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# $*$-JORDAN SEMI-TRIPLE DERIVABLE MAPPINGS

Lin Chen*,** and Jianhua Zhang*

*College of Mathematics and Information Science, Shaanxi Normal University, Xian, 710062, China

**Department of Mathematics and Physics, Anshun University, Anshun 561000 China

e-mails: Linchen198112@163.com; jhzhang@snnu.edu.cn

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In this paper, we characterize the $*$-Jordan semi-triple derivable mappings (i.e. a mapping $\Phi$ from $*$ algebra $\mathcal{A}$ into $\mathcal{A}$ satisfying $\Phi(AB^*A) = \Phi(A)B^*A + A\Phi(B)^*A + AB^*\Phi(A)$ for every $A, B \in \mathcal{A}$) in the finite dimensional case and infinite dimensional case.

**Key words** : Jordan semi-triple derivable mapping; derivation; matrix algebra.

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## 1. INTRODUCTION

It is a surprising result of Martindale [16] that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. This result was utilized by Šemrl in [20] to describe the form of the semigroup isomorphisms of standard operator algebras on Banach spaces. Some other results on the additivity of multiplicative mappings between operator algebras can be found in [3, 5, 13, 14, 17, 18]. Besides additivity of multiplicative mappings, additivity of derivable mappings is also an interesting problem.

Let $\mathcal{A}$ be an algebra. Recall that a mapping $\phi$ from $\mathcal{A}$ into $\mathcal{A}$ is called a *derivable mapping* if $\phi(AB) = \phi(A)B + A\phi(B)$ for all $A, B \in \mathcal{A}$ and a *Jordan derivable mapping* if $\phi(AB + BA) = \phi(A)B + A\phi(B) + \phi(B)A + B\phi(A)$ for all $A, B \in \mathcal{A}$. We say that additive derivable mappings are additive derivations, and additive Jordan derivable mappings are additive Jordan derivations. Lu [15] showed that each Jordan derivable mapping of a 2-torsion free prime ring containing a nontrivial idempotent is also additive. Let $[A, B] = AB - BA$ be the usual Lie product of $A$ and $B$. Recall that a mapping $\phi$ from $\mathcal{A}$ into $\mathcal{A}$ is called a *Lie derivable mapping* if $\phi([A, B]) = [\phi(A), B] + [A, \phi(B)]$ for all $A, B \in \mathcal{A}$. Lu [12] gave a characterization of Lie derivable mapping of operator algebra on Banach space. Some other results on derivable mappings can be found in [1, 7, 11].

In recent years, the additivity of *-derivable mappings has attracted the attentions of many researchers. Let $\mathcal{A}$ be an algebra with involution, a mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a $*$*-Lie derivable mapping* if for any $A, B \in \mathcal{A}$, $\phi([A, B]_*) = [\phi(A), B]_* + [A, \phi(B)]_*$, where $[A, B]_* = AB - BA^*$ is the skew Lie product of $A$ and $B$. In [21] Yu and Zhang showed that every $*$-Lie derivable mapping from a factor von Neumann algebra on an infinite dimensional complex Hilbert space into itself is an additive $*$-derivation. In [10], Li, Lu and Fang arrived the same conclusion on von Neumann algebra without central abelian projections. Jing [8] proved that every $*$-Lie derivable mapping of standard operator algebra on complex Hilbert space is an inner $*$-derivation. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a $*$*-Jordan triple multiplicative mapping* if for any$A, B \in \mathcal{A}$, $\phi(AB^*A) = \phi(A)\phi(B)^*\phi(A)$. In [3], Gao gave a full characterization of $*$-Jordan triple multiplicative surjective mappings. Notice that the operator algebras in above papers are all on complex Hilbert space, how about on real space?

In [1] a mapping $\phi$ satisfying $\phi(ABA) = \phi(A)\phi(B)\phi(A)$ is called a *Jordan semi-triple mapping*. Molnár showed in [17] that in the case of standard operator algebras acting on infinite dimensional Banach spaces every bijective semi-triple mapping is additive. Later, Lu in [13, 14] presented a purely algebraic proof. Gorazd Lešnjak and Nung-Sing Sze [4] gave a characterization of injective Jordan semi-triple mapping on matrix algebra $M_n(\mathbb{F})$ with entries in a field $\mathbb{F}$. Du and Zhang in [2] gave a characterization of *Jordan semi-triple derivable mapping* (i.e. a mapping $\phi$ satisfying $\phi(ABA) = \phi(A)BA + A\phi(B)A + AB\phi(A)$) on matrix algebra over 2-torsion free commutative ring with unity.

In this paper, motivated by [2-4], we follow this line of investigation and consider $*$*-Jordan semi-triple derivable mapping* (i.e. a mapping $\phi$ satisfying $\phi(AB^*A) = \phi(A)B^*A + A\phi(B)^*A + AB\phi(A)^*$). We shall give a full characterization of a $*$-Jordan semi-triple derivable mapping on matrix algebra over 2-torsion free commutative real ring with unity and on operator algebra $B(\mathcal{H})$ respectively.

Let us fix some notation. Throughout this paper, $\mathbb{R}$ denote 2-torsion free commutative real ring, $M_n(\mathbb{R})$ ($n \geq 2$) denote the algebra of $n \times n$ matrices over $\mathbb{R}$. For any $1 \leq j, k \leq n$ we write $E_{jk}$ for the matrix having 1 as its $(j, k)$th entry and zeros elsewhere. For a matrix $A \in M_n(\mathbb{R})$ and a homomorphism $\varphi$ of $\mathbb{R}$, let $A_\varphi$ be the matrix obtained by applying $\varphi$ entrywise, i.e. $[A_\varphi]_{jk} = \varphi(a_{jk})$.

Let $\mathcal{H}$ be a (real or complex) Hilbert space and denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$.

## 2. Main Results

Before giving the main results we collect some easy verifiable facts about $*$-Jordan semi-triple derivable mappings.

*Lemma* 2.1 — If $\Phi : \mathcal{A} \to \mathcal{A}$ is a $*$-Jordan semi-triple derivable mapping, then

(1) $\Phi(0) = 0$;

(2) $\Phi(I) = -\Phi(I)^*$.

Proof : In fact, take $A = B = 0$, we get $\Phi(0) = 0$. Take $A = B = I$, we have $\Phi(I) = \Phi(II^*I) = \Phi(I)I^*I + I\Phi(I)^*I + II^*\Phi(I) = 2\Phi(I) + \Phi(I)^*$, hence $\Phi(I) = -\Phi(I)^*$. □

*Lemma* 2.2 — Let $\mathbb{R}$ be a 2-torsion free commutative real ring with unity, and $M_2(\mathbb{R})$ be the algebra of $2 \times 2$ matrices over $\mathbb{R}$. If $\Phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a $*$-Jordan semi-triple derivable mapping (here $*$ denote the transpose), then there exist $T \in M_2(\mathbb{R})$, $T^* = -T$, and an additive derivation $\varphi$ of $\mathbb{R}$ such that

$$\Phi(A) = \frac{1}{2}\Phi(I)A + \frac{1}{2}A\Phi(I) + AT - TA + A_\varphi$$

for all $A \in M_n(\mathbb{R})$, where $A_\varphi = (\varphi(a_{ij}))$.

Proof : Suppose $\Phi(E_{11}) = (a_{ij})$, $\Phi(E_{12}) = (b_{ij})$, $\Phi(E_{21}) = (c_{ij})$, $\Phi(E_{22}) = (d_{ij})$, $a_{ij}, b_{ij}, c_{i,j}$, $d_{ij} \in \mathbb{R}$, $1 \leq i, j \leq 2$. Since $\Phi(E_{11}E_{11}^*E_{11}) = \Phi(E_{11})E_{11}^*E_{11} + E_{11}\Phi(E_{11})^*E_{11} + E_{11}E_{11}^*\Phi(E_{11})$, we get $a_{11} = 0, a_{22} = 0$, thus $\Phi(E_{11}) = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$. Similarly, we can get $\Phi(E_{12}) = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}$, $\Phi(E_{21}) = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix}$, $\Phi(E_{22}) = \begin{pmatrix} 0 & d_{12} \\ d_{21} & 0 \end{pmatrix}$.

By the definition of $*$-Jordan semi-triple derivable mapping, chose $A = E_{11}, B = E_{12}$ in the above equality, we can get $b_{11} = -a_{12}$. Similarly, chose $A = E_{11}, B = E_{21}$, we can get $c_{11} = -a_{21}$; chose $A = E_{12}, B = E_{22}$, we can get $d_{12} = -b_{22}$; chose $A = E_{21}, B = E_{22}$, we can get $d_{21} = -c_{22}$. Thus we have

$$\Phi(E_{11}) = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}, \qquad \Phi(E_{12}) = \begin{pmatrix} -a_{12} & 0 \\ 0 & b_{22} \end{pmatrix}, \tag{2.1}$$

$$\Phi(E_{21}) = \begin{pmatrix} -a_{21} & 0 \\ 0 & c_{22} \end{pmatrix} \quad \text{and} \quad \Phi(E_{22}) = \begin{pmatrix} 0 & -b_{22} \\ -c_{22} & 0 \end{pmatrix}. \tag{2.2}$$

For any $A \in M_2(\mathbb{R})$, define

$$\Psi(A) = \Phi(A) - \frac{1}{2}\Phi(I)A - \frac{1}{2}A\Phi(I). \tag{2.3}$$

It is easy to verify that $\Psi$ is a $*$-Jordan semi-triple derivable mapping with $\Psi(I) = 0$. So, for each $A \in M_2(\mathbb{R})$, $\Psi(A^2) = \Psi(A)A + A\Psi(A)$. Since

$$\begin{aligned} \Psi(A^*) &= \Phi(A^*) - \frac{1}{2}\Phi(I)A^* - \frac{1}{2}A^*\Phi(I) \\ &= \Phi(I)A^* + \Phi(A)^* + A^*\Phi(I) - \frac{1}{2}\Phi(I)A^* - \frac{1}{2}A^*\Phi(I) \\ &= \frac{1}{2}\Phi(I)A^* + \frac{1}{2}A^*\Phi(I) + \Phi(A)^* \\ &= -\frac{1}{2}\Phi(I)^*A^* - \frac{1}{2}A^*\Phi(I)^* + \Phi(A)^* \\ &= (\Phi(A) - \frac{1}{2}\Phi(I)A - \frac{1}{2}A\Phi(I))^* \\ &= \Psi(A)^*, \end{aligned}$$

hence $\Psi$ preserving $*$ operation. By Lemma 2.1, assume $\phi(I) = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$, for some $b \in \mathbb{R}$. It follows from Eq. (2.1) and Eqs. (2.2) and (2.3) that
$\Psi(E_{11}) = \begin{pmatrix} 0 & a_{12} + \frac{1}{2}b \\ a_{21} - \frac{1}{2}b & 0 \end{pmatrix}$, $\Psi(E_{12}) = \begin{pmatrix} -a_{12} - \frac{1}{2}b & 0 \\ 0 & b_{22} - \frac{1}{2}b \end{pmatrix}$, $\Psi(E_{21}) = \begin{pmatrix} -a_{21} + \frac{1}{2}b & 0 \\ 0 & c_{22} + \frac{1}{2}b \end{pmatrix}$ and $\Psi(E_{22}) = \begin{pmatrix} 0 & -b_{22} + \frac{1}{2}b \\ -c_{22} + \frac{1}{2}b & 0 \end{pmatrix}$.

Since $\Psi(E_{12}) = \Psi(E_{12}^*) = \Psi(E_{21})$, we have $b = a_{21} - a_{12} = b_{22} - c_{12}$. By Lemma 2.1, $0 = \Psi(E_{12}^2) = E_{12}\Psi(E_{12}) + E_{12}\Psi(E_{12})$, we get $b_{22} + c_{22} = -(a_{12} + a_{21})$. Let $a_{12} + a_{21} = 2x$. Then

$$\Psi(E_{11}) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \qquad \Psi(E_{12}) = \begin{pmatrix} -x & 0 \\ 0 & x \end{pmatrix},$$

$$\Psi(E_{21}) = \begin{pmatrix} -x & 0 \\ 0 & x \end{pmatrix} \quad \text{and} \quad \Psi(E_{22}) = \begin{pmatrix} 0 & -x \\ -x & 0 \end{pmatrix}.$$

Let $T = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$.

It is clear that $T = -T^*$ and $\Psi(E_{ij}) = E_{ij}T - TE_{ij}$ for all $1 \leq i, j \leq 2$. Now for any $A \in M_2(\mathbb{R})$, define

$$\Delta(A) = \Psi(A) - (AT - TA). \tag{2.4}$$

It is easy to verify that $\Delta$ is a $*$-Jordan semi-triple derivable mapping with $\Delta(E_{ij}) = 0$ for all $1 \leq i, j \leq 2$. For $A = (a_{ij}) \in M_2(\mathbb{R})$, let $\Delta(A) = (b_{ij})$. Then

$$b_{ij}E_{ji} = E_{ji}\Delta(A)E_{ji} = \Delta(E_{ij}AE_{ij}) = \Delta(a_{ji}E_{ji}).$$

Thus, the $(i, j)$th entry of $\Delta(A)$ depends on the $(j, i)$th entry of $A$ only. Therefore, we may write $\Delta(A) = \begin{pmatrix} \varphi_{11}(a_{11}) & \varphi_{12}(a_{21}) \\ \varphi_{21}(a_{12}) & \varphi_{22}(a_{22}) \end{pmatrix}$ for some maps $\varphi_{ij}$ on $\mathbb{R}$. Furthermore, from $\Delta(E_{ij}) = 0$ for all $i, j \in \{1, 2\}$ we conclude that $\varphi_{ij}(0) = 0$ and $\varphi_{ij}(1) = 0$. Let $J = E_{11} + E_{12} + E_{21} + E_{22}$. For any $a \in \mathbb{R}$, since $\Delta(I) = 0$, we have

$$\begin{aligned} \varphi(a_{11})J =& J(\varphi(a_{11}E_{11}))J = J\Delta(aE_{11})J \\ &= \Delta(aJE_{11}J) = \Delta(aJ) = \begin{pmatrix} \varphi_{11}(a_{11}) & \varphi_{12}(a_{21}) \\ \varphi_{21}(a_{12}) & \varphi_{22}(a_{22}) \end{pmatrix}. \end{aligned}$$

Therefore, $\varphi_{11} = \varphi_{12} = \varphi_{21} = \varphi_{22}$. We label this common mapping by $\varphi$ and it follows that $\Delta(A) = A_\varphi$ for every $A \in M_2(\mathbb{R})$. It remains to prove that $\varphi$ is an additive derivation of $\mathbb{R}$. For any $a, b \in \mathbb{R}$, let $A = aE_{11} + bE_{12}$. Then $\Delta(A) = \varphi(a)E_{11} + \varphi(b)E_{12}$. Since

$$\varphi(a)^2E_{11} + \varphi(a)\varphi(b)E_{12} = \Delta(A)^2 = \Delta(A^2) = \varphi(a^2)E_{11} + \varphi(ab)E_{12}$$

and

$$(\varphi(a) + \varphi(b))J = J\Delta(A)J = \Delta(JAJ) = \Delta((a + b)J) = \varphi(a + b)J,$$

we have $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a + b) = \varphi(a) + \varphi(b)$. Hence, by Eq. (2.3) and Eq. (2.4), $\Phi(A) = \frac{1}{2}\Phi(I)A + \frac{1}{2}A\Phi(I) + AT - TA + A_\varphi$ for all $A \in M_2(\mathbb{R})$. □

***Theorem* 2.3** — *Let $\mathbb{R}$ be a 2-torsion free commutative real ring with unity, and $M_n(\mathbb{R})$ ($n \geq 2$) be the algebra of $n \times n$ matrices over $\mathbb{R}$. If $\Phi$ is a $*$-Jordan semi-triple derivable mapping from $M_n(\mathbb{R})$ into $M_n(\mathbb{R})$ (here $*$ denote the transpose) if and only if there exist $T \in M_n(\mathbb{R}), T^* = -T$, and an additive derivation $\varphi$ of $\mathbb{R}$ such that*

$$\Phi(A) = \frac{1}{2}\Phi(I)A + \frac{1}{2}A\Phi(I) + AT - TA + A_\varphi$$

*for all $A \in M_n(\mathbb{R})$, where $A_\varphi$ is the image of $A$ under $\varphi$ applied entrywise.*

PROOF : For any $A \in M_n(\mathbb{R})$, define

$$\Psi(A) = \Phi(A) - \frac{1}{2}\Phi(I)A - \frac{1}{2}A\Phi(I). \tag{2.5}$$

By the proof of Lemma 2.1, we know that $\Psi(I) = 0$, $\Psi(A^2) = \Psi(A)A + A\Psi(A)$ and $\Psi(A^*) = \Psi(A)^*$. We proceed to prove the theorem by induction of $n$ for $\Psi$. By Lemma 2.1, the theorem is hold for $n = 2$. Now we assume that the theorem is hold for $n = m$. For $n = m+1$, let $P = I_m \oplus [0]$ and $P^\perp = I - P = [0]_m \oplus [1]$, $[0]_m$ is the zero matrix in $M_m(\mathbb{R})$. Since $\Psi(P^2) = \Psi(P)P + P\Psi(P)$, we have $P\Psi(P)P = P^\perp\Psi(P)P^\perp = 0$. Thus

$$\Psi(P) = P\Psi(P)P^\perp + P^\perp\Psi(P)P = PU - UP,$$

here $U = P\Psi(P)P^\perp - P^\perp\Psi(P)P \in M_{m+1}(\mathbb{R})$ and $U^* = -U$. For any $A \in M_{m+1}(\mathbb{R})$, replacing $\Psi$ by the mapping

$$A \mapsto \Psi(A) - (AU - UA) \tag{2.6}$$

we may assume that $\Psi(P) = 0$. For any $A_m \in M_m(\mathbb{R})$ let $A = A_m \oplus [0]$. Then $A = PAP \in M_{m+1}(\mathbb{R})$ and $\Psi(P) = 0$ implies

$$\Psi(A) = \Psi(PAP) = P\Psi(A)P = B_m \oplus [0]$$

for some matrix $B_m \in M_m(\mathbb{R})$. Define the mapping $\hat{\Psi}$ on $M_m(\mathbb{R})$ by $\hat{\Psi}(A_m) = B_m$. It is easy to check that $\hat{\Psi}$ is a $*$-Jordan semi-triple derivable mapping from $M_m(\mathbb{R})$ into $M_m(\mathbb{R})$. By the induction hypothesis there is a $S \in M_m(\mathbb{R})$ with $T^* = -T$ and an additive derivation from $\mathbb{R}$ into $\mathbb{R}$ such that $\hat{\Psi}(A_m) = A_mS - SA_m + A_\varphi$ for all $A_m \in M_m(\mathbb{R})$. Let $V = S \oplus [0]$. For any $X \in M_{m+1}(\mathbb{R})$, define

$$\Delta(X) = \Psi(X) - (XV - VX). \tag{2.7}$$

Thus we can get a $*$-Jordan semi-triple derivable mapping $\hat{\Delta}$ from $M_m(\mathbb{R})$ into $M_m(\mathbb{R})$ such that $\Delta(A_m \oplus [0]) = \hat{\Delta}(A_m) \oplus [0]$. This is equivalent to

$$\Delta(A_m \oplus 0) = A_\varphi \oplus 0. \tag{2.8}$$

Also, for any $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{m+1}(\mathbb{R})$ with $A_{11} \in M_m(\mathbb{R})$ we have $PAP = A_{11} \oplus [0]$.

Thus

$$P\Delta(A)P = \Delta(PAP) = \hat{\Delta}(A_{11} \oplus [0]) = (A_{11})_{\varphi} \oplus [0]. \tag{2.9}$$

Let us define matrices $D_i$ for each $i \in \{1, 2, \cdots, m\}$ by $D_i = I_{m+1} - E_{ii} - E_{(m+1)(m+1)} + E_{i(m+1)} + E_{(m+1)i}$. Let $i$ be arbitrary, but fixed. From Eq. (2.9) we have $P\Delta(D_i)P = 0$. Then there exists $x_i = (x_{i1}, x_{i2}, \cdots, x_{im}), (y_{i1}, y_{i2}, \cdots, y_{im}) \in \mathbb{R}^m$ and $z_i \in \mathbb{R}$ such that $\Delta(D_i) = \begin{pmatrix} 0_m & {x_i}^* \\ y_i & z_i \end{pmatrix}$.

For each fixed $i$, since $D_i^2 = I_{m+1}$, from the equality

$$D_i\Delta(D_i) + \Delta(D_i)D_i = \Delta(D_i^2) = \Delta(I_{m+1}) = 0$$

we get $x_{ii} = -y_{ii}, z_i = 0$ and $x_{ik} = y_{ik} = 0$ $(k \neq i)$. Hence,

$$\Delta(D_i) = x_{ii}E_{i(m+1)} - x_{ii}E_{(m+1)i}.$$

Let $j \in \{1, 2, \cdots, m\}$ and $j \neq i$. Then $D_iD_jD_i = I_{m+1} - E_{ii} - E_{jj} + E_{ij} + E_{ji}$. From the equality

$$\begin{aligned} 0 &= P\Delta(I_{m+1} - E_{ii} - E_{jj} + E_{ij} + E_{ji})P = P\Delta(D_iD_jD_i)P \\ &= P[\Delta(D_i)D_jD_i + D_i\Delta(D_j)D_i + D_iD_j\Delta(D_i)]P \\ &= P[(x_{ii} - x_{jj})E_{ij} + (x_{jj} - x_{ii})E_{ji}]P \\ &= (x_{ii} - x_{jj})E_{ij} + (x_{jj} - x_{ii})E_{ji} \end{aligned}$$

we get $x_{ii} = x_{jj}$ for all $i, j \in \{1, 2, \cdots, m\}$ and $i \neq j$. So, $\Delta(D_i) = x_{11}E_{i(m+1)} - x_{11}E_{(m+1)i}$ for each $i \in \{1, 2, \cdots, m\}$. Let $W = [0]_m \oplus x_{11}$. For any $X \in M_{m+1}(\mathbb{R})$, replacing $\Delta$ by the map

$$X \mapsto \Psi(X) - (XW - WX) \tag{2.10}$$

we may assume that $\Delta(D_i) = 0$ for all $i \in \{1, 2, \cdots, m\}$. Let us fix some $i \in \{1, 2, \cdots, m\}$ again. As $m \geq 2$, there is another $j \in \{1, 2, \cdots, m\}$ and $j \neq i$ such that $E_{i(m+1)} = D_jE_{ij}D_j$ and $E_{ji} = D_jE_{(m+1)iD_j}$. Then for any $a \in \mathbb{R}$,

$$\Delta(aE_{i(m+1)}) = \Delta(D_j(aE_{ij})D_j) = D_j\varphi(a)E_{ij}D_j = \varphi(a)E_{i(m+1)} \tag{2.11}$$

and

$$\Delta(aE_{(m+1)i}) = \Delta(D_j(aE_{ji})D_j) = D_j\varphi(a)E_{ji}D_j = \varphi(a)E_{(m+1)i}. \tag{2.12}$$

Also $E_{(m+1)(m+1)} = D_1E_{11}D_1$, we arrive

$$\Delta(aE_{(m+1)(m+1)}) = \Delta(D_1(aE_{11})D_1) = D_1\varphi(a)E_{11}D_1 = \varphi(a)E_{(m+1)(m+1)}. \tag{2.13}$$

Eq. (2.8), (2.11), (2.12), (2.13) imply that $\Delta(aE_{ij}) = \varphi(a)E_{ij}$ for all $i, j \in \{1, 2, \cdots m + 1\}$ and $a \in \mathbb{R}$. Finally, for any $A \in M_{m+1}(\mathbb{R})$, let $\Delta(A) = (b_{ij})$. Then

$$b_{ij}E_{ji} = Eji\Delta(A)Eji = \Delta(E_{ij}AE_{ij}) = \Delta(a_{ji}E_{ji}).$$

This shows $\Delta(A) = (\varphi(a_{ij})) = A_\varphi$ for all $A \in M_{m+1}(\mathbb{R})$. By Eq. (2.6), (2.7) and (2.10), we get

$$\Psi(A) = AT - TA + A_\varphi$$

for all $A \in M_{m+1}(\mathbb{R})$, here $T = U + V + W$ with $T^* = -T$. Thus, by Eq. (2.5)

$$\Phi(A) = \frac{1}{2}\Phi(I)A + \frac{1}{2}A\Phi(I) + AT - TA + A_\varphi$$

for all $A \in M_n(\mathbb{R})$, and hence the proof is completed. □

In the following theorem, we will characterize a $*$-Jordan semi-triple derivable mapping of $B(\mathcal{H})$. For $A \in B(\mathcal{H})$, $A^*$ denote self adjoint of $A$.

***Theorem* 2.4** — *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. If $\Phi$ is a $*$-Jordan semi-triple derivable mapping from $B(\mathcal{H})$ into $B(\mathcal{H})$ if and only if there exist $T \in B(\mathcal{H})$ with $T^* = -T$ such that*

$$\Phi(A) = \frac{1}{2}\Phi(I)A + \frac{1}{2}A\Phi(I) + AT - TA$$

*for all $A \in B(\mathcal{H})$.*

PROOF : For any $A \in B(\mathcal{H})$, define

$$\Psi(A) = \Phi(A) - \frac{1}{2}\Phi(I)A - \frac{1}{2}A\Phi(I). \tag{2.14}$$

By the proof of Lemma 2.1, we know that $\Psi(I) = 0$, $\Psi(A^2) = \Psi(A)A + A\Psi(A)$ and $\Psi(A^*) = \Psi(A)^*$. If $\Psi$ is additive, then $\Psi$ is an additive Jordan derivation. By [6, Theorem 3.1], $\Psi$ is a derivation. By the Kadison-Sakai theorem [9, 19], it is an inner derivation, thus by Eq. (2.14) the theorem is proved. So, it remains to show that $\Psi$ is additive.

Since dim$\mathcal{H} = \infty$, there exists a projection $P \in B(\mathcal{H})$ such that dim$(P\mathcal{H})$ = dim$(P^\perp\mathcal{H}) = \infty$. Let $P_1 = P, P_2 = P^\perp$ and $\mathcal{A}_{ij} = P_iB(\mathcal{H})P_j$, $1 \le i, j \le 2$. Then $B(\mathcal{H}) = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$.

For the convenience of citation and clarity of exposition, we shall organize the proof in a series of claims.

*Claim* 1 : There exists $S \in B(\mathcal{H})$ with $S^* = -S$ such that $\Psi(P_i) = P_iS - SP_i$, $i = 1, 2$.

Since $\Psi(P_i) = \Psi(P_i)P_i + P_i\Psi(P_i)$, we get $P_j\Psi(P_i)P_j = 0$, for $1 \leq i \neq j \leq 2$. Thus,

$$\Psi(P_i) = P_i\Psi(P_i)P_j + P_j\Psi(P_i)P_i$$

for $1 \leq i \neq j \leq 2$. Since

$$\begin{aligned}\Psi(P_1) &= \Psi((I - 2P_2)P_1(I - 2P_2)) \\ &= \Psi(I - 2P_2)P_1 + (I - 2P_2)\Psi(P_1)(I - 2P_2) + P_1\Psi(I - 2P_2)\end{aligned}$$

Multiplying both sides of the above equation by $P_1$ (or $P_2$) and $P_2$ (or $P_1$) from the left and right, respectively, we get that

$$2P_1\Psi(P_1)P_2 = P_1\Psi(I - 2P_2)P_2 \text{ and } 2P_2\Psi(P_1)P_1 = P_2\Psi(I - 2P_2)P_1.$$

Since

$$\begin{aligned}\Psi(P_2) &= \Psi((I - 2P_2)P_2(I - 2P_2)) \\ &= -\Psi(I - 2P_2)P_2 + (I - 2P_2)\Psi(P_2)(I - 2P_2) - P_2\Psi(I - 2P_2)\end{aligned}$$

Multiplying both sides of the above equation by $P_1$ (or $P_2$) and $P_2$ (or $P_1$) from the left and right, respectively, we get that

$$2P_1\Psi(P_2)P_2 = -P_1\Psi(I - 2P_2)P_2 \text{ and } 2P_2\Psi(P_2)P_1 = -P_2\Psi(I - 2P_2)P_1.$$

Hence, $\Psi(P_1) = -\Psi(P_2)$. Let $S = P_1\Psi(P_1)P_2 - P_2\Psi(P_1)P_1$. For $i = 1, 2$, $\Psi(P_i) = P_iS - SP_i$.

Now, for any $A \in B(\mathcal{H})$, define $\Delta(A) = \Psi(A) - (AS - SA)$. It is easy to verify that $\Delta$ is also a $*$-Jordan semi-triple derivable mapping and $\Delta(P_i) = 0$ for $i = 1, 2$.

*Claim* 2 : For any $A \in B(\mathcal{H})$ and $i, j = 1, 2$, we have $\Delta(P_iAP_j) = P_i\Delta(A)P_j$.

For any $A \in B(\mathcal{H})$ and $i = 1, 2$, it follows from $\Delta(P_i) = 0$ that

$$\Delta(P_iAP_i) = \Delta(P_i)A\Delta(P_i) + P_i\Delta(A)P_i + P_iA\Delta(P_i) = P_i\Delta(A)P_i. \tag{2.15}$$

Since dim$P_1\mathcal{H}$ = dim$P_2\mathcal{H}$, by polar decomposition theorem, there exists a partial isometry $U \in \mathcal{A}_{12}$ such that $UU^* = P_1, U^*U = P_2$. Since $P_2U = UP_1 = 0$, we have

$$\begin{aligned}0 &= \Delta(UP_1AP_2U) = \Delta(U)P_1AP_2U + U\Delta(P_1AP_2)U + UP_1AP_2\Delta(U) \\ &= U\Delta(P_1AP_2)U.\end{aligned}$$

Multiplying both sides of the above equation by $U^*$, we get $P_2\Delta(P_1AP_2)P_1 = 0$. This together with Eq. (2.15), we get $\Delta(P_1AP_2) = P_1\Delta(P_1AP_2)P_2$. Similarly, one can prove that $\Delta(P_2AP_1) = P_2\Delta(P_2AP_1)P_1$. In particularly, $\Delta(U^*) = P_2\Delta(U^*)P_1$. On the other hands, from the fact $U^*AU^* = U^*P_1AP_2U^*$ we have

$$\begin{aligned}\Delta(U^*P_1AP_2U^*) &= \Delta(U^*)P_1AP_2U^* + U^*\Delta(P_1AP_2)U^* + U^*P_1AP_2\Delta(U^*)\\ &= \Delta(U^*AU^*) = \Delta(U^*)AU^* + U^*\Delta(A)U^* + U^*A\Delta(U^*)\\ &= \Delta(U^*)P_1AP_2U^* + U^*\Delta(A)U^* + U^*P_1AP_2\Delta(U^*),\end{aligned}$$

this shows $U^*\Delta(P_1AP_2)U^* = U^*\Delta(A)U^*$. Hence, $P_1\Delta(P_1AP_2)P_2 = P_1\Delta(A)P_2$. This together with $\Delta(P_1AP_2) = P_1\Delta(P_1AP_2)P_2$ we get $\Delta(P_1AP_2) = P_1\Delta(A)P_2$. Similarly, one can prove that $\Delta(P_2AP_1) = P_2\Delta(A)P_1$.

*Claim* 3 : Let $A_{ij} \in \mathcal{A}_{ij}$, $1 \leq i, j \leq 2$. Then $\Delta(\sum_{i,j=1}^2 A_{ij}) = \sum_{i,j=1}^2 \Delta(A_{ij})$.

Suppose there exists $X \in B(\mathcal{H})$ such that $X = \Delta(\sum_{i,j=1}^2 A_{ij})$. By Claim 2,

$$X_{ij} = P_i\Delta\left(\sum_{i,j=1}^2 A_{ij}\right)P_j = \Delta\left(P_i\left(\sum_{i,j=1}^2 A_{ij}\right)P_j\right) = \Delta(A_{ij}).$$

Hence $\Delta(\sum_{i,j=1}^2 A_{ij}) = \sum_{i,j=1}^2 \Delta(A_{ij})$.

*Claim* 4 : Let $A_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$. Then $\Delta(2A_{ij}) = 2\Delta(A_{ij})$.

For any $A_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$, by Claim 1 and Claim 3 we have

$$\Delta(I + A_{ij}) = \Delta(P_1 + P_2 + A_{ij}) = \Delta(A_{ij}).$$

Thus

$$\begin{aligned}\Delta(2A_{ij}) &= \Delta((I + A_{ij})^2) = \Delta(I + A_{ij})(I + A_{ij}) + (I + A_{ij})\Delta(I + A_{ij})\\ &= \Delta(A_{ij})(I + A_{ij}) + (I + A_{ij})\Delta(A_{ij}) = 2\Delta(A_{ij}).\end{aligned}$$

*Claim* 5 : Let $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$. Then $\Delta(A_{ij} + B_{ij}) = \Delta(A_{ij}) + \Delta(B_{ij})$.

For any $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ and $1 \leq i \neq j \leq 2$, we have

$$(I + \frac{1}{2}A_{ij})(I + B_{ij})(I + \frac{1}{2}A_{ij}) = I + A_{ij} + B_{ij}.$$

By Claim 4,

$$\begin{aligned}\Delta(A_{ij}+B_{ij}) &= \Delta(I+A_{ij}+B_{ij})\\ &= \Delta((I+\frac{1}{2}A_{ij})(I+B_{ij})(I+\frac{1}{2}A_{ij}))\\ &= \frac{1}{2}\Delta(A_{ij})(I+B_{ij})(I+\frac{1}{2}A_{ij})+(I+\frac{1}{2}A_{ij})\Delta(B_{ij})(I+\frac{1}{2}A_{ij})\\ &+(I+\frac{1}{2}A_{ij})(I+B_{ij})\frac{1}{2}\Delta(A_{ij})\\ &= \frac{1}{2}\Delta(A_{ij})+\Delta(B_{ij})+\frac{1}{2}\Delta(A_{ij})\\ &= \Delta(A_{ij})+\Delta(B_{ij}).\end{aligned}$$

*Claim* 6 : Let $A_{ii}\in\mathcal{A}_{ii}, B_{ij}\in\mathcal{A}_{ij}$, $1\leq i\neq j\leq 2$. Then $\Delta(A_{ii}B_{ij})=\Delta(A_{ii})B_{ij}+A_{ii}\Delta(B_{ij})$.

For any $A_{ii}\in\mathcal{A}_{ii}, B_{ij}\in\mathcal{A}_{ij}$, $1\leq i\neq j\leq 2$, we have

$$A_{ii}+A_{ii}B_{ij}=(P_i+B_{ij})A_{ii}(P_i+B_{ij}).$$

By Claim 4 and Claim 5,

$$\begin{aligned}\Delta(A_{ii}+A_{ii}B_{ij}) &= \Delta(A_{ii}+\Delta(A_{ii}B_{ij})=\Delta((P_i+B_{ij})A_{ii}(P_i+B_{ij}))\\ &= \Delta(B_{ij})A_{ii}(P_i+B_{ij})+(P_i+B_{ij})\Delta(A_{ii})(P_i+B_{ij})\\ &+(P_i+B_{ij})A_{ii}\Delta(B_{ij})\\ &= (P_i+B_{ij})\Delta(A_{ii})(P_i+B_{ij})+(P_i+B_{ij})A_{ii}\Delta(B_{ij})\\ &= \Delta(A_{ii})+\Delta(A_{ii})B_{ij}+A_{ii}\Delta(B_{ij}).\end{aligned}$$

Hence, $\Delta(A_{ii}B_{ij})=\Delta(A_{ii})B_{ij}+A_{ii}\Delta(B_{ij})$.

*Claim* 7 : Let $A_{ii},B_{ii}\in\mathcal{A}_{ii}$, $i=1,2$. Then $\Delta(A_{ii}+B_{ii})=\Delta(A_{ii})+\Delta(B_{ii})$.

Suppose $1\leq j\neq i\leq 2$, for any $A_{ii},B_{ii}\in\mathcal{A}_{ii}$ and $C_{ij}\in\mathcal{A}_{ij}$, by Claim 6 we have

$$\Delta((A_{ii}+B_{ii})C_{ij})=\Delta(A_{ii}+B_{ii})C_{ij}+(A_{ii}+B_{ii})\Delta(C_{ij}). \tag{2.16}$$

On the other hands, by Claim 5 and Claim 6,

$$\begin{aligned}\Delta(A_{ii}+B_{ii})C_{ij}) &= \Delta(A_{ii}C_{ij})+\Delta(B_{ii}C_{ij})\\ &= \Delta(A_{ii})C_{ij}+A_{ii}\Delta(C_{ij})+\Delta(B_{ii})C_{ij}+B_{ii}\Delta(C_{ij})\\ &= (\Delta(A_{ii})+\Delta(B_{ii}))C_{ij}+A_{ii}\Delta(C_{ij})+B_{ii}\Delta(C_{ij}).\end{aligned}$$

This together with Eq. (2.16) we can get

$$\Delta(A_{ii}+B_{ii})C_{ij} = (\Delta(A_{ii})+\Delta(B_{ii}))C_{ij}$$

for all $C_{ij} \in \mathcal{A}_{ij}$. So, $\Delta(A_{ii}+B_{ii}) = \Delta(A_{ii})+\Delta(B_{ii})$ for all $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$.

*Claim* 8 : $\Delta$ is additive.

Let $A, B \in B(\mathcal{H})$. Then $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$, $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. By Claim 3, Claim 5, Claim 6 and Claim 7,

$$\begin{aligned}\Delta(A+B) &= \Delta(\sum_{i,j=1}^{2}(A_{ij}+B_{ij})) = \sum_{i,j=1}^{2}\Delta(A_{ij}+B_{ij}) \\ &= \sum_{i,j=1}^{2}\Delta(A_{ij})+\Delta(B_{ij}) = \sum_{i,j=1}^{2}\Delta(A_{ij}) + \sum_{i,j=1}^{2}\Delta(B_{ij}) \\ &= \Delta(A)+\Delta(B).\end{aligned}$$

Completing the proof. □

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