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CONVERGENCE RATES OF SOLUTIONS FOR ELLIPTIC REITERATED HOMOGENIZATION PROBLEMS

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In this paper, we study reiterated homogenization problems for equations $-div(A(x/\varepsilon, x/\varepsilon^2)\nabla u_\varepsilon) = f(x)$. We introduce auxiliary functions and obtain the representation formula satisfied by u_ε and homogenized solution u_0. Then we utilize this formula in combination with the asymptotic estimates of Neumann functions for operators and uniform regularity estimates of solutions to obtain convergence rates in L^p for solutions as well as gradient error estimates for Neumann problems.

Key words : Reiterated homogenization; convergence rates; Neumann functions; regularity estimate.

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1. INTRODUCTION

This paper concerns with the asymptotic behavior of solutions to reiterated homogenization equations with Neumann boundary conditions. More precisely, given a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$, we consider

$$L_\varepsilon u_\varepsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}(x/\varepsilon, x/\varepsilon^2)\frac{\partial u_\varepsilon}{\partial x_j}\right) = f \text{ in } \Omega \text{ and } \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = n_i a_{ij}\dfrac{\partial u_\varepsilon}{\partial x_j}$ denotes the conormal derivative with L_ε and $n(x)$ is the outward unit normal to $\partial\Omega$ at the point x.

Throughout this paper, the summation convention is used. We assume that the coefficient matrix $A(y,z) = (a_{ij}(y,z))$ with $1 \le i,j \le n$ is real symmetric and satisfies the ellipticity condition

$$\lambda \mid \xi \mid^2 \le a_{ij}(y,z)\xi_i\xi_j \le \frac{1}{\lambda} \mid \xi \mid^2, \quad \text{for } y,z \in \mathbb{R}^n \text{ and } \xi = (\xi_i) \in \mathbb{R}^n, \tag{2}$$

where $\lambda > 0$, and the periodicity condition

$$A(y+l, z+h) = A(y,z), \quad \text{for } y, z \in \mathbb{R}^n \text{ and } l, h \in \mathbb{Z}^n. \tag{3}$$

We impose the smoothness condition

$$\| A(y,z) \|_{C^{1,\alpha}(\mathbb{R}^n \times \mathbb{R}^n)} \leq \Lambda, \quad \text{for some } \alpha \in (0,1) \text{ and } \Lambda > 0. \tag{4}$$

Without loss of generality, we also assume the compatibility condition

$$\int_{\partial\Omega} u_\varepsilon(x) d\sigma(x) = \int_\Omega f(x) dx = 0. \tag{5}$$

Error estimates of solutions is one of the main questions in homogenization theory. There are many papers about convergence of solutions for elliptic homogenization problems. Assume that all of functions are smooth enough, the $O(\varepsilon)$ error estimate in L^∞ was presented by Bensoussan, Lions and Papanicolaou [5]. In 1987, Avelcaneda and Lin [3] proved L^p convergence by the method of maximum principle. At the same year, they [4] obtained L^∞ error estimate when f is less regular than Bensoussan, Lions and Papanicolaou's. After that, Griso [11, 12] studied interior error estimates by using the periodic unfolding method. In 2012, Kenig, Lin and Shen [16] obtained convergence of solutions in L^2 and $H^{\frac{1}{2}}$ in Lipschitz domains with Dirichlet or Neumann boundary conditions. In 2014, they [17] have also studied the asymptotic behavior of the Green and Neumann functions obtaining some error estimates of solutions.

One may consult several outstanding sources [1, 5, 7, 9, 10, 13, 20] for background and overview of the homogenization theory.

In this work, we study convergence rates of solutions for reiterated homogenization equations with Neumann boundary conditions. The concept of reiterated homogenization was first introduced by Bensoussan, Lions, Papanicolaou [5]. The nonlinear case was studied by Bradies and Lukkassen [6] for elliptic and convex problems. The nonlinear case for periodic monotone operators was obtained in [19] by using the method of energy and multiscales convergence, see also [2] for this method. Meunier and Schaftingen [21, 22] studied convergence weakly in $W_0^{1,p}$ for solutions via the periodic unfolding method. It is important to consider reiterated homogenization problem which has been found applied to advection and diffusion of passive tracers in fluids.

It should be noted that reiterated homogenization problems are much more difficult to deal with than homogenization problems, the main reason is that there are two cell problems and we have to introduce two auxiliary functions in order to get homogenized equation, which causes new difficulties in the estimation of the representation formula satisfied by u_ε and homogenized solution u_0. As the same time, we consider the Neumann boundary value problem. Unlike Dirichlet problem, Neumann problem requires more restrictions on the boundary conditions and is more complicated. As it is well known, the Neumann problem has played a significant role in mathematical physics, for instance, equilibrium problems concerning beams, columns, or strings and engineering problems such as in thermodynamics, and hence has attracted the attention of many researchers over the last two decades. In practice, Neumann boundary conditions are important in applications of homogenization [14, 18, 23].

The procedure we used for obtaining convergence rates estimates is somewhat analogous to the process Kenig, Lin and Shen [17] used for the most classical homogenization problems. The main purpose of this paper is to extend their results [17] to the reiterated homogenization problem. This would be more interesting and technical.

The rest of the paper is organized as follows. Section 2 contains some basic estimates and useful propositions which are very important to obtain error estimates. In section 3, we show that u_ε converges in $L^p(\Omega)$ to the solutions of the corresponding homogenized problems based on obtaining estimates for the Neumann functions of operators. In Section 4, we use uniform regularity estimate to obtain the Lipschitz convergence rate estimate for solutions.

2. Preliminaries

In this section, we show some basic formulas and useful propositions, which are more or less known (see for example [5]). For the sake of completeness we include them here.

Associated with (1.1), the homogenized problem is

$$L_0 u_0 = -q_{ij}\frac{\partial^2 u_0}{\partial x_i \partial x_j} = f \text{ in } \Omega \text{ and } \frac{\partial u_0}{\partial \nu_0} = 0 \text{ on } \partial\Omega,$$

where L_0 is a constant coefficient operator which is also called homogenized operator. The constant matrix is given by

$$\begin{aligned} q_{ij} = & \int\int_{Y\times Z} [a_{ij}(y,z) - a_{ik}(y,z)\frac{\partial \chi_y^j(z)}{\partial z_k} \\ & -a_{ik}(y,z)\frac{\partial \chi^j(y)}{\partial y_k} + a_{ik}(y,z)\frac{\partial \chi_y^l(z)}{\partial z_k}\frac{\partial \chi^j(y)}{\partial y_l}]dydz, \end{aligned}$$

where $Y = Z = [0,1)^n \simeq \mathbb{R}^n/\mathbb{Z}^n$ and $\dfrac{\partial u_0}{\partial \nu_0} = n_i q_{ij}\dfrac{\partial u_0}{\partial x_j}$.

Functions $\chi(y) = (\chi^j(y))$ and $\chi_y(z) = (\chi_y^j(z)) = (\chi^j(y,z))$ are solutions of the following two cell problems,

$$\begin{cases} -\dfrac{\partial}{\partial z_i}\left[a_{ik}(y,z)\dfrac{\partial \chi_y^j(z)}{\partial z_k} - a_{ij}(y,z)\right] = 0 \quad \text{in } \ Z, \\ \chi_y^j(z+h) = \chi_y^j(z) \qquad \text{for } z \in \mathbb{R}^n, \ \ h \in \mathbb{Z}^n, \\ \displaystyle\int_Z \chi_y^j(z)dz = 0 \end{cases}$$

and

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left[\displaystyle\int_Z (a_{il}(y,z) - a_{ik}(y,z)\dfrac{\partial \chi_y^l(z)}{\partial z_k})dz(\dfrac{\partial \chi^j(y)}{\partial y_l} - \delta_l^j)\right] = 0 \quad \text{in } \ Y, \\ \chi^j(y+l) = \chi^j(y) \qquad \text{for } y \in \mathbb{R}^n, \ \ l \in \mathbb{Z}^n, \\ \displaystyle\int_Y \chi^j(y)dy = 0, \end{cases}$$

for any $1 \leq j \leq n$, where $\delta_l^j = 1$ if $l = j$, otherwise, $\delta_l^j = 0$.

Recall that Neumann functions $N_\varepsilon(x,y)$ in a bounded $C^{1,1}$ domain Ω such that

$$\begin{cases} L_\varepsilon N_\varepsilon(\cdot,y) = \delta_y(x) \qquad \text{in} \quad \Omega, \\ \dfrac{\partial N_\varepsilon(\cdot,y)}{\partial \nu_\varepsilon} = -\dfrac{1}{\mid \partial\Omega \mid} \quad \text{on} \ \ \partial\Omega, \\ \displaystyle\int_{\partial\Omega} N_\varepsilon(x,y)d\sigma = 0, \end{cases}$$

where $\delta_y(x)$ denotes the Dirac delta function with pole at y.

It follows essentially the same steps as [15], we can obtain the estimates of Neumann functions which are more or less standard. More precisely, assume that Ω is a bounded $C^{1,\eta}$ domain for some $0 < \eta < 1$, then for any $x, y \in \Omega$, $x \neq y$ and $n \geq 3$,

$$\begin{cases} \mid N_\varepsilon(x,y) \mid \leq \dfrac{C}{\mid x-y \mid^{n-2}}, \\ \mid \nabla_x N_\varepsilon(x,y) \mid \leq \dfrac{C}{\mid x-y \mid^{n-1}}, \\ \mid \nabla_y N_\varepsilon(x,y) \mid \leq \dfrac{C}{\mid x-y \mid^{n-1}}, \\ \mid \nabla_x \nabla_y N_\varepsilon(x,y) \mid \leq \dfrac{C}{\mid x-y \mid^{n}}. \end{cases} \tag{6}$$

Proposition 2.1 — Let $F_{ij}(y) \in L^2(Y)$ with $1 \leq i,j \leq n$. Suppose that $\int_Y F_{ij}(y)dy = 0$ and $\dfrac{\partial}{\partial y_i}(F_{ij}(y)) = 0$. Then there exists $\Phi_{kij} \in H^1(Y)$ such that $F_{ij} = \dfrac{\partial \Phi_{kij}}{\partial y_k}$ and $\Phi_{kij} = -\Phi_{ikj}$.

PROOF : This proposition had been proved by Kenig, Lin and Shen [16]. □

Proposition 2.2 — Let $F_{ij}(y,z) \in L^2(Y\times Z)$ with $1 \le i, j \le n$. Suppose that $\int\int_{Y\times Z} F_{ij}(y,z)dydz = 0$, $\frac{\partial}{\partial y_i}\left(\int_Z F_{ij}dz\right) = 0$ and $\frac{\partial}{\partial z_i}(F_{ij}) = 0$. Then there exist $\Psi_{kij} \in H^1(Y\times Z)$ and $\Phi_{kij} \in H^1(Y)$ such that

$$\Psi_{kij} = -\Psi_{ikj}, \Phi_{kij} = -\Phi_{ikj} \text{ and } F_{ij} = \frac{\partial \Phi_{kij}}{\partial y_k} + \frac{\partial \Psi_{kij}}{\partial z_k}.$$

PROOF : From Proposition 2.1, we find that there exists function Φ_{kij} such that $\int_Z F_{ij}dz = \frac{\partial \Phi_{kij}}{\partial y_k}$ and $\Phi_{kij} = -\Phi_{ikj}$. It follows that $\int_Z \left(F_{ij} - \frac{\partial \Phi_{kij}}{\partial y_k}\right) dz = 0$.

Let $W_{ij} = F_{ij} - \frac{\partial \Phi_{kij}}{\partial y_k}$. This gives $\int_Z W_{ij}dz = 0$ and $\frac{\partial}{\partial z_i}(W_{ij}) = 0$. Using Proposition 2.1, we obtain that there exists function Ψ_{kij} such that $W_{ij} = \frac{\partial \Psi_{kij}}{\partial z_k}$ and $\Psi_{kij} = -\Psi_{ikj}$. Hence

$$F_{ij} = \frac{\partial \Phi_{kij}}{\partial y_k} + \frac{\partial \Psi_{kij}}{\partial z_k},$$

where Φ_{kij}, Ψ_{kij} satisfy $\Phi_{kij} = -\Phi_{ikj}$ and $\Psi_{kij} = -\Psi_{ikj}$. This completes the proof. □

Remark 2.3 : Let

$$B_{ij}(y,z) = q_{ij} - a_{ij}(y,z) + a_{ik}(y,z)\frac{\partial \chi_y^j(z)}{\partial z_k} + a_{ik}(y,z)\frac{\partial \chi^j(y)}{\partial y_k} - a_{ik}(y,z)\frac{\partial \chi_y^l(z)}{\partial z_k}\frac{\partial \chi^j(y)}{\partial y_l}. \quad (7)$$

Note that $\int\int_{Y\times Z} B_{ij}(y,z)dydz = 0$, $\frac{\partial}{\partial y_i}\left(\int_Z B_{ij}dz\right) = 0$ and $\frac{\partial}{\partial z_i}(B_{ij}) = 0$. It follows from Proposition 2.1 that there exist two functions $\Phi_{kij}(y)$ and $\Psi_{kij}(y,z)$ such that $\Phi_{kij} = -\Phi_{ikj}$, $\Psi_{kij} = -\Psi_{ikj}$ and

$$B_{ij}(y,z) = \frac{\partial \Phi_{kij}(y)}{\partial y_k} + \frac{\partial \Psi_{kij}(y,z)}{\partial z_k}. \quad (8)$$

Remark 2.4 : Under the assumption $A(y,z) \in C^{1,\alpha}(\mathbb{R}^n\times\mathbb{R}^n)$, it is known that $\triangledown\chi(y) \in C^{1,\alpha}(Y)$ and $\triangledown\chi_y(z) \in C^{1,\alpha}(Z)$. This implies that $\triangledown\Phi(y) \in C^{1,\alpha}(Y)$ and $\triangledown\Psi(z) \in C^{1,\alpha}(Z)$. In particular,

$$\|\chi^j(y)\|_{W^{2,\infty}(Y)} + \|\chi_y^j(z)\|_{W^{1,\infty}(Z)} + \|\Phi_{kij}(y)\|_{L^\infty(Y)} + \|\Psi_{kij}(z)\|_{L^\infty(Z)} \le C, \quad (9)$$

where constant C depends only on $n, \alpha, \lambda, \Lambda$.

Proposition 2.5 — Suppose that $u_\varepsilon \in H^1(\Omega)$, $u_0 \in H^2(\Omega)$ and $L_\varepsilon(u_\varepsilon) = L_0(u_0)$ in Ω. Let

$$\omega_\varepsilon(x) = u_\varepsilon(x) - u_0(x) + \varepsilon\chi^j(x/\varepsilon)\frac{\partial u_0}{\partial x_j} + \varepsilon^2\chi_y^j(x/\varepsilon^2)\left(\frac{\partial u_0}{\partial x_j} - \frac{\partial \chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k}\right).$$

Then

$$L_\varepsilon(\omega_\varepsilon) = \varepsilon\frac{\partial}{\partial x_i}[a_{ik}\left(\varepsilon\chi_y^l\frac{\partial\chi^j}{\partial y_l} - \chi^j - \varepsilon\chi_y^j\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k} + (\Phi_{kij}+\varepsilon\Psi_{kij})\frac{\partial^2 u_0}{\partial x_j\partial x_k} + a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j} - a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} + a_{ik}\frac{\partial\chi^l}{\partial y_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}]. \quad (10)$$

PROOF : Note that

$$\begin{aligned} a_{ij}\frac{\partial\omega_\varepsilon}{\partial x_j} = {} & a_{ij}\frac{\partial u_\varepsilon}{\partial x_j} - a_{ij}\frac{\partial u_0}{\partial x_j} + a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} + \varepsilon a_{ik}\chi^j\frac{\partial^2 u_0}{\partial x_j\partial x_k} + a_{ik}\frac{\partial\chi_y^j}{\partial z_k}\frac{\partial u_0}{\partial x_j} \\ & + \varepsilon^2 a_{ik}\chi_y^j\frac{\partial^2 u_0}{\partial x_j\partial x_k} - \varepsilon^2 a_{ik}\chi_y^l\frac{\partial\chi^j}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j\partial x_k} - a_{ik}\frac{\partial\chi_y^l}{\partial z_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j} \\ & - \varepsilon a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j} + \varepsilon a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} - \varepsilon a_{ik}\frac{\partial\chi^l}{\partial y_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}. \end{aligned}$$

This together with $L_\varepsilon(u_\varepsilon) = L_0(u_0)$, gives

$$\begin{aligned} L_\varepsilon(\omega_\varepsilon) = {} & -\frac{\partial}{\partial x_i}\left[\left(q_{ij} - a_{ij} + a_{ik}\frac{\partial\chi_y^j}{\partial z_k} + a_{ik}\frac{\partial\chi^j}{\partial y_k} - a_{ik}\frac{\partial\chi_y^l}{\partial z_k}\frac{\partial\chi^j}{\partial y_l}\right)\frac{\partial u_0}{\partial x_j}\right] \\ & -\varepsilon\frac{\partial}{\partial x_i}[a_{ik}\left(\chi^j + \varepsilon\chi_y^j - \varepsilon\chi_y^l\frac{\partial\chi^j}{\partial y_l}\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k} - a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & + a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} - a_{ik}\frac{\partial\chi^l}{\partial y_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}] \\ = {} & -\frac{\partial}{\partial x_i}\left(B_{ij}\frac{\partial u_0}{\partial x_j}\right) - \varepsilon\frac{\partial}{\partial x_i}[a_{ik}\left(\chi^j + \varepsilon\chi_y^j - \varepsilon\chi_y^l\frac{\partial\chi^j}{\partial y_l}\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k} - a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & + a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} - a_{ik}\frac{\partial\chi^l}{\partial y_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}] \\ = {} & -\frac{\partial}{\partial x_i}\left[\left(\varepsilon\frac{\partial\Phi_{kij}}{\partial x_k} + \varepsilon^2\frac{\partial\Psi_{kij}}{\partial x_k}\right)\frac{\partial u_0}{\partial x_j}\right] \\ & -\varepsilon\frac{\partial}{\partial x_i}[a_{ik}\left(\chi^j + \varepsilon\chi_y^j - \varepsilon\chi_y^l\frac{\partial\chi^j}{\partial y_l}\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k} - a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & + a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} - a_{ik}\frac{\partial\chi^l}{\partial y_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}], \end{aligned}$$

where we have used (2.2) and (2.3). From the antisymmetry of Φ_{kij} and Ψ_{kij}, we obtain (2.5). □

Next proposition we shall establish the conormal derivative with L_ε.

Proposition 2.6 — Suppose that $u_\varepsilon \in H^1(\Omega)$ and $u_0 \in H^2(\Omega)$. Let

$$\omega_\varepsilon(x) = u_\varepsilon(x) - u_0(x) + \varepsilon\chi^j(x/\varepsilon)\frac{\partial u_0}{\partial x_j} + \varepsilon^2\chi_y^j(x/\varepsilon^2)\left(\frac{\partial u_0}{\partial x_j} - \frac{\partial\chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k}\right).$$

Then

$$\begin{aligned}\frac{\partial \omega_\varepsilon}{\partial \nu_\varepsilon} = \; & \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \frac{\partial u_0}{\partial \nu_0} + \frac{\varepsilon}{2}\left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}\right)\left(\Phi_{kij}\frac{\partial u_0}{\partial x_j}\right) \\ & + \frac{\varepsilon^2}{2}\left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}\right)\left(\Psi_{kij}\frac{\partial u_0}{\partial x_j}\right) \\ & - \varepsilon n_i [a_{ik}(\varepsilon \chi_y^l \frac{\partial \chi^j}{\partial y_l} - \chi^j - \varepsilon \chi_y^j)\frac{\partial^2 u_0}{\partial x_j \partial x_k} - a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & + (\Phi_{kij} + \varepsilon \Psi_{kij})\frac{\partial^2 u_0}{\partial x_j \partial x_k} + a_{ik}\chi_y^l \frac{\partial^2 \chi^j}{\partial y_l \partial y_k}\frac{\partial u_0}{\partial x_j} + a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}],\end{aligned}$$

where $n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}$ is a tangential derivative for $1 \leq i, k \leq n$.

PROOF : A direct computation shows that

$$\begin{aligned} n_i a_{ij}(x/\varepsilon)\frac{\partial \omega_\varepsilon}{\partial x_j} = \; & n_i a_{ij}\frac{\partial u_\varepsilon}{\partial x_j} - n_i a_{ij}\frac{\partial u_0}{\partial x_j} + n_i a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} + \varepsilon n_i a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & + \varepsilon n_i a_{ik}\chi^j \frac{\partial^2 u_0}{\partial x_k \partial x_j} n_i a_{il}\frac{\partial \chi_y^j}{\partial z_l}\left(\frac{\partial u_0}{\partial x_j} - \frac{\partial \chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k}\right) - \varepsilon n_i a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j} \\ & + \varepsilon^2 n_i a_{ik}\chi_y^j \left(\frac{\partial^2 u_0}{\partial x_j \partial x_k} - \frac{1}{\varepsilon}\frac{\partial^2 \chi^l}{\partial y_j \partial y_k}\frac{\partial u_0}{\partial x_l} - \frac{\partial \chi^l}{\partial y_j}\frac{\partial^2 u_0}{\partial x_l \partial x_k}\right) \\ = \; & \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \frac{\partial u_0}{\partial \nu_0} + n_i B_{ij}\frac{\partial u_0}{\partial x_j} + \varepsilon n_i a_{ik}\chi^j \frac{\partial^2 u_0}{\partial x_k \partial x_j} + \varepsilon n_i a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} - \varepsilon n_i a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j} \\ & + \varepsilon^2 n_i a_{ik}\chi_y^j \left(\frac{\partial^2 u_0}{\partial x_j \partial x_k} - \frac{1}{\varepsilon}\frac{\partial^2 \chi^l}{\partial y_j \partial y_k}\frac{\partial u_0}{\partial x_l} - \frac{\partial \chi^l}{\partial y_j}\frac{\partial^2 u_0}{\partial x_l \partial x_k}\right) \\ = \; & \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \frac{\partial u_0}{\partial \nu_0} + \varepsilon n_i \frac{\partial}{\partial x_k}\left[(\Phi_{kij} + \varepsilon \Psi_{kij})\frac{\partial u_0}{\partial x_j}\right] + \varepsilon n_i a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & - \varepsilon n_i (\Phi_{kij} + \varepsilon \Psi_{kij})\frac{\partial^2 u_0}{\partial x_j \partial x_k} + \varepsilon n_i a_{ik}\chi^j \frac{\partial^2 u_0}{\partial x_k \partial x_j} - \varepsilon n_i a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j} \\ & + \varepsilon^2 n_i a_{ik}\chi_y^j \left(\frac{\partial^2 u_0}{\partial x_j \partial x_k} - \frac{1}{\varepsilon}\frac{\partial^2 \chi^l}{\partial y_j \partial y_k}\frac{\partial u_0}{\partial x_l} - \frac{\partial \chi^l}{\partial y_j}\frac{\partial^2 u_0}{\partial x_l \partial x_k}\right).\end{aligned}$$

This gets the desired result. □

3. CONVERGENCE RATES IN L^P

The goal of this section is to establish error estimates of $\| u_\varepsilon - u_0 \|_{L^p(\Omega)}$ for any $1 < p \leq \infty$. In the rest of this paper, we set

$$D_r = D_r(x_0) = B_r(x_0) \cap \Omega \;\; and \;\; \Gamma_r = \Gamma_r(x_0) = B_r(x_0) \cap \partial\Omega,$$

for some $x_0 \in \overline{\Omega}$ and $0 < r < r_0$, where $B_r(x_0)$ is the open ball of radius r centered at x_0 and r_0 is a constant.

Lemma 3.1 — Suppose that u_ε satisfies

$$L_\varepsilon(u_\varepsilon)=0 \text{ in } D_{2r} \text{ and } \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}=0 \text{ on } \Gamma_{2r}.$$

Then

$$\| u_\varepsilon \|_{L^\infty(D_r)} \le C r^{n(1-q)/q} \| u_\varepsilon \|_{L^{q/(q-1)}(D_{2r})},$$

if $q>n$, where C depends on $q,\alpha,\lambda,n,\Lambda$, and Ω.

PROOF : This estimate had been proved by Kenig, Lin and Shen [15]. □

Next we establish an L^∞ estimate for local solutions.

Lemma 3.2 — Let $u_\varepsilon \in H^1(D_{4r})$ and $u_0 \in W^{2,q}(D_{4r})$ for some $n<q\le\infty$. Suppose that

$$L_\varepsilon(u_\varepsilon)=L_0(u_0) \text{ in } D_{4r} \text{ and } \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}=\frac{\partial u_0}{\partial \nu_0} \text{ on } \Gamma_{4r}.$$

Then

$$\begin{aligned} \| u_\varepsilon - u_0 \|_{L^\infty(D_r)} \quad & \le C\varepsilon \ln[(r/\varepsilon)+2] \| \nabla u_0 \|_{L^\infty(D_{4r})} + C\varepsilon r^{1-n/q} \| \nabla^2 u_0 \|_{L^q(D_{4r})} \\ & + C r^{n(1-q)/q} \| u_\varepsilon - u_0 \|_{L^{q/(q-1)}(D_{4r})}, \end{aligned} \tag{11}$$

where C depends only on $\Lambda,n,q,\alpha,\lambda$.

PROOF : Note that if $L_\varepsilon(u_\varepsilon)=f$, then $L_{\varepsilon/r}(v)=\widetilde{f}$, where $v(x)=r^{-2}u_\varepsilon(rx)$ and $\widetilde{f}=f(rx)$. Thus by scaling we may assume that $r=1$. Let

$$\omega_\varepsilon(x)=u_\varepsilon(x)-u_0(x)+\varepsilon\chi^j(x/\varepsilon)\frac{\partial u_0}{\partial x_j}+\varepsilon^2\chi_y^j(x/\varepsilon^2)\left(\frac{\partial u_0}{\partial x_j}-\frac{\partial \chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k}\right) \quad \text{in } D_3.$$

In view of Proposition 2.5 and Proposition 2.6, we obtain

$$\begin{aligned} L_\varepsilon(\omega_\varepsilon)= \quad & \varepsilon\frac{\partial}{\partial x_i}[a_{ik}\left(\varepsilon\chi_y^l\frac{\partial \chi^j}{\partial y_l}-\chi^j-\varepsilon\chi_y^j\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k}+(\Phi_{kij}+\varepsilon\Psi_{kij})\frac{\partial^2 u_0}{\partial x_j\partial x_k} \\ & + a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j}-a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j}+a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}], \\ \frac{\partial \omega_\varepsilon}{\partial \nu_\varepsilon}= \quad & \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}-\frac{\partial u_0}{\partial \nu_0}+\frac{\varepsilon}{2}\left(n_i\frac{\partial}{\partial x_k}-n_k\frac{\partial}{\partial x_i}\right)\left(\Phi_{kij}\frac{\partial u_0}{\partial x_j}\right) \\ & +\frac{\varepsilon^2}{2}\left(n_i\frac{\partial}{\partial x_k}-n_k\frac{\partial}{\partial x_i}\right)\left(\Psi_{kij}\frac{\partial u_0}{\partial x_j}\right) \\ & -\varepsilon n_i[a_{ik}\left(\varepsilon\chi_y^l\frac{\partial \chi^j}{\partial y_l}-\chi^j-\varepsilon\chi_y^j\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k}-a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & +(\Phi_{kij}+\varepsilon\Psi_{kij})\frac{\partial^2 u_0}{\partial x_j\partial x_k}+a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j}+a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}]. \end{aligned}$$

Next let $\omega_\varepsilon \doteq \omega_\varepsilon^{(1)} + \omega_\varepsilon^{(2)} + \omega_\varepsilon^{(3)} + \omega_\varepsilon^{(4)}$, where

$$\begin{cases} L_\varepsilon(\omega_\varepsilon^{(1)}) = \varepsilon \dfrac{\partial}{\partial x_i}[a_{ik}\left(\varepsilon\chi_y^l \dfrac{\partial \chi^j}{\partial y_l} - \chi^j - \varepsilon\chi_y^j\right)\dfrac{\partial^2 u_0}{\partial x_j \partial x_k} + (\Phi_{kij} + \varepsilon\Psi_{kij})\dfrac{\partial^2 u_0}{\partial x_j \partial x_k} \\ \qquad + a_{ik}\chi_y^l \dfrac{\partial^2 \chi^j}{\partial y_l \partial y_k}\dfrac{\partial u_0}{\partial x_j} - a_{ik}\dfrac{\partial \chi^j}{\partial y_k}\dfrac{\partial u_0}{\partial x_j} + a_{ik}\dfrac{\partial \chi^l}{\partial y_k}\dfrac{\partial \chi^j}{\partial y_l}\dfrac{\partial u_0}{\partial x_j}] \qquad \text{in} \quad D_3, \\ \dfrac{\partial \omega_\varepsilon^{(1)}}{\partial \nu_\varepsilon} \quad = -\varepsilon n_i[a_{ik}\left(\varepsilon\chi_y^l \dfrac{\partial \chi^j}{\partial y_l} - \chi^j - \varepsilon\chi_y^j\right)\dfrac{\partial^2 u_0}{\partial x_j \partial x_k} - a_{ik}\dfrac{\partial \chi^j}{\partial y_k}\dfrac{\partial u_0}{\partial x_j} + (\Phi_{kij} + \varepsilon\Psi_{kij})\dfrac{\partial^2 u_0}{\partial x_j \partial x_k} \\ \qquad + a_{ik}\chi_y^l \dfrac{\partial^2 \chi^j}{\partial y_l \partial y_k}\dfrac{\partial u_0}{\partial x_j} + a_{ik}\dfrac{\partial \chi^l}{\partial y_k}\dfrac{\partial \chi^j}{\partial y_l}\dfrac{\partial u_0}{\partial x_j}] \qquad \text{on} \ \partial D_3, \end{cases} \tag{12}$$

$$\begin{cases} L_\varepsilon(\omega_\varepsilon^{(2)}) = 0 & \text{in} \quad D_3, \\ \dfrac{\partial \omega_\varepsilon^{(2)}}{\partial \nu_\varepsilon} \quad = \dfrac{\varepsilon}{2}\left(n_i \dfrac{\partial}{\partial x_k} - n_k \dfrac{\partial}{\partial x_i}\right)\left(\Phi_{kij}\dfrac{\partial u_0}{\partial x_j}\right) & \text{on} \ \partial D_3, \end{cases} \tag{13}$$

$$\begin{cases} L_\varepsilon(\omega_\varepsilon^{(3)}) = 0 & \text{in} \quad D_3, \\ \dfrac{\partial \omega_\varepsilon^{(3)}}{\partial \nu_\varepsilon} \quad = \dfrac{\varepsilon^2}{2}\left(n_i \dfrac{\partial}{\partial x_k} - n_k \dfrac{\partial}{\partial x_i}\right)\left(\Psi_{kij}\dfrac{\partial u_0}{\partial x_j}\right) & \text{on} \ \partial D_3, \end{cases} \tag{14}$$

and

$$\begin{cases} L_\varepsilon(\omega_\varepsilon^{(4)}) = 0 & \text{in} \quad D_3, \\ \dfrac{\partial \omega_\varepsilon^{(4)}}{\partial \nu_\varepsilon} \quad = \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \dfrac{\partial u_0}{\partial \nu_0} & \text{on} \ \partial D_3. \end{cases} \tag{15}$$

To estimate $\omega_\varepsilon^{(1)}$, we use the Neumann functions representation

$$| \omega_\varepsilon^{(1)}(x) | \leq C\varepsilon \int_{D_3} | \bigtriangledown_y \widehat{N}_\varepsilon(x,y) | (| \bigtriangledown^2 u_0 | + | \bigtriangledown u_0 |) dy,$$

where $\widehat{N}_\varepsilon(x,y)$ denotes the Neumann functions for L_ε in D_3. By $H\ddot{o}lder$ inequality, this gives

$$\| \omega_\varepsilon^{(1)} \|_{L^\infty(D_2)} \leq C\varepsilon(\| \bigtriangledown^2 u_0 \|_{L^q(D_3)} + \| \bigtriangledown u_0 \|_{L^q(D_3)}), \tag{16}$$

where $q > n$.

The existence of the solution of equations (3.3) and (3.4) is according to the compatibility condition as well as the antisymmetry of Φ_{kij} from Proposition 2.2.

To estimate $\omega_\varepsilon^{(2)}$, it follows from (3.3) that

$$\omega_\varepsilon^{(2)}(x) = -\frac{\varepsilon}{2}\int_{\partial D_3}\left(n_i \frac{\partial}{\partial y_k} - n_k \frac{\partial}{\partial y_i}\right)(\widehat{N}_\varepsilon(x,y)) \cdot \Phi_{kij}\frac{\partial u_0}{\partial y_j} d\sigma(y).$$

Then for any $x \in D_3$, choose $\widehat{x} \in \partial D_3$ such that $\mid x - \widehat{x} \mid = dist(x, \partial D_3)$. Hence for any $y \in \partial D_3$, $\mid \widehat{x} - y \mid \leq \mid x - y \mid + \mid \widehat{x} - x \mid \leq 2 \mid x - y \mid$. If we set $R(y) = (R_{ki}(y)) = \Phi_{kij} \dfrac{\partial u_0}{\partial y_j}$, then

$$\begin{aligned} \mid \omega_\varepsilon^{(2)}(x) \mid &= \left| \frac{\varepsilon}{2} \int_{\partial D_3} \left(n_i \frac{\partial}{\partial y_k} - n_k \frac{\partial}{\partial y_i} \right) (\widehat{N}_\varepsilon(x, y)) \cdot [R(\widehat{x}) - R(y)] d\sigma(y) \right| \\ &\leq C\varepsilon \int_{\partial D_3} \frac{\mid R(\widehat{x}) - R(y) \mid}{\mid \widehat{x} - y \mid^{n-1}} d\sigma(y). \end{aligned}$$

Since

$$\begin{aligned} \| R \|_{L^\infty(D_3)} &\leq C \| \triangledown u_0 \|_{L^\infty(D_3)}, \\ \mid R(\widehat{x}) - R(y) \mid &\leq C(\| \triangledown u_0 \|_{C^{0,\alpha}(D_3)} + \varepsilon^{-\alpha} \| \triangledown u_0 \|_{L^\infty(D_3)}) \mid \widehat{x} - y \mid^\alpha, \end{aligned}$$

this implies that

$$\begin{aligned} \mid \omega_\varepsilon^{(2)}(x) \mid &\leq C\varepsilon \| \triangledown u_0 \|_{L^\infty(D_3)} \int_{\partial D_3 \setminus B_\varepsilon(\widehat{x})} \frac{1}{\mid \widehat{x} - y \mid^{n-1}} d\sigma(y) \\ &\quad + C\varepsilon^{1-\alpha} \| \triangledown u_0 \|_{L^\infty(D_3)} \int_{\partial D_3 \cap B_\varepsilon(\widehat{x})} \frac{1}{\mid \widehat{x} - y \mid^{n-1-\alpha}} d\sigma(y) \\ &\quad + C\varepsilon \| \triangledown u_0 \|_{C^{0,\alpha}(D_3)} \int_{\partial D_3 \cap B_\varepsilon(\widehat{x})} \frac{1}{\mid \widehat{x} - y \mid^{n-1-\alpha}} d\sigma(y) \\ &\leq C\varepsilon \ln[1/\varepsilon + 2] \| \triangledown u_0 \|_{L^\infty(D_3)} + C\varepsilon^{1+\alpha} \| \triangledown u_0 \|_{C^{0,\alpha}(D_3)} . \end{aligned} \tag{17}$$

The estimate about $\omega_\varepsilon^{(3)}$ is similar to $\omega_\varepsilon^{(2)}$. Finally to estimate $\omega_\varepsilon^{(4)}$, it follows from Lemma 3.1 and (3.5) that

$$\begin{aligned} \| \omega_\varepsilon^{(4)} \|_{L^\infty(D_1)} &\leq C \| \omega_\varepsilon^{(4)} \|_{L^{q/(q-1)}(D_2)} \\ &\leq C \| u_\varepsilon - u_0 \|_{L^{q/(q-1)}(D_2)} + C\varepsilon \| \triangledown u_0 \|_{L^\infty(D_2)} \\ &\quad + C \| \omega_\varepsilon^{(1)} \|_{L^\infty(D_2)} + C \| \omega_\varepsilon^{(2)} \|_{L^\infty(D_2)} + C \| \omega_\varepsilon^{(3)} \|_{L^\infty(D_2)} . \end{aligned}$$

This together with (3.6), (3.7) and imbedding theorem, gives (3.1). This completes the proof. □

The next Lemma is exactly same with result due to Pastukhova in [24], which was used to establish the asymptotic behavior of Neumann functions.

Lemma 3.3 — Suppose that $f \in L^2(\Omega)$. Let $u_\varepsilon \in H^1(\Omega)$ be a solution of

$$L_\varepsilon(u_\varepsilon) = f \text{ in } \Omega \text{ and } \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0 \text{ on } \partial\Omega.$$

Then

$$\| u_\varepsilon - u_0 \|_{L^2(\Omega)} \leq C\varepsilon \| f \|_{L^2(\Omega)},$$

where C depends only on $\Lambda, n, q, \alpha, \lambda$ and Ω.

PROOF : The proof of Lemma 3.3 follows the same line of argument as Lemma 3.2. Let

$$\omega_\varepsilon(x) = u_\varepsilon(x) - u_0(x) + \varepsilon\chi^j(x/\varepsilon)\frac{\partial u_0}{\partial x_j} + \varepsilon^2\chi_y^j(x/\varepsilon^2)\left(\frac{\partial u_0}{\partial x_j} - \frac{\partial \chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k}\right).$$

Then

$$\begin{aligned} L_\varepsilon(\omega_\varepsilon) = {} & \varepsilon\frac{\partial}{\partial x_i}[a_{ik}\left(\varepsilon\chi_y^l\frac{\partial\chi^j}{\partial y_l} - \chi^j - \varepsilon\chi_y^j\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k} + (\Phi_{kij} + \varepsilon\Psi_{kij})\frac{\partial^2 u_0}{\partial x_j\partial x_k} \\ & + a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j} - a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} + a_{ik}\frac{\partial\chi^l}{\partial y_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}], \\ \frac{\partial\omega_\varepsilon}{\partial\nu_\varepsilon} = {} & \frac{\partial u_\varepsilon}{\partial\nu_\varepsilon} - \frac{\partial u_0}{\partial\nu_0} + \frac{\varepsilon}{2}\left(n_i\frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial x_i}\right)\left(\Phi_{kij}\frac{\partial u_0}{\partial x_j}\right) \\ & + \frac{\varepsilon^2}{2}\left(n_i\frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial x_i}\right)\left(\Psi_{kij}\frac{\partial u_0}{\partial x_j}\right) \\ & - \varepsilon n_i[a_{ik}\left(\varepsilon\chi_y^l\frac{\partial\chi^j}{\partial y_l} - \chi^j - \varepsilon\chi_y^j\right)\frac{\partial^2 u_0}{\partial x_j\partial x_k} - a_{ik}\frac{\partial\chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & + (\Phi_{kij} + \varepsilon\Psi_{kij})\frac{\partial^2 u_0}{\partial x_j\partial x_k} + a_{ik}\chi_y^l\frac{\partial^2\chi^j}{\partial y_l\partial y_k}\frac{\partial u_0}{\partial x_j} + a_{ik}\frac{\partial\chi^l}{\partial y_k}\frac{\partial\chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}]. \end{aligned}$$

Next let $\omega_\varepsilon \doteq \omega_\varepsilon^{(1)} + \omega_\varepsilon^{(2)} + \omega_\varepsilon^{(3)} + \omega_\varepsilon^{(4)}$, where

$$\begin{cases} L_\varepsilon(\omega_\varepsilon^{(1)}) = \varepsilon\dfrac{\partial}{\partial x_i}[a_{ik}\left(\varepsilon\chi_y^l\dfrac{\partial\chi^j}{\partial y_l} - \chi^j - \varepsilon\chi_y^j\right)\dfrac{\partial^2 u_0}{\partial x_j\partial x_k} + (\Phi_{kij} + \varepsilon\Psi_{kij})\dfrac{\partial^2 u_0}{\partial x_j\partial x_k} \\ \qquad + a_{ik}\chi_y^l\dfrac{\partial^2\chi^j}{\partial y_l\partial y_k}\dfrac{\partial u_0}{\partial x_j} - a_{ik}\dfrac{\partial\chi^j}{\partial y_k}\dfrac{\partial u_0}{\partial x_j} + a_{ik}\dfrac{\partial\chi^l}{\partial y_k}\dfrac{\partial\chi^j}{\partial y_l}\dfrac{\partial u_0}{\partial x_j}] \quad \text{in} \quad \Omega, \\ \dfrac{\partial\omega_\varepsilon^{(1)}}{\partial\nu_\varepsilon} = -\varepsilon n_i[a_{ik}\left(\varepsilon\chi_y^l\dfrac{\partial\chi^j}{\partial y_l} - \chi^j - \varepsilon\chi_y^j\right)\dfrac{\partial^2 u_0}{\partial x_j\partial x_k} - a_{ik}\dfrac{\partial\chi^j}{\partial y_k}\dfrac{\partial u_0}{\partial x_j} + (\Phi_{kij} + \varepsilon\Psi_{kij})\dfrac{\partial^2 u_0}{\partial x_j\partial x_k} \\ \qquad + a_{ik}\chi_y^l\dfrac{\partial^2\chi^j}{\partial y_l\partial y_k}\dfrac{\partial u_0}{\partial x_j} + a_{ik}\dfrac{\partial\chi^l}{\partial y_k}\dfrac{\partial\chi^j}{\partial y_l}\dfrac{\partial u_0}{\partial x_j}] \quad \text{on} \ \partial\Omega, \\ \displaystyle\int_\Omega \omega_\varepsilon^{(1)} dy = 0, \end{cases}$$

$$\begin{cases} L_\varepsilon(\omega_\varepsilon^{(2)}) = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial\omega_\varepsilon^{(2)}}{\partial\nu_\varepsilon} = \dfrac{\varepsilon}{2}\left(n_i\dfrac{\partial}{\partial x_k} - n_k\dfrac{\partial}{\partial x_i}\right)\left(\Phi_{kij}\dfrac{\partial u_0}{\partial x_j}\right) & \text{on} \ \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \omega_\varepsilon^{(2)} d\sigma = 0, \end{cases}$$

$$\begin{cases} L_\varepsilon(\omega_\varepsilon^{(3)}) = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial \omega_\varepsilon^{(3)}}{\partial \nu_\varepsilon} = \dfrac{\varepsilon^2}{2}\left(n_i \dfrac{\partial}{\partial x_k} - n_k \dfrac{\partial}{\partial x_i}\right)\left(\Psi_{kij}\dfrac{\partial u_0}{\partial x_j}\right) & \text{on} \ \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \omega_\varepsilon^{(3)} d\sigma = 0, \end{cases}$$

and

$$\omega_\varepsilon^{(4)} = \frac{1}{|\ \partial\Omega\ |}\int_{\partial\Omega}(\omega_\varepsilon - \omega_\varepsilon^{(1)})d\sigma.$$

From the usual energy estimate, we have

$$\| \ \omega_\varepsilon^{(1)} \ \|_{L^2(\Omega)} \leq C\varepsilon \ \| \ u_0 \ \|_{H^2(\Omega)} \ . \tag{18}$$

Next, we use a duality argument to estimate $\omega_\varepsilon^{(2)}$. Assume that F_ε is a solution to Neumann problem

$$\begin{cases} L_\varepsilon(F_\varepsilon) = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial F_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on} \ \partial\Omega, \\ \displaystyle\int_{\partial\Omega} F_\varepsilon d\sigma = 0, \quad F_\varepsilon \in H^1(\Omega), \end{cases}$$

where $g \in H^2(\Omega)$ and $\int_{\partial\Omega} g d\sigma = 0$. It follows from integration by parts that

$$\begin{aligned} \left|\int_{\partial\Omega} \omega_\varepsilon^{(2)} g d\sigma\right| &= \left|\int_{\partial\Omega} F_\varepsilon \cdot \frac{\partial \omega_\varepsilon^{(2)}}{\partial \nu_\varepsilon} d\sigma\right| \\ &\leq \frac{\varepsilon}{2}\left|\int_{\partial\Omega}\left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}\right) F_\varepsilon \cdot \Phi_{kij}\frac{\partial u_0}{\partial x_j})\right| \\ &\leq C\varepsilon \ \| \ \triangledown F_\varepsilon \ \|_{L^2(\partial\Omega)} \| \ \triangledown u_0 \ \|_{L^2(\partial\Omega)} \ . \end{aligned}$$

In view of the estimate $\| \ \triangledown F_\varepsilon \ \|_{L^2(\partial\Omega)} \leq C \ \| \ g \ \|_{L^2(\partial\Omega)}$ and the square function estimate for the L^2 Neumann problem [16], we obtain

$$\| \ \omega_\varepsilon^{(2)} \ \|_{L^2(\Omega)} \leq C\varepsilon \ \| \ \triangledown u_0 \ \|_{L^2(\partial\Omega)} \ . \tag{19}$$

Similar, we obtain

$$\| \ \omega_\varepsilon^{(3)} \ \|_{L^2(\Omega)} \leq C\varepsilon^2 \ \| \ \triangledown u_0 \ \|_{L^2(\partial\Omega)} \ . \tag{20}$$

Finally to estimate $\omega_\varepsilon^{(4)}$, we note that

$$| \omega_\varepsilon^{(4)} | \leq C \int_{\partial\Omega} | \omega_\varepsilon^{(1)} | \, d\sigma + C\varepsilon \int_{\partial\Omega} | \bigtriangledown u_0 | \, d\sigma \leq C\varepsilon \parallel u_0 \parallel_{H^2(\Omega)} . \tag{21}$$

This together with estimates of $\omega_\varepsilon^{(1)}, \omega_\varepsilon^{(2)}, \omega_\varepsilon^{(3)}$ and the well-known inequality $\parallel u_0 \parallel_{H^2(\Omega)} \leq C \parallel f \parallel_{L^2(\Omega)}$, completes the proof of Lemma 3.3.

Now, we obtain the convergence rate of Neumann functions from the following theorem.

***Theorem* 3.4** — *Suppose that $N_\varepsilon(x,y)$,$N_0(x,y)$ denotes Neumann functions for operators L_ε, L_o in Ω respectively. Then for any $x, y \in \Omega$,*

$$| N_\varepsilon(x,y) - N_0(x,y) | \leq \frac{C\varepsilon \ln(\varepsilon^{-1} | x-y | +2)}{| x-y |^{n-1}}, \tag{22}$$

where C depends on $n, \Lambda, \alpha, \lambda$ and Ω.

PROOF : By scaling we may assume that $diam(\Omega) = 1$. Fix $x_0, y_0 \in \Omega$ and let $r = | x_0 - y_0 | /4$. We only need to prove $\varepsilon \leq r$, since the case $\varepsilon > r$ is trivial and follows from the estimate of (2.1). Let $f \in C_0^\infty(D_r(y_0))$ and $\int_{\partial\Omega} u_\varepsilon = 0$. It follows from the Neumann functions representation that

$$u_\varepsilon(x) = \int_{D_r(y_0)} N_\varepsilon(x,y) f(y) dy \quad \text{and} \quad u_0(x) = \int_{D_r(y_0)} N_0(x,y) f(y) dy. \tag{23}$$

Since Ω is $C^{1,1}$, we have the following estimates [8],

$$\begin{cases} \parallel \bigtriangledown^2 u_0 \parallel_{L^q(\Omega)} \leq C \parallel f \parallel_{L^q(D_r(y_0))}, & \text{for any } 1 < q < \infty, \\ \parallel \bigtriangledown u_0 \parallel_{L^\infty(\Omega)} \leq C r^{1-n/q} \parallel f \parallel_{L^q(D_r(y_0))}, & \text{for any} \quad q > n. \end{cases}$$

It follows from Lemma 3.2, Lemma 3.3 and duality theory that

$$\parallel N_\varepsilon(x_0,y) - N_0(x_0,y) \parallel_{L^{q/(q-1)}(D_r(y_0))} \leq C\varepsilon \ln[(r/\varepsilon)+2] r^{1-n/q},$$

where $q > n$.

Since $L_\varepsilon(N_\varepsilon(x_0,y)) = L_0(N_0(x_0,y)) = 0$ in $D_r(y_0)$ and $\dfrac{\partial N_\varepsilon(x,y)}{\partial \nu_\varepsilon} = \dfrac{\partial N_0(x,y)}{\partial \nu_0} = -\dfrac{1}{| \partial\Omega |}$ on $\partial\Omega$. It follows from Lemma 3.2 that

$$\begin{aligned} | N_\varepsilon(x_0,y_0) - N_0(x_0,y_0) | \ & \leq C\varepsilon \ln[(r/\varepsilon)+2] \parallel \bigtriangledown_y N_0(x_0,y) \parallel_{L^\infty(D_r(y_0))} \\ & \quad + C\varepsilon r^{1-n/q} \parallel \bigtriangledown_y^2 N_0(x_0,y) \parallel_{L^q(D_r(y_0))} \\ & \quad + C r^{n/q-n} \parallel N_\varepsilon - N_0 \parallel_{L^{q/(q-1)}(D_r(y_0))} \\ & \leq C\varepsilon \ln[(r/\varepsilon)+2] r^{1-n}, \end{aligned}$$

where we have used (2.1). This completes the proof.

As an application of Theorem 3.4, we obtain error estimates of $\| u_\varepsilon - u_0 \|_{L^p(\Omega)}$ for any $1 < p \leq \infty$.

***Theorem* 3.5** — *Assume that $u_\varepsilon \in H^1(\Omega)$ and $f \in L^q(\Omega)$. Suppose that u_ε is the solution of Neumann problem*

$$L_\varepsilon(u_\varepsilon) = f \ \ in\, \Omega \ \ and \ \ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0 \ \ on\, \partial\Omega.$$

Then these estimates

$$\begin{cases} \| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \leq C\varepsilon[\ln(\widetilde{d}/\varepsilon + 2)]^{2-1/n} \| f \|_{L^n(\Omega)}, \ \ where \ \ \widetilde{d} = diam(\Omega), \\ \| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \leq C\varepsilon^\beta[\ln(\widetilde{d}/\varepsilon + 2)] \| f \|_{L^{n-\delta}(\Omega)}, \quad for\ n/(n-\delta) + \beta = 2, \ if\ 0 < \delta < n/2, \\ \| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \leq C\varepsilon[\ln(\widetilde{d}/\varepsilon + 2)] \| f \|_{L^q(\Omega)}, \quad if \ \ q > n, \\ \| u_\varepsilon - u_0 \|_{L^p(\Omega)} \leq C\varepsilon[\ln(\widetilde{d}/\varepsilon + 2)] \| f \|_{L^q(\Omega)}, \quad for\ 1/p = 1/q - 1/n, \ \ if \ \ 1 < q < n \end{cases}$$

hold, where C depends on $n, q, \alpha, \Lambda, \lambda$ and Ω.

PROOF : In view of (2.1) and (3.12), we obtain the convergence rates for Neumann functions

$$| N_\varepsilon(x,y) - N_0(x,y) | \leq \frac{C}{| x-y |^{n-2}} \quad \text{and} \quad | N_\varepsilon(x,y) - N_0(x,y) | \leq \frac{C\varepsilon \ln(\varepsilon^{-1} \mid x-y \mid +2)}{| x-y |^{n-1}}.$$

By the Neumann functions representation and $H\ddot{o}lder's$ inequality, it gives

$$\begin{aligned} | u_\varepsilon(x) - u_0(x) | &\leq C \int_{D_\varepsilon(x)} \frac{| f(y) |}{| x-y |^{n-2}} dy + C\varepsilon \ln(\widetilde{d}/\varepsilon + 2) \int_{\Omega \setminus D_\varepsilon(x)} \frac{| f(y) |}{| x-y |^{n-1}} dy \\ &\leq C\varepsilon \| f \|_{L^n(\Omega)} + C\varepsilon[\ln(\widetilde{d}/\varepsilon + 2)]^{2-1/n} \| f \|_{L^n(\Omega)} \\ &\leq C\varepsilon[\ln(\widetilde{d}/\varepsilon + 2)]^{2-1/n} \| f \|_{L^n(\Omega)}, \end{aligned}$$

where $\widetilde{d} = diam(\Omega)$. This gets the first estimate.

The second estimate is the same as the first estimate.

$$\begin{aligned} | u_\varepsilon(x) - u_0(x) | &\leq C \int_{D_\varepsilon(x)} \frac{| f(y) |}{| x-y |^{n-2}} dy + C\varepsilon \ln(\widetilde{d}/\varepsilon + 2) \int_{\Omega \setminus D_\varepsilon(x)} \frac{| f(y) |}{| x-y |^{n-1}} dy \\ &\leq C\varepsilon^{(n-2\delta)/(n-\delta)} \| f \|_{L^{n-\delta}(\Omega)} \\ &\quad + C\varepsilon^{(n-2\delta)/(n-\delta)}[\ln(\widetilde{d}/\varepsilon + 2)] \| f \|_{L^{n-\delta}(\Omega)} \\ &\leq C\varepsilon^\beta[\ln(\widetilde{d}/\varepsilon + 2)] \| f \|_{L^{n-\delta}(\Omega)}, \end{aligned}$$

where $n/(n-\delta) + \beta = 2$ and $0 < \delta < n/2$.

The third estimate follows from $H\ddot{o}lder's$ inequality directly.

The last inequality follows from Hardy-Littlewood-Sobolev theorem of fractional integration (Chapter 5, Theorem 1 in [25]). This completes the proof. □

4. GRADIENT ERROR ESTIMATE

In this section, we consider the gradient error estimate for solutions. This can be obtained via the uniform regularity estimate.

We introduce a boundary correction function $\xi_\varepsilon \in H^1(\Omega)$, which is the solution to

$$\begin{cases} L_\varepsilon(\xi_\varepsilon) = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial \xi_\varepsilon}{\partial \nu_\varepsilon} = -\left(n_i \dfrac{\partial}{\partial x_k} - n_k \dfrac{\partial}{\partial x_i}\right)\left[\left(\dfrac{\varepsilon}{2}\Phi_{kij} + \dfrac{\varepsilon^2}{2}\Psi_{kij}\right)\dfrac{\partial u_0}{\partial x_j}\right] & \text{on} \ \partial\Omega. \end{cases}$$

Let

$$W_\varepsilon = u_\varepsilon - u_0 + \varepsilon \chi^j \frac{\partial u_0}{\partial x_j} + \varepsilon^2 \chi_y^j \left(\frac{\partial u_0}{\partial x_j} - \frac{\partial \chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k}\right) + \xi_\varepsilon. \tag{24}$$

It follows from Proposition 2.5 and Proposition 2.6 that

$$\begin{aligned} L_\varepsilon W_\varepsilon = {} & \varepsilon \frac{\partial}{\partial x_i}[a_{ik}\left(\varepsilon \chi_y^l \frac{\partial \chi^j}{\partial y_l} - \chi^j - \varepsilon \chi_y^j\right)\frac{\partial^2 u_0}{\partial x_j \partial x_k} + (\Phi_{kij} + \varepsilon \Psi_{kij})\frac{\partial^2 u_0}{\partial x_j \partial x_k} \\ & + a_{ik}\chi_y^l \frac{\partial^2 \chi^j}{\partial y_l \partial y_k}\frac{\partial u_0}{\partial x_j} - a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} + a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}], \\ \frac{\partial W_\varepsilon}{\partial \nu_\varepsilon} = {} & -\varepsilon n_i[a_{ik}\left(\varepsilon \chi_y^l \frac{\partial \chi^j}{\partial y_l} - \chi^j - \varepsilon \chi_y^j\right)\frac{\partial^2 u_0}{\partial x_j \partial x_k} - a_{ik}\frac{\partial \chi^j}{\partial y_k}\frac{\partial u_0}{\partial x_j} \\ & + (\Phi_{kij} + \varepsilon \Psi_{kij})\frac{\partial^2 u_0}{\partial x_j \partial x_k} + a_{ik}\chi_y^l \frac{\partial^2 \chi^j}{\partial y_l \partial y_k}\frac{\partial u_0}{\partial x_j} + a_{ik}\frac{\partial \chi^l}{\partial y_k}\frac{\partial \chi^j}{\partial y_l}\frac{\partial u_0}{\partial x_j}]. \end{aligned} \tag{25}$$

Lemma 4.1 — Assume that Ω is a bounded $C^{1,1}$ domain. Let matrix A satisfy (1.2) (1.3) and (1.4). Suppose that $u_\varepsilon \in H^1(\Omega)$ is a solution of the Neumann problem

$$\begin{cases} L_\varepsilon(u_\varepsilon) = div(A(x/\varepsilon, x/\varepsilon^2)F) & \text{in} \quad \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -n \cdot A(x/\varepsilon, x/\varepsilon^2)F & \text{on} \ \partial\Omega, \end{cases} \tag{26}$$

where $F \in C^{0,\rho}(\Omega)$. Then $\bigtriangledown u_\varepsilon \in L^\infty(\Omega)$, and

$$\| \bigtriangledown u_\varepsilon \|_{L^\infty(\Omega)} \leq C \| F \|_{C^{0,\rho}(\Omega)},$$

where C depends only on $\lambda, \rho, \alpha, \Lambda, n$, and Ω.

PROOF : In view of (4.3) and the Neumann functions representation, we obtain

$$u_\varepsilon(x) = -\int_\Omega \frac{\partial}{\partial y_i}\left(N_\varepsilon(x,y)\right) a_{ij}(y/\varepsilon, y/\varepsilon^2) F_j(y) dy.$$

It follows that for any $x \in \Omega$,

$$\begin{aligned} \bigtriangledown u_\varepsilon(x) &= -\int_\Omega \frac{\partial}{\partial y_i}(\bigtriangledown_x N_\varepsilon(x,y))[a_{ij}(y/\varepsilon, y/\varepsilon^2)F_j(y) - a_{ij}(x/\varepsilon, x/\varepsilon^2)F_j(x)]dy \\ &\quad -a_{ij}(x/\varepsilon, x/\varepsilon^2)F_j(x)\int_{\partial\Omega} n_i(y) \bigtriangledown_x N_\varepsilon(x,y)d\sigma(y) \\ &= -\int_\Omega \frac{\partial}{\partial y_i}(\bigtriangledown_x N_\varepsilon(x,y))a_{ij}(y/\varepsilon, y/\varepsilon^2)[F_j(y) - F_j(x)]dy \\ &\quad -F_j(x)\int_\Omega \frac{\partial}{\partial y_i}(\bigtriangledown_x N_\varepsilon(x,y))[a_{ij}(y/\varepsilon, y/\varepsilon^2) - a_{ij}(x/\varepsilon, x/\varepsilon^2)]dy \\ &\quad -a_{ij}(x/\varepsilon, x/\varepsilon^2)F_j(x)\int_{\partial\Omega} n_i(y) \bigtriangledown_x N_\varepsilon(x,y)d\sigma(y). \end{aligned} \tag{27}$$

Note that if $F_j(x) = -\delta_{jk}$, then $u_\varepsilon(x) = x_k$ is a solution of (4.3). In view of (4.4), we obtain

$$\begin{aligned} \bigtriangledown(x_k) &= \delta_{jk}\int_\Omega \frac{\partial}{\partial y_i}(\bigtriangledown_x N_\varepsilon(x,y))[a_{ij}(y/\varepsilon, y/\varepsilon^2) - a_{ij}(x/\varepsilon, x/\varepsilon^2)]dy \\ &\quad +a_{ij}(x/\varepsilon, x/\varepsilon^2)\delta_{jk}\int_{\partial\Omega} n_i(y) \bigtriangledown_x N_\varepsilon(x,y)d\sigma(y). \end{aligned} \tag{28}$$

It follows from (4.4) and (4.5) that

$$\bigtriangledown u_\varepsilon(x) + F_j(x) \bigtriangledown (x_j) = -\int_\Omega \frac{\partial}{\partial y_i}(\bigtriangledown_x N_\varepsilon(x,y))a_{ij}(y/\varepsilon, y/\varepsilon^2)[F_j(y) - F_j(x)]dy.$$

Hence,

$$\begin{aligned} | \bigtriangledown u_\varepsilon(x) | &\le C \parallel F \parallel_{L^\infty(\Omega)} + C \parallel F \parallel_{C^{0,\rho}(\Omega)} \int_\Omega \frac{dy}{| x - y |^{n-\rho}} \\ &\le C \parallel F \parallel_{C^{0,\rho}(\Omega)} . \end{aligned}$$

This completes the proof. □

As an application of Lemma 4.3, we obtain an error estimate of $\parallel \bigtriangledown[u_\varepsilon - u_0 + \varepsilon\chi^j \frac{\partial u_0}{\partial x_j} + \varepsilon^2\chi_y^j(\frac{\partial u_0}{\partial x_j} - \frac{\partial \chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k}) + \xi_\varepsilon] \parallel_{L^\infty(\Omega)}$.

***Theorem* 4.2** — *Assume that $u_\varepsilon \in H^1(\Omega)$. Let $f \in C^{0,\rho}(\Omega)$ for some $0 < \rho < 1/2$. Suppose that u_ε is the solution of Neumann problem*

$$L_\varepsilon(u_\varepsilon) = f \ \ in\ \Omega \ \ and \ \ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0 \ \ on\ \partial\Omega.$$

Then the estimate

$$\left\| \bigtriangledown \left[u_\varepsilon - u_0 + \varepsilon\chi^j \frac{\partial u_0}{\partial x_j} + \varepsilon^2\chi_y^j \left(\frac{\partial u_0}{\partial x_j} - \frac{\partial \chi^k}{\partial y_j}\frac{\partial u_0}{\partial x_k} \right) + \xi_\varepsilon \right] \right\|_{L^\infty(\Omega)} \le C\varepsilon^{1-2\rho} \parallel f \parallel_{C^{0,\rho}(\Omega)} \tag{29}$$

holds, where C depends only on $\lambda, \alpha, n, \Lambda$ and Ω.

PROOF : In view of (4.1) (4.2) and Lemma 4.1, we obtain

$$\begin{aligned} | \bigtriangledown W_\varepsilon(x) | \quad &\leq C\varepsilon \parallel A(x/\varepsilon, x/\varepsilon^2) \parallel_{C^{0,\rho}(\Omega)} \| \bigtriangledown^2 u_0 \parallel_{C^{0,\rho}(\Omega)} \\ &\leq C\varepsilon^{1-2\rho} \parallel f \parallel_{C^{0,\rho}(\Omega)} . \end{aligned}$$

This completes the proof. □

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