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A NOTE ON THE LINEAR STABILITY OF THE STEADY STATE OF A NONLINEAR RENEWAL EQUATION WITH A PARAMETER

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In this article we consider a variant of age-structured nonlinear Lebowitz-Rubinow equation. We study the linear stability of this equation near the nontrivial steady state by analyzing the corresponding characteristic equation. In particular, we provide some sufficient conditions under which the nonzero steady state is linearly stable.

Key words : Existence of steady states; linear stability; nonlinear renewal equation.

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1. Introduction

Age structured models play a central role in the study of population dynamics. The mathematical theory of linear age structured population models was originally studied by Lotka and Sharpe (see [8, 11]). Among the age structured models the simplest one is due to McKendrick–Von Foerster (see [8, 11, 16]). Mathematical study of this model can be found in [5, 16]. In this model, the mortality rate d and the fertility rate B for the population are just age dependent functions. So this model does not take into account the aging process which is very important for cell population (see [2]). Moreover, in the McKendrick–Von Foerster model it is assumed that the birth process does not change aging properties of off-spring. However, experimental results show that different cells, under identical conditions, have exhibited highly variable intermitotic intervals (see [16]).

Therefore, Lebowitz *et al*. have proposed a modified McKendrick-Von Foerster model, which is widely known as the Lebowitz-Rubinow model, where population is structured not only in age x but also in terms of generation time a. Existence, uniqueness and long time behavior of a solution to the Lebowitz-Rubinow model has been discussed in [10].

However, this model or McKendrick-Von Foerster model do not incorporate competition among cells to acquire/utilize resources (like nutrients). This competition term induces nonlinearity in vital rates (see [4]). For existence, uniqueness results and asymptotic behavior of solution to the nonlinear McKendrick-Von Foerster equations, one can refer [5, 12, 16].

In this article, we consider the following partial differential equation which is a variant of a Lebowitz-Rubinow model with competition term in the fertility rate, i.e.,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x;a) + \frac{\partial}{\partial x}u(t,x;a) + d(x;a)u(t,x;a) = 0,\ t>0,\ x>0, a\geq 0, \\ u(t,0;a) = \displaystyle\int_0^\infty B(x,S(t);a)u(t,x;a)dx, t>0,\ a\geq 0, \\ u(0,x;a) = u^0(x;a) \geq 0,\ x>0,\ a\geq 0, \end{cases} \tag{1}$$

with a coupling

$$S(t;a) = \int_0^\infty \psi(x;a)u(t,x;a)dx,\ t>0,\ x>0. \tag{2}$$

Throughout this article we assume that the vital rates B and d satisfy the following conditions. We assume that B is a smooth, positive, bounded and integrable function for every nonnegative parameter a. Moreover, we assume

$$\frac{\partial B(x,S;a)}{\partial S} \leq 0,\ x,S,a\geq 0, \tag{3}$$

$$1 < \int_0^\infty B(x,0;a)e^{-\int_0^x d(y;a)dy}dx < \infty,\ a\geq 0, \tag{4}$$

$$\exists\Sigma > 0 \text{ such that } 0 < \int_0^\infty B(x,\Sigma;a)e^{-\int_0^x d(y;a)dy}dx < 1,\ a\geq 0. \tag{5}$$

Furthermore, we assume that there exist two constants $M>0, \underline{d}>0$ such that

$$|B_S(x,S;a)| < M,\ x,S,a\geq 0, \qquad \underline{d} \leq d(x;a),\ x>0,\ a\geq 0. \tag{6}$$

We also assume that ψ is a nonnegative bounded function. For an existence and uniqueness result for a weak solution to (1)-(2), readers can refer [12].

This paper is organized as follows. In Section 2, we define the steady state equations corresponding to (1)-(2) and discuss the behavior of trivial steady state. We conclude Section 2 by establishing existence, uniqueness of a nontrivial steady state. In Section 3, we derive the characteristic equation whose roots characterize the linear stability of the steady state. Later, we provide some sufficient conditions that vital rates satisfy such that the nonzero steady state exhibits the linear stability.

2. NONTRIVIAL STEADY STATE

One of the guiding principles in demographic evolution of population is that *any population eventually forgets its initial age distribution* (see [5]). Therefore we expect that the population density converges to the steady state, which is of course in dependent of its initial profile, as time tends to infinity in appropriate sense.

We now define the system of steady state equations corresponding to (1)-(2) as follows: for $a \geq 0$,

$$\begin{cases} \frac{d}{dx} U(x;a) + d(x;a)U(x;a) = 0, \ x \geq 0, \\ U(0;a) = \displaystyle\int_0^\infty B(x, \bar{S}(a); a)U(x;a)dx, \\ \bar{S}(a) = \displaystyle\int_0^\infty \psi(x;a)U(x;a)dx. \end{cases} \tag{7}$$

It is easy to observe that $\bar{U} \equiv 0$ is always a solution to [1], i.e., a steady state. We now consider the linearized version of the renewal equation (1)-(2) around this trivial steady state which reads as, for $a \geq 0$,

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u}(t,x;a) + \frac{\partial}{\partial x}\tilde{u}(t,x;a) + d(x;a)\tilde{u}(t,x;a) = 0, \ t \geq 0, \ x \geq 0, \\ \tilde{u}(t,0;a) = \displaystyle\int_0^\infty B(x,0;a)\tilde{u}(t,x;a)dx, \\ \tilde{u}(0,x;a) = u^0(x;a). \end{cases} \tag{8}$$

Thanks to (4), it follows that the first eigenvalue of the steady state equation corresponding to (8) is positive. Therefore the zero steady state $\bar{U} \equiv 0$ is globally unstable. It can be proved that, in some particular cases, the dynamics of (1)-(2) never come close to zero (see [1, 12]).

We now show the existence and uniqueness of a nontrivial solution to (7). To this end, first notice that any solution to (7) is given by

$$U(x;a) = U_0 e^{-D(x;a)}, \ D(x;a) = \int_0^x d(y;a)dy, \tag{9}$$

where $U_0, \ \bar{S}$ solve the system

$$\bar{S}(a) = U_0 \int_0^\infty \psi(x;a)e^{-D(x;a)}dx, \ \int_0^\infty B(x, \bar{S}(a); a)e^{-D(x;a)}dx = 1. \tag{10}$$

Thanks to (3)-(5), using the standard monotonicity arguments one can obtain that there exists a unique solution to (10) for every $a \geq 0$. This immediately gives us existence and uniqueness of a nonzero solution to (7).

3. The Linear Stability

In this section, we study the asymptotic behavior of the solution to the equations that we obtain by linearizing (1)-(2) around the nontrivial steady state. In [16] the author has proposed and proved 'the principle of linear stability', which states that, a steady state is exponentially stable if all the spectral values of the infinitesimal generator of the linearized semi group has negative real part and is unstable if there is at least one spectral value with positive real part. Integrated semi groups are used to prove the principle of linearized stability for a variety of nonlinear age structured models (see [14]). In [15] analytical methods are used to study the linear stability of a nonlinear McKendrick–Von Foerster model.

3.1 *Linearization*

In this subsection, we linearize (1)-(2) around the nonzero steady state U and derive the characteristic equation corresponding to the linearized problem. The characteristic equation is indispensable because its roots determine the linear stability (see [3]). Define w by $u(t,x;a) = w(t,x;a) + U(x;a)$ for every $a,\ x,\ t \geq 0$. The linearized system of (1)-(2) around the steady state U is given by

$$\begin{cases} \frac{\partial}{\partial t}w(t,x;a) + \frac{\partial}{\partial x}w(t,x;a) + d(x;a)w(t,x;a) = 0,\ t>0, x>0, a\geq 0, \\ w(t,0;a)=\displaystyle\int_0^\infty B(x,\bar{S}(a);a)w(t,x;a)dx + S_w(t;a)\int_0^\infty B_S(x,\bar{S}(a);a)U(x;a)dx, \\ S_w(t;a) = \displaystyle\int_0^\infty \psi(x;a)w(t,x;a)dx,\ t>0,\ a\geq 0. \end{cases} \tag{11}$$

To investigate the long time behavior of the solution to (11), we seek the standard particular solution of the form $w(t,x;a) = e^{\lambda t}W(x;a)$ with $\lambda \in \mathbb{C}$. Then we substitute this ansatz in (11) to get the corresponding eigenvalue problem

$$\begin{cases} \frac{d}{dx}W(x;a) + \big(d(x;a)+\lambda\big)W(x;a) = 0,\ x>0,\ a\geq 0, \\ W(0;a) = \displaystyle\int_0^\infty B(x,\bar{S}(a);a)W(x;a)dx + S_W\int_0^\infty B_S(x,\bar{S}(a);a)U(x;a)dx, \\ S_W(a) = \displaystyle\int_0^\infty \psi(x;a)W(x;a)dx,\ a\geq 0. \end{cases} \tag{12}$$

One can easily solve (12) and use (9) to get

$$W(x;a) = W(0;a)e^{-D(x;a)-\lambda x}, S_W(a) = W(0;a)\int_0^\infty \psi(x;a)e^{-D(x;a)-\lambda x}dx. \tag{13}$$

We introduce the notations

$$\begin{cases} L(\lambda;a) = \int_0^\infty \psi(x;a)e^{-D(x;a)-\lambda x}dx, \\ M(\lambda;a) = \int_0^\infty B(x,\bar{S}(a);a)e^{-D(x;a)-\lambda x}dx, \\ N(a) = \frac{\bar{S}(a)}{L(0;a)}\int_0^\infty e^{-D(x;a)}B_S(x,\bar{S}(a);a)dx. \end{cases} \tag{14}$$

We substitute W in the boundary condition of (12) to obtain

$$W(0;a)L(\lambda;a) - S_W(a) = 0, \tag{15}$$

$$W(0;a)(M(\lambda;a)-1) + S_W(a)N(a) = 0. \tag{16}$$

A necessary and sufficient condition for the existence of a nontrivial solution of the form $e^{\lambda t}W(x;a)$ to (11) is that equations (14)-(15) admit a nontrivial solution, i.e.,

$$\Gamma(\lambda;a) := L(\lambda;a)N(a) + M(\lambda;a) = 1. \tag{17}$$

This is the characteristic equation corresponding to (11). The main difficulty to analyze this condition is that the equation (11) is very intricate.

Remark 1 : It is easy to observe that there is no positive root to (16). For, notice that $N(a) < 0,\ L(\lambda;a) > 0$ and $M(\lambda;a) \leq 1,\ \lambda \geq 0, a \geq 0$. Therefore we obtain $\Gamma(\lambda;a) < 1,\ \lambda \geq 0, a \geq 0$.

We conclude this subsection with the following characterization of linearly stable steady states.

Lemma 2 — The steady state U is linearly asymptotically stable if all the roots (λ's) of equation (10), (16) have negative real part and unstable if there exists at least one root which has positive real part. □

Main result

Our objective in this subsection is to study the characteristic equation and deduce some sufficient conditions for the linear stability of the steady state. In [19], the author has given a sufficient condition for global nonlinear stability, linear stability when mortality is solely age dependent and the birth rate is of a particular form. In the next result that we present here, we give some other set of sufficient conditions to have the linear stability.

***Theorem* 3** — *Assume (3)-(6) and there exists $k > 0$ such that*

$$\psi \in L^1(e^{kx}dx),\ a \geq 0, \qquad J := \int_0^\infty \psi(x;a)e^{kx}dx. \tag{18}$$

Moreover, assume that

$$\frac{\partial}{\partial a}B(x,S;0)<0,\ \frac{\partial}{\partial a}B_S(x,\bar{S}(0);0)<0,\ \frac{\partial}{\partial a}D(x;0)\geq 0,\ x,S>0, \tag{19}$$

$$B_S(x,\bar{S}(0);0)=0,\ x>0, \tag{20}$$

then there exists $\varepsilon>0$ *such that for all* $a\in(0,\varepsilon]$ *the nontrivial steady state is linearly stable.*

PROOF : We begin with setting

$$F(\lambda,a):=\frac{\bar{S}(a)}{L(0;a)}\int_0^\infty \psi(x;a)e^{-D(x;a)-\lambda x}dx\int_0^\infty e^{-D(x;a)}B_S(x,\bar{S}(a);a)dx,$$

$$G(\lambda,a)=\int_0^\infty B(x,\bar{S}(a);a)e^{-D(x;a)-\lambda x}dx.\square$$

Then characteristic equation (19) becomes

$$1=F(\lambda,a)+G(\lambda,a). \tag{21}$$

Step 1 : In this step we prove that there exists $\alpha<0$ such that there is at most one solution to

$$G(\lambda,a)=1, \tag{22}$$

satisfying Re$(\lambda)\geq\alpha$. For, first we observe that $\lambda=0$ is a unique real valued solution to (21). Assume that $\lambda\neq 0$ is another solution to (21) in $\mathbb{C}$. Then we have

$$1=\int_0^\infty B(x,\bar{S}(a);a)e^{-D(x;a)-\lambda x}dx<\int_0^\infty B(x,\bar{S}(a);a)e^{-D(x;a)-\mathrm{Re}(\lambda)x}dx,$$

which readily implies that Re$(\lambda)<0$. On the other hand, notice that there are only finite number of roots in any strip $\alpha_1<$Re$(\lambda)<0$ because $G(a,\lambda)\to 0$ uniformly in a as $|\lambda|\to\infty$. Hence there exist only finitely many roots in the strip $\alpha_1<$Re$(\lambda)<0$. Therefore there exists $\alpha<0$ independent of a such that there is at most one solution (namely $\lambda=0$) to (21) satisfying Re$(\lambda)\geq\alpha$.

Step 2 : In this step we show that for each $a\geq 0$, (20) has at most one solution in the half plane Re$(\lambda)\geq\widetilde{\alpha}:=\max\{\alpha,-k\}$. In order to do that first choose $r>0$ large enough such that

$$|1-G(\lambda,a)|>\tfrac{1}{2},\ |\lambda|>r,\ \mathrm{Re}(\lambda)\geq\widetilde{\alpha},\ a\geq 0.$$

We now set

$$m:=\inf_{\substack{\beta\in\mathbb{R}\\ a\geq 0}}\left|1-G(\widetilde{\alpha}+\mathrm{i}\beta,a)\right|\neq 0,$$

$$E(r) := \left[\{\lambda : Re(\lambda) > \widetilde{\alpha}\} \cap C(0,r)\right] \cup \{\lambda : \mathrm{Re}(\lambda) = \widetilde{\alpha}\},$$

where $C(0,r)$ is the circle in the complex plane with center at (0,0) and radius r. Therefore we get

$$\min\{\tfrac{1}{2}, m\} < |1 - G(\lambda, a)| \text{ for } \lambda \in E(r),\ a \geq 0,$$

for r large enough.

We next turn our attention towards $F(\lambda, a)$. Consider

$$L(0;a) = \int_0^\infty \psi(x;a) e^{-D(x;a)} dx > e^{-D(1;a)} \int_0^1 \psi(x;a) dx.$$

Moreover, for $\mathrm{Re}(\lambda) \geq \widetilde{\alpha}$ we have

$$\int_0^\infty \psi(x;a) e^{-D(x;a) - \lambda x} dx < J.$$

Furthermore, from (12) it is evident that $\bar{S}(.)$ is bounded and we set $\gamma := \|\bar{S}\|_\infty$. We combine all these facts to get

$$|F(\lambda, a)| < \frac{J e^{D(1;a)} \gamma}{\int_0^1 \psi(x;a) dx} \int_0^\infty |B_S(x, \bar{S}(a); a)| e^{-D(x;a)} dx.$$

From (22) and the Lebesgue dominated convergence theorem we obtain

$$\int_0^\infty B_S(x, \bar{S}(a); a) e^{-D(x;a)} dx \to 0 \text{ as } a \to 0. \tag{23}$$

Hence there exists $\bar{\varepsilon} > 0$ such that every $a \in [0, \bar{\varepsilon}]$ satisfies

$$|F(\lambda, a)| < \min\{\frac{1}{2}, m\} < |1 - G(\lambda, a)| \text{ for } \lambda \in E(r). \tag{24}$$

Therefore by Rouche's theorem, (21) has at most one root in the half plane $\mathrm{Re}(\lambda) \geq \widetilde{\alpha}$ for each $a \in [0, \bar{\varepsilon}]$.

Step 3 : In this step we locate the root in the complex plane. Thanks to (19), we have $F(0,0) = 0,\ G(0,0) = 1$ and $\lambda(0) = 0$. For small values of a, we denote the root of (20) by $\lambda(a)$. Observe that $\lambda(a)$ defines a differentiable path in the complex plane, starting from $\lambda(0) = 0$. We differentiate (20) with respect to a to obtain (24)

$$\mathrm{Re}\Big[\frac{\partial}{\partial a} F(\lambda, a) + \frac{d\lambda(a)}{da} \frac{\partial}{\partial \lambda} F(\lambda, a) + \frac{\partial}{\partial a} G(\lambda, a) + \frac{d\lambda(a)}{da} \frac{\partial}{\partial \lambda} G(\lambda, a)\Big]_{a=0} = 0.$$

A straightforward computation using (21) yields us

$$\frac{\partial}{\partial \lambda}F(\lambda,a)\big|_{a=0}=0,$$
$$\frac{\partial}{\partial a}F(\lambda,a)\big|_{a=0}=\bar{S}(0)\int_0^\infty \frac{\partial}{\partial a}B_S(x,\bar{S}(0);0)e^{-D(x;0)}dx,$$
$$\frac{\partial}{\partial \lambda}G(\lambda,a)\big|_{a=0}=-\int_0^\infty xB(x,\bar{S}(0);0)e^{-D(x;0)}dx,$$
$$\frac{\partial}{\partial a}G(\lambda,a)\big|_{a=0}=\int_0^\infty \big(B_a(x,\bar{S}(0);0)-B(x,\bar{S}(0);0)D_a(x;0)\big)e^{-D(x;0)}dx.$$

Therefore from (6), (24) we obtain

$$\begin{aligned}\mathrm{Re}\frac{d\lambda(0)}{da} &= \frac{\displaystyle\int_0^\infty \big(\bar{S}(0)\frac{\partial B_S}{\partial a}+B_a-D_a(x;0)B\big)(x,\bar{S}(0);0)e^{-D(x;0)}dx}{\displaystyle\int_0^\infty xB(x,\bar{S}(0);0)e^{-D(x;0)}dx}\\ &< 0.\end{aligned}$$

Hence there exits $\varepsilon>0$ such that for every $a\in(0,\varepsilon]$, the corresponding first eigenvalue $\lambda(a)$ has the property that Re$(\lambda(a))$ is negative. Hence we proved the announced result. □

We conclude this section by presenting an example of fertility rate B, mortality rate d, and competition weight ψ such that these vital rates satisfy hypotheses of Theorem 3.

Example 4 : First set $B(x,S;a) := (e^{-a}+M_1e^{-aS})B_1(x)$ where M_1 is a positive constant and B_1 is a nonnegative function to be chosen later. We take $d(x;a) := d_1(x)$ such that the net reproduction number $R := \int_0^\infty B_1(x)e^{-d_1(x)}dx$ is a finite quantity. Furthermore, we take $\psi(x;a) = e^{-a^2-x}$. We now choose $M_1>0$ large enough such that (1)-(2) hold. Now it is easy to choose B_1, d_1 such that hypotheses of Theorem 3 is satisfied and we omit the details.

4. Conclusions

We have discussed the existence of steady states for the Lebowitz-Rubinow type equations when the fertility rate decreases in a, S. We have derived the characteristic equation whose location of roots completely characterize the linear stability. Furthermore, we have demonstrated that, if the birth rate B decreases in each of its variables in the particular way described in Theorem 3, then the corresponding steady states will exhibit the linear stability.

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