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A NUMERICAL STUDY OF EIGENVALUES AND EIGENFUNCTIONS OF FRACTIONAL STURM-LIOUVILLE PROBLEMS VIA LAPLACE TRANSFORM

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In this paper, we consider a class of fractional Sturm-Liouville problems, in which the second order derivative is replaced by the Caputo fractional derivative. The Laplace transform method is applied to obtain algebraic equations. Then, the eigenvalues and the eigenfunctions of the fractional Sturm-Liouville problems are obtained numerically. We provide a convergence analysis for given method. Finally, the simplicity and efficiency of the numerical method is shown by some examples.

Key words : Fractional Sturm-Liouville problem; Caputo fractional derivative; Laplace transform.

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1. Introduction

Sturm-Liouville problems have been known since 1836. The concept of Sturm-Liouville problems plays an important role in mathematics and physics [4]. In recent years, researchers have focused on a certain generalization type of the classical Sturm-Liouville problem to the fractional one. Many authors have studied the Sturm-Liouville problem in which the second order derivative is replaced by a Caputo fractional derivative. Study of this type of Sturm-Liouville problem is important for solving diffusion equations and oscillator problems [11].

We consider the Caputo fractional Sturm-Liouville problem (CFSLP) of the form

$$ {}_0^C D_t^\alpha y(t) + (\lambda r(t) - q(t))\, y(t) = 0, \quad t \in [0,1], \tag{1}$$

where q and r are real-valued constant functions, $1 < \alpha \leq 2$ and ${}_0^C D_t^\alpha$ is the Caputo fractional derivative. The boundary conditions are

$$\begin{aligned} ay(0) + by'(0) &= 0, \\ cy(1) + dy'(1) &= 0, \end{aligned} \tag{2}$$

where $a, b, c, d \in \mathbb{R}$, $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$.

The problems (1)-(2) are solved using some numerical schemes such as Adomian decomposition method [2], Predictor-corrector algorithm with Newton method [8], Homotopy analysis method [1], Homotopy Perturbation Method [3, 11], and Augmented-RBF Method [5].

The aim is to solve the problems (1)-(2) with constant coefficients. As a comparison with other methods, we use the Laplace transform to obtain algebraic equations and introduce a simple and effective method for approximating the eigenvalues and the eigenfunctions of the CFSLPs with a high convergence rate.

The structure of the paper is as follows:

In Section 2, we recall some definitions and results related to fractional calculus and Laplace transform. In Section 3, we construct the method using the Laplace transform. In Section 4, a convergence analysis is obtained. In Section 5, some examples is provided to show simplicity and efficiency of the method.

2. PRELIMINARIES

In this section we recall some of the basic definitions and results related to fractional calculus [6, 9, 10, 12]. In order to define the Caputo-fractional derivative, we first define the Riemann-Liouville fractional integral.

Definition 1 — A real-valued function $f(t)$ defined in $(0, \infty)$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0, \infty)$, and it is said to be in the space C_μ^n if $f^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2 — The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$${}_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(x)}{(t-x)^{1-\alpha}} dx \quad t > 0. \tag{3}$$

Definition 3 — Let $f \in C_{-1}^{m}$, $m \in \mathbb{N}$. Then, the Caputo-fractional derivative of f is defined as

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \begin{cases} {}_{t_0}I_t^{m-\alpha} f^{(m)}(t), & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{4} $$

Thus, for $1 < \alpha < 2$, we have

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{t} \frac{f^{(2)}(x)}{(t-x)^{\alpha-1}} dx. \tag{5} $$

Definition 4 — The one parameter Mittag-Leffler function is defined by

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \text{Re}(\alpha) > 0, $$

and the two-parameter Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \;\; \alpha, \beta \in \mathbb{C}, \quad \text{Re}(\alpha) > 0. \tag{6} $$

Definition 5 — A function $f(t)$ defined for $0 \leq t < \infty$ is said to be exponentially bounded of order $\alpha \in \mathbb{R}$ if it satisfies an inequality of the form

$$ \|f(t)\| \leq M e^{\alpha t} $$

for some real constant $M > 0$ and for all sufficiently large t.

Definition 6 — The Laplace transform is defined by

$$ F(s) = L(f(t); s) := \int_0^{\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}, \tag{7} $$

where $f(t)$ is a exponentially bounded function of order α.

Lemma 1 — (see [12]). The Laplace transform of the Caputo derivative operator of order α is given by

$$ L({}_{0}^{C}D_t^{\alpha} f(t); s) = s^{\alpha} L(f(t); s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \;\; n-1 < \alpha \leq n. $$

Lemma 2 — (see [12]). For $k \in \mathbb{N}_0$, $\alpha > 0$ and $\beta > 0$, we have

$$ L(t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^{\alpha}); s) = \left(\frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}} \right), \qquad (Re(s) > |a|^{\frac{1}{\alpha}}). \tag{8} $$

Corollary 1 — The particular case of (8) for $k = 0$ is in the following form

$$ L(t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha}); s) = s^{\alpha-\beta} (s^{\alpha} \mp a)^{-1}. \tag{9} $$

3. Constructing the Method

Let $q(t) = q$ and $r(t) = r$ be constant functions. Taking the Laplace transform of both side of (1), we obtain

$$s^{\alpha} L(y; s) - s^{\alpha-1} y(0) - s^{\alpha-2} y'(0) + (\lambda r - q) L(y; s) = 0,$$

hence,

$$L(y; s) = s^{\alpha-1}(s^{\alpha} + \lambda r - q)^{-1} y(0) + s^{\alpha-2}(s^{\alpha} + \lambda r - q)^{-1} y'(0).$$

The inverse Laplace transform using (9) yields

$$y(t) = E_{\alpha,1}(-(\lambda r - q)t^{\alpha}) y(0) + t E_{\alpha,2}(-(\lambda r - q)t^{\alpha}) y'(0). \tag{10}$$

By differentiating equation (6), we obtain

$$\begin{aligned} E'_{\alpha,1}(-(\lambda r - q)t^{\alpha}) &= \frac{1}{t} \sum_{k=1}^{\infty} \frac{(-(\lambda r - q)t^{\alpha})^k}{\Gamma(\alpha k)} \\ &= \frac{1}{t} \sum_{k=0}^{\infty} \frac{(-(\lambda r - q)t^{\alpha})^{k+1}}{\Gamma(\alpha k + \alpha)} \\ &= -(\lambda r - q) t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-(\lambda r - q)t^{\alpha})^k}{\Gamma(\alpha k + \alpha)} \\ &= -(\lambda r - q) t^{\alpha-1} E_{\alpha,\alpha}(-(\lambda r - q)t^{\alpha}). \end{aligned} \tag{11}$$

Similarly, we have

$$(t E_{\alpha,2}(-(\lambda r - q)t^{\alpha}))' = E_{\alpha,1}(-(\lambda r - q)t^{\alpha}). \tag{12}$$

Therefore,

$$y'(t) = -(\lambda r - q) t^{\alpha-1} E_{\alpha,\alpha}(-(\lambda r - q)t^{\alpha}) y(0) + E_{\alpha,1}(-(\lambda r - q)t^{\alpha}) y'(0). \tag{13}$$

Thus,

$$y'(1) = -(\lambda r - q) E_{\alpha,\alpha}(-(\lambda r - q)) y(0) + E_{\alpha,1}(-(\lambda r - q)) y'(0), \tag{14}$$

and

$$y(1) = E_{\alpha,1}(-(\lambda r - q)) y(0) + E_{\alpha,2}(-(\lambda r - q)) y'(0). \tag{15}$$

Using the equations (14-15) and the second boundary condition (2), we obtain

$$\begin{aligned}&\{cE_{\alpha,1}(-(\lambda r-q))-d(\lambda r-q)E_{\alpha,\alpha}(-(\lambda r-q))\}\,y(0)+\\&\{cE_{\alpha,2}(-(\lambda r-q))+dE_{\alpha,1}(-(\lambda r-q))\}\,y'(0)=0.\end{aligned}\tag{16}$$

By applying the first boundary condition (2), we have the following cases:

Case I : Let $y(0)=0$ and $y'(0)\neq 0$, then the eigenvalues of CFSLP is obtained by solving the following equation:

$$cE_{\alpha,2}(-(\lambda r-q))+dE_{\alpha,1}(-(\lambda r-q))=0.\tag{17}$$

Case II : Let $y'(0)=0$ and $y(0)\neq 0$, then the eigenvalues of CFSLP is obtained by solving the following equation:

$$cE_{\alpha,1}(-(\lambda r-q))-d(\lambda r-q)E_{\alpha,\alpha}(-(\lambda r-q))=0.\tag{18}$$

Case III : Let $y'(0)\neq 0$ and $y(0)\neq 0$, then the eigenvalues of CFSLP is obtained by solving the following equations:

$$\begin{aligned}&-\frac{bc}{a}E_{\alpha,1}(-(\lambda r-q))+\frac{bd}{a}(\lambda r-q)E_{\alpha,\alpha}(-(\lambda r-q))\\&+cE_{\alpha,2}(-(\lambda r-q))+dE_{\alpha,1}(-(\lambda r-q))=0,\end{aligned}\tag{19}$$

whenever $a\neq 0, y(0)=\frac{-b}{a}y'(0)$ and

$$\begin{aligned}&cE_{\alpha,1}(-(\lambda r-q))-d(\lambda r-q)E_{\alpha,\alpha}(-(\lambda r-q))\\&-\frac{ca}{b}E_{\alpha,2}(-(\lambda r-q))-\frac{da}{b}E_{\alpha,1}(-(\lambda r-q))=0,\end{aligned}\tag{20}$$

whenever $b\neq 0, y'(0)=\frac{-a}{b}y(0)$.

Now, we can use the first N terms of the corresponding power series of (17)-(20) to approximate the eigenvalues. The main object of the current work is solving problems (1)-(2) with the constant coefficients. However, the proposed method can be applied to the non-constant functions. In this case, one can approximate the functions $q(t)$ and $r(t)$ with suitable constant functions, and then solve the approximate problem with proposed method. A convergence analysis is provided in the next section.

4. Convergence Analysis

The equations (17)-(20) show that the eigenvalues of CFSLP are zeros of Mittag-Leffler functions. We know that the power series in the definition of the Mittag-Leffler functions are uniformly convergent and holomorphic on $\mathbb{C}$. Hence, we can use the following theorems for convergence analysis.

***Theorem* 1** — *[Hurwitz's theorem] [7]. Let f_k be a sequence of holomorphic functions on a connected open set G that converge uniformly on compact subsets of G to a holomorphic function f which is not constantly zero on G. If f has a zero of order m at z_0 then for every small enough $\epsilon > 0$ and for sufficiently large $k \in \mathbb{N}$, (depending on ϵ), f_k has precisely m zeros in the disk defined by $|z - z_0| < \epsilon$, including multiplicity. Furthermore, these zeros converge to z_0 as $k \to \infty$.*

***Theorem* 2** — *Let (a_n) be a sequence of real numbers and*

$$f_N = \sum_{n=0}^{N} a_n z^n, \quad N = 0, 1, 2, \cdots .$$

Suppose f_N uniformly converges to f on $\mathbb{C}$ and $f(z) = \lim_{N\to\infty} f_N(z)$. Define

$$R_N(z) := \sum_{n=N+1}^{\infty} a_n z^n.$$

Suppose that u_N is the first positive zero of f_N and $\widehat{z}$ is the first positive zero of f of order m. Then, u_N converges to $\widehat{z}$ and for sufficiently large N, the convergence rate is proportional to $|R_N(\widehat{z})|$.

PROOF : By Hurwitz's theorem, for every small enough $\epsilon > 0$ there exists $M \in \mathbb{N}$, such that for $N > M$, f_N has precisely m zeroes in the disk defined by $|z - \widehat{z}| < \epsilon$, including multiplicity, and

$$\lim_{N\to\infty} u_N = \widehat{z}.$$

Therefore, for $|z - \widehat{z}| < \epsilon$ there exist $g(z) \neq 0$ that

$$f_N(z) = (z - u_N)^m g(z).$$

Since

$$f(\widehat{z}) = \sum_{n=0}^{N} a_n \widehat{z}^n + R_N(\widehat{z}) = 0,$$

we can write

$$(\widehat{z} - u_N)^m g(\widehat{z}) + R_N(\widehat{z}) = 0.$$

Consequently,

$$|(\widehat{z} - u_N)^m| = \frac{|R_N(\widehat{z})|}{|g(\widehat{z})|},$$

and hence convergence rate depend on $|R_N(\widehat{z})|$.

5. EXAMPLES

Example 1 : Consider the fractional eigenvalue problem

$$
\begin{aligned}
&{}_0^C D_t^\alpha y(t) + \lambda r y(t) = 0, \quad t \in [0,1], \\
&y(0) = y(1) = 0.
\end{aligned}
\tag{21}
$$

By using the boundary condition at $t = 0$, according with (10), we have

$$y(t) = AtE_{\alpha,2}(-\lambda r t^\alpha), \quad A \neq 0,$$

and by using the boundary condition at $t = 1$, λ must be a zero of

$$E_{\alpha,2}(-\lambda r) = \sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{\lambda r}^{2k}}{\Gamma(\alpha k + 2)}.$$

By introducing

$$\sin_\alpha(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\Gamma(\alpha k + 2)},$$

we can write

$$E_{\alpha,2}(-\lambda r) = \frac{\sin_\alpha(\sqrt{\lambda r})}{\sqrt{\lambda r}}.$$

Therefore, if we denote the positive roots of $\sin_\alpha(z)$ by $0 < z_{\alpha,1} < z_{\alpha,2} < z_{\alpha,3} < \ldots$ We have

$$\lambda_{\alpha,k} = \frac{z_{\alpha,k}^2}{r}.$$

We may approximate the roots of $\sin_\alpha(z)$ by solving

$$S_N(z) := \sum_{k=0}^{N} \frac{(-1)^k z^{2k+1}}{\Gamma(\alpha k + 2)} = 0.$$

We note that,

$$\lim_{\alpha \to 2^-} tE_{\alpha,2}(-\lambda r t^\alpha) = tE_{2,2}(-\lambda r t^2) = \frac{1}{\sqrt{\lambda r}} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{\lambda r} t)^{2k+1}}{(2k+1)!} = \frac{\sin(\sqrt{\lambda r} t)}{\sqrt{\lambda r}},$$

which shows that the behavior of eigenvalues of fractional case is similar to the ordinary one. The numerical results for different values of α and N are presented in Table 1. The rapid convergence rate was observed with increasing N. Also, we observed that as $\alpha \to 2$, $z_{\alpha,k} \to \pi$. In Table 2, we

Table 1: The first positive zero of the S_N for different values of N and α. We note that $\pi \approx 3.1415926535897932$

N	$z_{1.85,1}$	$z_{1.9,1}$	$z_{2,1}$
$N = 10$	3.123296202303711	3.108466841833564	3.148690071466963
$N = 20$	3.072480596282568	3.084500457816756	3.141592653060877
$N = 30$	3.072480637058272	3.084500466674975	3.141592653589788
$N = 40$	3.072480637058277	3.084500466674982	3.141592653589798

Table 2: Numerical values of the first three eigenvalues of CFSLP in example 1 with $N = 50$

$\lambda_{\alpha,k}$	$k = 1$	$k = 2$	$k = 3$
$\lambda_{1.75,k}$	9.59774287120276	25.9580498036576	59.494550310686
$\lambda_{1.85,k}$	9.44013726509802	30.9825058644835	66.944793840096
$\lambda_{1.90,k}$	9.51414312891812	33.5956714097741	73.0390172208613
$\lambda_{1.95,k}$	9.66077192405220	36.4045244994683	80.3290435879122
$\lambda_{2,k}$	9.86960440108938	39.4784176043584	88.8264396097779

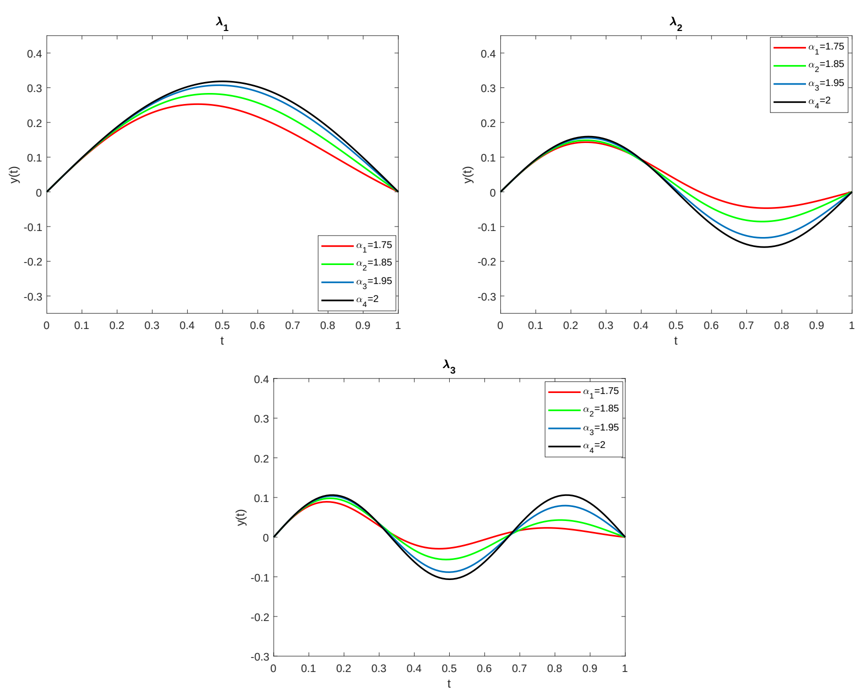

Figure 1: Eigenfunctions corresponding to the first three eigenvalues of CFSLP in Example 1 with $N = 50$

obtained $\lambda_{\alpha,k}$, $k = 1, 2, 3$, for different values of α, $N = 50$, and $r = 1$. As we expected, if $\alpha \to 2$, then $\lambda_{\alpha,k} \to (k\pi)^2$. Fig. 1 shows the first three eigenfunctions for $\alpha = 1.75, 1.85, 1.95, 2$.

Example 2 : [2]. Consider the fractional eigenvalue problem

$$\begin{aligned} &{}_0^C D_t^\alpha y(t) + \lambda r y(t) = 0, \quad t \in [0,1], \\ &y'(0) = y(1) = 0. \end{aligned} \tag{22}$$

By using boundary condition at $t = 0$, according with (10), we have

$$y(t) = A E_{\alpha,1}(-\lambda r t^\alpha), \quad A \neq 0.$$

By using boundary condition at $t = 1$, λ must be a zero of

$$E_{\alpha,1}(-\lambda r) = \sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{\lambda r}^{2k}}{\Gamma(\alpha k + 1)}.$$

By introducing

$$\cos_\alpha(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(\alpha k + 1)},$$

we can write

$$E_{\alpha,1}(-\lambda r) = \cos_\alpha(\sqrt{\lambda r}).$$

Let $z_{\alpha,k}$ be an ascending sequence corresponding to the positive zeros of

$$S_N(z) = \sum_{k=0}^{N} \frac{(-1)^k z^{2k}}{\Gamma(\alpha k + 1)}.$$

Then, the corresponding eigenvalues are $\lambda_{\alpha,k} \simeq \frac{z_{\alpha,k}^2}{r}$. Table 3 shows the results for different values of N, $r = 1$, and $\alpha = 1.5$. This is in the agreement with the results of [1] and [2]. We note that,

$$\lim_{\alpha \to 2} E_{\alpha,1}(-\lambda r t^\alpha) = E_{2,1}(-\lambda r t^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{\lambda r} t)^{2k}}{(2k)!} = \cos(\sqrt{\lambda r} t),$$

which shows that the behavior of eigenvalues of fractional case is similar to ordinary one. Eigenfunctions corresponding to the first three eigenvalues of this problem is shown in Fig. 2.

Example 3 : [3]. Consider the fractional eigenvalue problem

$$\begin{aligned} &{}_0^C D_t^\alpha y(t) + \lambda r y(t) = 0, \quad t \in [0,1], \\ &y(0) = 0, \\ &y(1) + y'(1) = 0. \end{aligned} \tag{23}$$

Table 3: Numerical values of the first three eigenvalues of CFSLP in example 2 with $N = 50$

N	$\lambda_{1.5,1}$	$\lambda_{1.5,2}$	$\lambda_{1.5,3}$
$N = 40$	2.110277084326249	13.765382245224549	24.239418829655065
$N = 50$	2.110277084326246	13.765382232319869	24.243286866542256
$N = 60$	2.110277084326245	13.765382232321333	24.243286760999158
$N = 70$	2.110277084326244	13.765382232326859	24.243286760870578

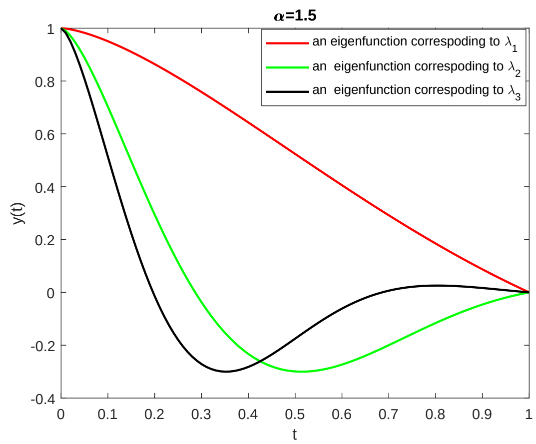

Figure 2: Eigenfunctions corresponding to the first three eigenvalues of CFSLP in Example 2 with $N = 50$

By using boundary conditions, according with (10), we have

$$y(t) = AtE_{\alpha,2}(-\lambda r t^{\alpha}), \quad A \neq 0,$$

and

$$E_{\alpha,2}(-\lambda r) + E_{\alpha,1}(-\lambda r) = \frac{\sin_{\alpha}(\sqrt{\lambda r}) + \sqrt{\lambda r}\cos_{\alpha}(\sqrt{\lambda r})}{\sqrt{\lambda r}} = 0. \tag{24}$$

Let $z_{\alpha,k}$ be an ascending sequence corresponding to the positive zeros of

$$\sin_{\alpha}(z_{\alpha,k}) + z_{\alpha,k}\cos_{\alpha}(z_{\alpha,k}) = 0,$$

then,

$$\lambda_{\alpha,k} = \frac{z_{\alpha,k}^2}{r},$$

are the corresponding eigenvalues of the problem (25). Table 4 is in agreement with the paper [3].

Table 4: Numerical values of the first three eigenvalues of CFSLP in example 3 with $N = 50$, $r = 1$

N	$k = 1$	$k = 2$	$k = 3$
$\lambda_{1.75,k}$	1.94361270610105	4.10575243360411	6.29852004880917
$\lambda_{1.85,k}$	1.96481739311505	4.39162805370592	6.88087415928556
$\lambda_{1.90,k}$	1.98224337335402	4.55231397760251	7.21605953982922
$\lambda_{1.95,k}$	2.00365194220657	4.72591879694141	7.58151184601632
$\lambda_{2,k}$	2.02875783811044	4.91318043943491	7.97866571241289

6. Conclusion

In this study, we presented a simple and efficient algorithm to numerically compute the eigenvalues and eigenfunctions of fractional Sturm-Liouville problems. For this purpose, we obtained the general solution of the fractional Sturm-Liouville equation using the Laplace transform. By using the boundary conditions, we achieved algebraic equations of eigenvalues and eigenfunctions. The method was examined with several examples. The convergence of the method was proven as N is increased. We confirmed the accuracy of this method by comparing our results with other reported results. We showed that the solutions of fractional Sturm-Liouville problems converge to the classical one as $\alpha \to 2$.

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