $u, v \in L^\infty(\Omega)$, with $u(x) \leq v(x)$ for a.e. $x \in \Omega$, we denote by $[u,v]$ the set $\{w \in L^\infty(\Omega) : u(x) \leq w(x) \leq v(x) \text{ for a.e. } x \in \Omega\}$. Using Lemma 2.2 and the method as in the proof of Theorem 2.4 of [16] (see also Section 4 of [10]), we can establish a version of the abstract sub- and supersolution method for our class of the operators as follows.

Proposition 2.3 — Let $M_1, M_2 : \mathbb{R}_0^+ \to \mathbb{R}^+$ be two functions satisfying the condition $(H3)$. Assume that the functions h, k satisfy the conditions $(HK1)$ and $(HK2)$. Assume that $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ are, respectively, a weak subsolution and a weak supersolution of system (2.15) with $\underline{u}(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq \overline{v}(x)$ for a.e. $x \in \Omega$. Then there exists a minimal (u_*, v_*) (and, respectively, a maximal (u^*, v^*)) weak solution for system (2.15) in the set $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. In particular, every weak solution $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of system (2.15) satisfies $u_*(x) \leq u(x) \leq u^*(x)$ and $v_*(x) \leq v(x) \leq v^*(x)$ for a.e. $x \in \Omega$.

Now we are ready to state our existence result.

***Theorem* 2.4** — *Let $(H1)$-$(H6)$ hold. Then, there exists a positive solution of (1.1) for every $\lambda \in [\underline{\lambda}(\epsilon), \overline{\lambda}(\epsilon)]$ and $\mu \in [\underline{\mu}(\epsilon), \overline{\mu}(\epsilon)]$, where*

$$\overline{\lambda}(\epsilon) = \min\left\{\frac{m}{2a_0\epsilon^{d_1-1}}, \frac{m}{2b_0\epsilon^{d_2-1}}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\},$$

$$\underline{\lambda}(\epsilon) = \max\left\{\frac{\lambda_{1,p}}{2c_1\epsilon^{d_1-1}}, \frac{\lambda_{1,q}}{2c_2\epsilon^{d_2-1}}\right\},$$

$$\overline{\mu}(\epsilon) = \min\left\{\frac{m\epsilon}{2a_0 f\left(\epsilon^{\frac{1}{p-1}}\right)}, \frac{m\epsilon}{2b_0 g\left(\epsilon^{\frac{1}{q-1}}\right)}, \frac{1}{\|a\|_\infty}, \frac{1}{\|b\|_\infty}\right\},$$

$$\underline{\mu}(\epsilon) = \max\left\{\frac{\lambda_{1,p}\epsilon}{2a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)}, \frac{\lambda_{1,q}\epsilon}{2b_1 g\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)}\right\}.$$

PROOF : Let

$$(\psi_1, \psi_2) = \left(\frac{p-1}{pm_1}\epsilon^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p}{p-1}}, \ \frac{q-1}{qm_2}\epsilon^{\frac{1}{q-1}}\phi_{1,q}^{\frac{q}{q-1}}\right).$$

We shall verify that (ψ_1, ψ_2) is a sub-solution of problem (1.1). Let $w \in W$. Then a calculation shows that

$$\begin{aligned} M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right) & \int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx = M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\frac{\epsilon}{m_1}\int_\Omega |\nabla\phi_{1,p}|^{p-2}\phi_{1,p} \\ & \nabla\phi_{1,p}\cdot\nabla w\,dx \le \epsilon\left[\int_\Omega |\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\cdot\nabla(\phi_{1,p}w)\,dx - \int_\Omega |\nabla\phi_{1,p}|^p w\,dx\right] \\ & \le \epsilon\int_\Omega\left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] w\,dx. \end{aligned} \tag{2.11}$$

A similar calculation shows that

$$\begin{aligned} & M_2\left(\int_\Omega |\nabla\psi_2|^p\,dx\right)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_1\cdot\nabla w\,dx = M_2\left(\int_\Omega |\nabla\psi_2|^p\,dx\right) \\ & \frac{\epsilon}{m_2}\int_\Omega\left[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q\right] w\,dx. \end{aligned} \tag{2.12}$$

First we consider the case when $x \in \overline{\Omega}_\eta$. We have $|\nabla\phi_{1,r}|^r - \lambda_{1,r}\phi_{1,r}^r \le m$ on $\overline{\Omega}_\eta$ for $r = p, q$. Since $\lambda \le \overline{\lambda}(\epsilon)$ and $\mu \le \overline{\mu}(\epsilon)$ we have $\lambda \le \frac{m}{2a_0\epsilon^{d_1-1}}$ and $\mu \le \frac{m\epsilon}{2a_0 f(\epsilon^{\frac{1}{p-1}})}$. Hence,

$$-\frac{m\epsilon}{2} \le -\lambda a_0\epsilon^{\frac{\alpha}{p-1}+\frac{\gamma}{q-1}}, \qquad -\frac{m\epsilon}{2} \le -\mu a_0 f(\epsilon^{\frac{1}{p-1}}),$$

and

$$\begin{aligned} & \epsilon\left[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p\right] \\ & \le -m\epsilon \le -a_0\left(\lambda\epsilon^{\frac{\alpha}{p-1}+\frac{\gamma}{q-1}} + \mu f(\epsilon^{\frac{1}{p-1}})\right) \\ & \le -a_0\left\{\lambda\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}||\phi_{1,p}||_\infty^{\frac{p}{p-1}}\right)^\alpha\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}||\phi_{1,q}||_\infty^{\frac{q}{q-1}}\right)^\gamma + \mu f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p}{p-1}}\right)\right\} \\ & \le a(x)\left(\lambda\psi_1^\alpha\psi_2^\gamma + \mu f(\psi_1)\right). \end{aligned} \tag{2.13}$$

A similar argument shows that

$$\epsilon\left[\lambda_{1,q}\phi_{1,q}^{q}-|\nabla\phi_{1,q}|^{q}\right]\leq b(x)\left(\lambda\psi_1^{\delta}\psi_2^{\beta}+\mu g(\psi_2)\right). \tag{2.14}$$

On the other hand, on $\Omega\setminus\overline{\Omega}_\eta$, we note that $\phi_{1,r}\geq\sigma_r$ for $r=p,q$, $a(x)\geq a_1$, $b(x)\geq b_1$. Since $\lambda\geq\underline{\lambda}(\epsilon)$ and $\mu\geq\underline{\mu}(\epsilon)$, we have $\lambda\geq\frac{\lambda_{1,p}}{2c_1\epsilon^{d_1-1}}$ and $\mu\geq\frac{\lambda_{1,p}\epsilon}{2a_1 f\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)}$. Hence,

$$\frac{1}{2}\epsilon\lambda_{1,p}\leq\lambda a_1\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)^{\alpha}\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)^{\gamma},$$

$$\frac{1}{2}\epsilon\lambda_{1,p}\leq\mu a_1 f\Big(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\Big),$$

and

$$\begin{aligned} &M_1\left(\int_\Omega|\nabla\psi_1|^p\,dx\right)\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx\\ &\leq\epsilon\left[\int_\Omega|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\cdot\nabla(\phi_{1,p}w)\,dx-\int_\Omega|\nabla\phi_{1,p}|^{p}w\,dx\right]\\ &\leq\epsilon\int_\Omega\left[\lambda_{1,p}\phi_{1,p}^{p}-|\nabla\phi_{1,p}|^{p}\right]w\,dx. \end{aligned} \tag{2.15}$$

So,

$$\begin{aligned} \epsilon\left[\lambda_{1,p}\phi_{1,p}^{p}-|\nabla\phi_{1,p}|^{p}\right] &\leq \epsilon\lambda_{1,p}\phi_{1,p}^{p}\\ &\leq \epsilon\lambda_{1,p}\|\phi_{1,p}\|_\infty^{p}\\ &\leq \epsilon\lambda_{1,p}\\ &\leq \lambda a_1\left(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\right)^{\alpha}\left(\frac{q-1}{q}\epsilon^{\frac{1}{q-1}}\sigma_q^{\frac{q}{q-1}}\right)^{\gamma}+\mu a_1 f\Big(\frac{p-1}{p}\epsilon^{\frac{1}{p-1}}\sigma_p^{\frac{p}{p-1}}\Big)\\ &\leq a(x)\left(\lambda\psi_1^{\alpha}\psi_2^{\gamma}+\mu f(\psi_1)\right), \end{aligned} \tag{2.16}$$

on $\Omega\setminus\overline{\Omega}_\eta$. A similar argument shows that

$$\epsilon\left[\lambda_{1,q}\phi_{1,q}^{q}-|\nabla\phi_{1,q}|^{q}\right]\leq b(x)\left(\lambda\psi_1^{\delta}\psi_2^{\beta}+\mu g(\psi_2)\right), \tag{2.17}$$

on $\Omega\setminus\overline{\Omega}_\eta$. From (2.12)-(2.18) we have

$$M_1\left(\int_\Omega|\nabla\psi_1|^p\,dx\right)\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\,dx\leq\int_\Omega a(x)\left(\lambda\psi_1^{\alpha}\psi_2^{\gamma}+\mu f(\psi_1)\right)w\,dx,\qquad\forall w\in W,$$

and

$$M_2\left(\int_\Omega|\nabla\psi_2|^p\,dx\right)\int_\Omega|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\,dx\leq\int_\Omega b(x)\left(\lambda\psi_1^{\delta}\psi_2^{\beta}+\mu g(\psi_2)\right)w\,dx,\qquad\forall w\in W,$$

where $W := \{w \in C_0^\infty(\Omega) : \ w \geq 0 \text{ in } \Omega\}$. Therefore, (ψ_1, ψ_2) is a sub-solution of problem (1.1).

Now, we will construct a supersolution (z_1, z_2) of problem (1.1). We denote $z_1(x) = Ae_p(x)$ and $z_2(x) = Be_q(x)$, where $A, B > 1$ are large and to be chosen later. It is clear that

$$-\Delta_p z_1 = A, \qquad -\Delta_q z_2 = B, \qquad \text{on } \Omega. \tag{2.18}$$

Then, we have

$$\begin{aligned}
& M_1\left(\int_\Omega |\nabla z_1|^p \, dx\right) \int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla w \, dx \\
& = A^{p-1} M_1\left(\int_\Omega |\nabla z_1|^p \, dx\right) \int_\Omega |\nabla e_p|^{p-2} \nabla e_p \cdot \nabla w \, dx \\
& = A^{p-1} M_1\left(\int_\Omega |\nabla z_1|^p \, dx\right) \int_\Omega w \, dx \\
& \geq m_1 A^{p-1} \int_\Omega w \, dx,
\end{aligned}$$

and

$$\begin{aligned}
& M_2\left(\int_\Omega |\nabla z_2|^q \, dx\right) \int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla w \, dx \\
& = B^{q-1} M_2\left(\int_\Omega |\nabla z_2|^q \, dx\right) \int_\Omega |\nabla e_q|^{q-2} \nabla e_q \cdot \nabla w \, dx \\
& = B^{q-1} M_2\left(\int_\Omega |\nabla z_2|^q \, dx\right) \int_\Omega w \, dx \\
& \geq m_2 B^{q-1} \int_\Omega w \, dx.
\end{aligned}$$

Since $\lambda \leq \overline{\lambda}(\epsilon)$ and $\mu \leq \overline{\mu}(\epsilon)$, we have $\lambda \leq \frac{1}{\|a\|_\infty}$, $\lambda \leq \frac{1}{\|b\|_\infty}$, $\mu \leq \frac{1}{\|a\|_\infty}$ and $\mu \leq \frac{1}{\|b\|_\infty}$. Let $l_p = \|e_p\|_\infty$ and $l_q = \|e_q\|_\infty$. Since $\theta > 0$, there exist positive large constants $A, B > 1$ such that

$$A \geq \left(\frac{2B^\gamma l_p^\alpha l_q^\gamma}{m_1}\right)^{\frac{1}{p-1-\alpha}}, \tag{2.19}$$

and

$$B \geq \left(\frac{2A^\delta l_p^\delta l_q^\beta}{m_2}\right)^{\frac{1}{q-1-\beta}}. \tag{2.20}$$

Moreover, from the hypothesis $(H6)$, for $A, B > 1$ large enough we have

$$\frac{m_1}{2} A^{p-1} \geq f(A l_p), \tag{2.21}$$

and

$$\frac{m_2}{2}B^{q-1} \geq g(Bl_q). \tag{2.22}$$

Then we can chose $A, B > 1$ large enough such that

$$\begin{aligned} m_1 A^{p-1} &\geq (Al_p)^\alpha (Bl_q)^\gamma + f(Al_p) \\ &\geq \lambda \|a\|_\infty (Al_p)^\alpha (Bl_q)^\gamma + \mu \|a\|_\infty f(Al_p) \\ &\geq a(x) \left(\lambda z_1^\alpha z_2^\gamma + \mu f(z_1)\right), \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} m_2 B^{q-1} &\geq (Al_p)^\delta (Bl_q)^\beta + g(Bl_q) \\ &\geq \lambda \|a\|_\infty (Al_p)^\delta (Bl_q)^\beta + \mu \|a\|_\infty g(Bl_q) \\ &\geq b(x)\big(\lambda z_1^\delta z_2^\beta + \mu g(z_2)\big), \end{aligned} \tag{2.24}$$

for all $x \in \Omega$.

From (2.19)-(2.25), we have

$$M_1\left(\int_\Omega |\nabla z_1|^p\, dx\right) \int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla w\, dx \geq \int_\Omega a(x) \left(\lambda z_1^\alpha z_2^\gamma + \mu f(z_1)\right) w\, dx, \qquad \forall w \in W,$$

and

$$M_2\left(\int_\Omega |\nabla z_2|^q\, dx\right) \int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla w\, dx \geq \int_\Omega b(x)\big(\lambda z_1^\delta z_2^\beta + \mu g(z_2)\big) w\, dx, \qquad \forall w \in W.$$

Then (z_1, z_2) is a supersolution of problem (1.1). Obviously, $\psi_i(x) \leq z_i(x)$ for all $x \in \Omega$, $i = 1, 2$, with $A, B > 1$ large enough. Thus, by the comparison principle, there exists a solution (u, v) of problem (1.1) such that $\psi_1 \leq u \leq z_1$ and $\psi_2 \leq v \leq z_2$. This completes the proof of Theorem 2.1.

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