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EQUICHORDAL TIGHT FUSION FRAMES

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A Grassmannian fusion frame is an optimal configuration of subspaces of a given vector space, that are useful in some applications related to representing data in signal processing. Grassmannian fusion frames are robust against noise and erasures when the signal is reconstructed. In this paper, we present an approach to construct optimal Grassmannian fusion frames based on a given Grassmannian frame. We also analyse an algorithm for sparse fusion frames which was introduced by Calderbank *et al*. and present necessary and sufficient conditions for the output of that algorithm to be an optimal Grassmannian fusion frame.

Key words : Grassmannian frame; fusion frame; chordal distance; optimal Grassmannian fusion frame.

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1. Introduction

When data is transmitted over the communication channel, it might be corrupted by noise or be lost. Grassmannian frames provide a representation which is robust against noise and multiple erasures [16, 17]. They are characterized by a property that the maximal cross correlation of the frame elements have minimum value among a given class of frames [16, 20]. The definition of Grassmannian frame can be generalized to the Grassmannian fusion frame which is a set of subspaces that the minimal chordal distance between subspaces has the maximum value among a given class of fusion frames [16, 17]. Similar to Grasmanian frame, Grassmannian fusion frame is robust against noise and multiple erasures. In more details, Calderbank *et al*. [17] show that a fusion frame is optimally robust against noise if the fusion frame is tight and a tight fusion frame is optimally robust against one subspace erasure if the dimension of the subspaces are equal. They also proved that a tight fusion frame is optimally robust against multiple erasures if the subspaces are equidistant.

Sparse recovery from combined fusion frame measurements is an interesting problem for scientist during the last decade. Grassmannian fusion frames has such a property since the mutual coherence of subspaces meets the lower bound. Hence, these fusion frames minimize the coherence and use a very few projection measurements to recover the signal [5].

The notion of sparse fusion frame is a fusion frame whose subspaces are generated by orthonormal basis vectors that are sparse in a uniform basis over all subspaces, thereby enabling low complexity fusion frame decompositions [8]. Moreover, an algorithmic construction to compute fusion frames with desired fusion frame operators, including tight fusion frame are provided [6]. We will discuss the conditions that this fusion frame is Grassmannian.

This paper contains a new approach to construct a Grassmannian fusion frames base on a given Grassmannian frame or Grassmannian fusion frame, along with some illustrative examples. The following section is devoted to state a brief summary of the frame and the fusion frame. In Section 3, we recall the notion of optimal Grassmannian frames, Grassmannian fusion frames, and present an approach to construct an optimal Grassmannian fusion frame. Then we provide some examples of Grassmannian fusion frames. In the final section, we impose some conditions on an algorithm by Calderbank *et al.* such that the output becomes a sparse Grassmannian fusion frame.

2. Preliminaries and Notations

Fusion frames originally were called frames of subspaces, which are introduced by Casazza and Kutyniok [9]. It is a generalization of frame theory. In more details, in the frame theory, the signal is represented by magnitude of projections of the signal into frame vectors, while in the fusion frame, a signal is presented by a collection of vectors, where elements are the inner product of the signal and orthogonal bases of subspaces.

Fusion frames are new notion which may be applied in a number of different fields, such as sampling theory [13], data quantization [4], coding [3], image processing [7], time frequency analysis [12], and speech recognition [2].

Definition 1 — A fusion fame for $\mathbb{F}^m$ is a finite family of n dimensional subspaces $\{W_i\}_{i=1}^N$ in

$\mathbb{F}^m$ such that there exist two constants $0 < A \leq B < \infty$ satisfying

$$\forall x \in \mathbb{F}^m \quad A\|x\|^2 \leq \sum_{i=1}^{N} \|P_i x\|^2 \leq B\|x\|^2, \tag{2.1}$$

where P_i is the orthogonal projection to W_i.

The constants A and B are called the fusion frame bounds. Furthermore, when $A = B$ the fusion frame is tight. The inequality (2.1) is equivalent to

$$AI \leq \sum_{i=1}^{N} P_i \leq BI.$$

In this case, every $x \in \mathbb{F}^m$ can be reconstructed by the reconstruction formula as follows [10].

$$x = \sum_{i=1}^{N} S^{-1} P_i x,$$

where $S = \sum_{i=1}^{N} P_i$ is the fusion frame operator and it is positive and self adjoint.

3. Grassmannian Fusion Frame

In this section some necessary notations and theorems related to the Grassmannian frame are provided. To do so, first we explain the notion of maximal frame correlation, that is

$$\mathcal{M}_\infty(\{f_i\}_{i=1}^{N}) = \max_{1 \leq i < j \leq N} \{|\langle f_i, f_j \rangle|\},$$

where $\{f_i\}_{i=1}^{N}$ is unit norm frame for $\mathbb{F}^m$.

Definition 2 — A sequence of unit-norm vectors $\{u_i\}_{i=1}^{N} \subseteq \mathbb{F}^m$ is called a Grassmannian frame if

$$\mathcal{M}_\infty(\{u_i\}_{i=1}^{N}) = min\{\mathcal{M}_\infty(\{f_i\}_{i=1}^{N})\},$$

where the minimum is taken over all unit-norm frames $\{f_i\}_{i=1}^{N}$ in $\mathbb{F}^m$.

In fact, the frame with minimum correlation is equal to the frame with the maximum angle between each two elements be as small as possible. It is clear whenever $N = m$, Grassmannian frames are precisely the orthonormal bases for $\mathbb{F}^m$. There is a bound for $\mathcal{M}_\infty(\{f_i\}_{i=1}^{N})$ for a given N and m [20]. For $\{f_i\}_{i=1}^{N}$,which is a frame for $\mathbb{F}^m$, the following inequality is satisfied:

$$\mathcal{M}_\infty(\{f_i\}_{i=1}^{N}) \geq \sqrt{\frac{N-m}{m(N-1)}}. \tag{3.1}$$

The equality holds in (2) if and only if $\{f_i\}_{i=1}^N$ is an equiangular tight frame. Furthermore, if $\mathbb{F} = \mathbb{R}$ the equality (2) satisfies whenever $N \leq \frac{m(m+1)}{2}$ and if $\mathbb{F} = \mathbb{C}$, the equality (2) satisfies whenever $N \leq m^2$.

A unit-norm frame that meets bound (2) with equality is called the optimal Grassmannian frame. The following theorem shows the characterisation of Gram matrix of an optimal Grassmannian frame [20].

***Theorem* 3** — *Consider $m, N \in \mathbb{N}$ be natural numbers with $N \geq m$. Moreover, R is an Hermitian $N \times N$ matrix with entries $R_{k,k} = 1$ and $R_{k,l} = \pm\sqrt{\frac{N-m}{m(N-1)}}$ for $\mathbb{F} = \mathbb{R}$ and $R_{k,l} = \pm i\sqrt{\frac{N-m}{m(N-1)}}$ for $\mathbb{F} = \mathbb{C}$ and $k, l = 1, \ldots, N$ and $k \neq l$. If the eigenvalues $\lambda_1, \ldots, \lambda_N$ of R are such that $\lambda_1 = \ldots = \lambda_m = \frac{N}{m}$ and $\lambda_{m+1} = \ldots = \lambda_N = 0$, then there exists an optimal Grassmannian frame $\{f_i\}_{i=1}^N$ in $\mathbb{F}^m$.*

The entries $R_{k,l}$s can not be equal to the values $+\sqrt{\frac{N-m}{m(N-1)}}$ for $1 \leq k, l \leq N$ and $k \neq l$, since the eigenvalues of this matrix are $\lambda_1 = 1 + (N-1)\sqrt{\frac{N-m}{m(N-1)}}$ and $\lambda_2 = \ldots = \lambda_m = 1 - \sqrt{\frac{N-m}{m(N-1)}}$. Hence, the matrix R can not be the Gram matrix of an optimal Grassmannian frame, but if $R_{k,l}$ are equal to the values $-\sqrt{\frac{N-m}{m(N-1)}}$ for $1 \leq k, l \leq N$ and $k \neq l$, it is the Gram matrix of an optimal Grassmannian frame if and only if $N = m + 1$.

It is worthwhile to mention that there are some examples which are not optimal Grassmannian frame. For instance, the five vectors in $\mathbb{R}^3$ that minimize (3.1) are equiangular, but the maximal inner product is $\frac{1}{\sqrt{5}}$ (not $\frac{1}{\sqrt{6}}$), and it is not an optimal Grassmannian frame.

In order to clarify the notion of Grassmannian fusion frames, it is necessary to explain the notion of Grassmannian space $G(m, n)$.

Definition 4 — The Grassmannian space $G(m, n)$ is the collection of subspaces W_i of dimension n in $\mathbb{F}^m$ equipped with a metric known as chordal distance defined by

$$d_c(i, j) = [n - tr(P_i P_j)]^{\frac{1}{2}}\ , \tag{3.2}$$

where P_i is the orthogonal projection on W_i.

The Grassmannian packing problem is the problem of finding N elements in $G(m, n)$ such that the minimal distance between any two elements is as large as possible. This problem for n=1 is precisely equivalent to the construction of Grassmannian frames.

It is easy to check an equivalent for chordal distance:

$$d_c(i,j) = [n - \sum_{k,l=1}^{n} |\langle e_{ik}, e_{jl}\rangle|^2]^{\frac{1}{2}}, \tag{3.3}$$

where $\{e_{ik}\}_{k=1}^n$ is an orthonormal basis for W_i and $\{e_{jl}\}_{l=1}^n$ is an orthonormal basis for W_j.

The following theorem provide a bound on chordal distance. Another proof of this theorem is given in [11], which is more complicated than what is presented here.

***Theorem* 5** — *For every packing of subspaces $\{W_i\}_{i=1}^N$ of dimension n in $\mathbb{F}^m$,*

$$d_c^2 \leq \frac{n(m-n)}{m}\frac{N}{N-1}, \tag{3.4}$$

where d_c is the minimum chordal distance between any two elements.

PROOF : Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of $\sum_{i=1}^N P_i$, where P_i denotes the orthogonal projection to W_i. we have

$$\sum_{i=1}^{m} \lambda_i = tr(\sum_{i=1}^{N} P_i) = Nn.$$

On the other hand

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i^2 &= tr(\sum_{i=1}^{N} P_i \sum_{j=1}^{N} P_j) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} tr(P_i P_j) \\
&= (\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} tr(P_i P_j)) + Nn \\
&= \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} (n - d_c^2(i,j)) + Nn \\
&= N(N-1)n - \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} d_c^2(i,j) + Nn \\
&\geq \frac{N^2 n^2}{m}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} d_c^2 &\leq \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} d_c^2(i,j) \\
&\leq N(N-1)n + Nn - \frac{N^2 n^2}{m}.
\end{aligned}$$

Then

$$d_c^2 \leq \frac{1}{N(N-1)}[N(N-1)n + Nn - \frac{N^2n^2}{m}]$$
$$= \frac{(m-n)Nn}{m(N-1)}.$$

The bound in (5) is called simplex bound. Theorem 5 states if the minimal chordal distance between any two elements meets simplex bound, the fusion frame is the Grassmannian fusion frame. This happens, because the minimal distance can not grow any further.

Definition 6 — If the minimal chordal distance of a given fusion frame meets the simplex bound, it is called an optimal Grassmannian fusion frame.

The following theorem is useful to construct an optimal Grassmannian fusion frames [17].

***Theorem* 7** — *Assume* $\{W_i\}_{i=1}^N$ *is a fusion frame with equidimensional subspaces and equal chordal distance* d_c*. Then the fusion frame is tight if and only if* d_c^2 *equals to the simplex bound.*

The following algorithm shows a novel approach to construct an optimal Grassmannian fusion frame using other optimal Grassmannian fusion frames or Grassmannian frames.

Algorithm 1 : Constructing Optimal Grassmannian Fusion Frame (COGFF)

Parameters:

- Dimension $m \in \mathbb{N}$, number of subspaces N and dimension of subspaces m.
- Fusion frame $\{W_i'\}_{i=1}^N$ with $W_i' := span\{e_{jt}'; t = 1, \cdots, n\}$.

Algorithm set $\ell := (0, \cdots, 0)$

For $j = 1, \cdots, k$ do
 For $t = 1, \cdots, n$ do
  set $e_{jt}^i := \ell$
  For $s = 1, \cdots, m$ do
   $e_{jt}^i(jn + s) := e_{jt}'^i(s)$
   end;
  end;
 end;

Output:

- Fusion frame $\{W_i\}_{i=1}^N$ with $W_i := span\{e_{jt}^i; t = 1, \cdots, n, j = 1, \cdots, k\}$.

The following theorem shows that the output of the COGFF algorithm is an optimal Grassmannian fusion frame.

***Theorem* 8** — *Assume the natural numbers m, n, N and k are given. If there exists an optimal Grassmannian fusion frame for $\mathbb{F}^m$ with N, n-dimensional subspaces, then there exists an optimal Grassmannian fusion frame with N subspaces of dimension kn in $\mathbb{F}^{km}$.*

PROOF : Let $\{W'_i\}_{i=1}^N$ be an optimal Grassmannian fusion frame for $\mathbb{F}^m$, and $\{e'^i_{jt}\}_{t=1}^n$ be an orthonormal basis for W'_i. Consider $i : \mathbb{F}^m \to \mathbb{F}^{km}$ with $i\,(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$ and set $e^i_{1t} = i\left(e'^i_{1t}\right)$ for $t = 1, \cdots, n$ and $i = 1, \cdots, N$. We can construct e^i_{jt} for $j = 2, \cdots, k$ in the similar way. Assume W_i is generated by $\{e^i_{jt}\}_{j=1,t=1}^{k,n}$. Then

$$\begin{aligned} d_c^2(i,j) &= kn - (\sum_{j,l=1}^{k} \sum_{t,t'=1}^{N} |\langle e^i_{jt}, e^s_{lt'}\rangle|^2) \\ &= kn - \frac{kn(nN-m)}{m(N-1)} \\ &= \frac{n(m-n)}{m}\frac{N}{N-1}k, \end{aligned}$$

since $\{W'_i\}_{i=1}^N$ is an optimal Grassmannian fusion frame for $\mathbb{F}^m$ and meets the simplex bound. Therefore, $\{W_i\}_{i=1}^N$ is an optimal Grassmannian fusion frame for $\mathbb{F}^{km}$. □

Theorem 8 is used to construct Grassmannian fusion frames for special cases m, n, N. The following corollary is a direct conclusion of Theorem 8.

Corollary 9 — Assume m, N and k are given. There is an optimal Grassmannian fusion frame with N subspaces of dimension k in $\mathbb{F}^{km}$, when there exists an optimal Grassmannian frame for $\mathbb{F}^m$ with N vectors.

By using the Theorem 8 we provide some examples to construct optimal Grassmannian fusion frames.

Example 10 : For the first example, consider a result obtained by Strohmer *et al*. [20]. It states if p is a prime number and set $m = p^l + 1$ for $l \in \mathbb{N}$ and $N = m^2 - m + 1$. Then, there exist integers $0 \le d_1 \le d_2 \le \ldots \le d_m \le N$ such that all numbers $1, \ldots, N-1$ occur as residues mod N of the $n(n-1)$ differences $d_i - d_j$, $i \ne j$. For $k = 1, \ldots, N$, we define

$$f_k := \{\frac{1}{\sqrt{m}} e^{2\pi i k d_j / N}\}_{j=1}^m.$$

It follows that $\{f_k\}_{k=1}^N$ is an optimal Grassmannian frame.

Then, using Theorem 8, an optimal Grassmannian fusion frame is obtained with N subspaces of dimension k in $\mathbb{F}^{km}$, where m, N are as above.

Example 11 : This example consider a result of King [16] that states, if H is an $n \times n$ Hadamard matrix indexed by $0, \ldots, n-1$ which has been normalized so that the first row consists only of 1. Then,

$$\{f_i = \frac{1}{\sqrt{n-1}}(H(j.i))_{1 \le j \le n-1} : 0 \le i \le n-1\}$$

is an optimal Grassmannian frame for $\mathbb{F}^{n-1}$.

By Theorem 8 there exists an optimal Grassmannian fusion frame with $N = n-1$ subspaces of dimension k in $\mathbb{F}^{k(n-1)}$, where n is as above.

4. Sparse and Grassmannian Fusion Frame

In this section, the algorithm constructed by Calderbank *et al*. [6] is provided, where the output is a sparse fusion frame with prescribed fusion frame operator. Note that the notion of sparse fusion frames and the role of them in signal processing is mentioned in Section 1. Now, we are looking for some necessary and sufficient conditions to impose on this algorithm that makes the output an optimal Grassmannian fusion frame. Thereby, the resulted fusion frame operator is sparse in the sense of [6] and robust against noise and multiple erasures simultaneously.

Assume $\lambda_1, \cdots, \lambda_m$ are as follows: $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 2$ and $\sum_{i=1}^m \lambda_i = nN$. The algorithm is:

The following theorem shows how the algorithm 4 works [6].

***Theorem* 12** — *Suppose the real values* $N \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 2$ *and* $n \in \mathbb{N}$ *satisfy* $\sum_{i=1}^m \lambda_i = nN$. *In addition, if* j_0 *is the first integer in* $\{1, \cdots, m\}$, *for which* λ_{j_0} *is not an integer, then* $\lfloor \lambda_{j_0} \rfloor \le N-3$.

Then, the fusion frame constructed by SFFCRE algorithm satisfies: $\dim W_i = n$ *for all* $i = 1, \cdots, N$ *and the associate fusion frame operator has* $\{\lambda_i\}_{i=1}^m$ *as its eigenvalues.*

In the following theorem, we characterize the Gram matrix of an optimal Grassmannian fusion frame.

***Theorem* 13** — *Let* N, m, n *be natural numbers with* $m \ge n$. *Assume* R *is a Hermitian* $Nn \times Nn$ *matrix with* $R = (R_{ij})_{1 \le i,j \le N}$ *where* R_{ij} *is a square matrix of dimension* n *for all* $1 \le i, j \le N$ *and* $R_{ii} = I_n$ *for all* $1 \le i \le N$ *and* $\|R\|_{Fr}^2 = n - d_c^2$ *where* $d_c^2 = \frac{n(m-n)}{m}\frac{N}{N-1}$. *If the eigenvalues* $\lambda_1, \cdots, \lambda_m$ *of* R *are such that* $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \frac{Nn}{m}$ *and* $\lambda_{m+1} = \cdots = \lambda_{Nn} = 0$, *then there exists an optimal Grassmannian fusion frame* $\{W_i\}_{i=1}^N$ *that achieves the simplex bound.*

Algorithm 2 : Sparse Fusion Frame Construction for Real Eigenvalues (SFFCRE)

Parameters:

- Dimension $m \in \mathbb{N}$.
- Eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m > 0$. Number of subspaces N and dimension of subspaces m satisfying $\sum_{i=1}^{m} \lambda_i = Nn \in \mathbb{N}$.

Algorithm

set $k := 1$

For $j = 1, \cdots, m$ do

Repeat

If $\lambda_j < 1$ then

$w_k := \sqrt{\frac{\lambda_j}{2}} e_j + \sqrt{1 - \frac{\lambda_j}{2}} e_{j+1}$

$w_{k+1} := \sqrt{\frac{\lambda_j}{2}} e_j - \sqrt{1 - \frac{\lambda_j}{2}} e_{j+1}$

$k := k + 2$

$\lambda_{j+1} = \lambda_{j+1} - (2 - \lambda_j)$

$\lambda_j := 0$

else

$w_k := e_j$

$k := k + 1$

$\lambda_j := \lambda_j - 1$

end;

until $\lambda_j = 0$.

end;

Output:

- Fusion frame $\{W_i\}_{i=1}^{N}$ with $W_i := span\{w_{i+kn}; k = 0, \cdots, n-1\}$.

PROOF : Since R is Hermitian it has a spectral factorization of the form $R = WAW^*$ where the columns of W are the eigenvectors of R and A is the diagonal matrix with eigenvalues of R as the diagonal entries. Without loss of generality, assume that the non-zero eigenvalues of R are contained in the first m diagonal entries of R. Set $u_k := \sqrt{\frac{Nn}{m}}\{W_{kl}\}_{l=1}^m$ for $k = 1, \cdots, Nn$ and $W_i = span\{u_{kN+i}; 0 \leq k \leq m-1\}$. Hence, $d_c(i,j) = n - d_c^2$ for all $1 \leq i, j \leq N$, $i \neq j$. Then, $\{W_i\}_{i=1}^N$ is equidistance. Since all the nonzero eigenvalues of R are equal, $\{W_i\}_{i=1}^N$ is tight. So, by Theorem 7, $\{W_i\}_{i=1}^N$ is an optimal Grassmannian fusion frame.

The following theorem asserts necessary and sufficient conditions, that is the fusion frame constructed by the SFFCRE algorithm is an optimal Grassmannian fusion frame.

***Theorem* 14** — *Assume in the algorithm SFFCRE the eigenvalues $\lambda_1 = \cdots = \lambda_m = \frac{Nn}{m}$ are integers. Then the fusion frame constructed by SFFCRE algorithm is an optimal Grassmannian fusion frame if and only if $\frac{Nn}{m} = N - 1$.*

PROOF : From SFFCRE algorithm and by Theorem 13, the matrix R associated to the analysis operator for the fusion frame $\{W_i\}_{i=1}^N$ constructed by this algorithm has a form $R_{ii} = I_n$ and $\|R\|_{Fr}^2 = 2(N-1)(n - d_c^2) + N$.

Hence, by Theorem 7 it is enough to show that $\{W_i\}_{i=1}^N$ is an optimal Grassmannian fusion frame if and only if the number of common vectors e_i for any two subspaces of $\{W_i\}_{i=1}^N$ are equal.

In order to do this, consider $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \frac{Nn}{m} = N - 1$. The number of common vectors e_i for any two subspaces of $\{W_i\}_{i=1}^N$ are equal. Set $A_i = \{e_{i_t}; W_i = span\{e_{i_t} for \quad t = 1, \cdots, n\}\}$. Without loss of generality, consider $(m, n) = 1$.

We want to show that $|A_i \cap A_k| = n - 1$ for all $1 \leq i, k \leq N$. Hence, assume there exists two subspaces, for example W_i and W_k, such that $|A_i \cap A_k|$ is equal to $n - a$ where $a > 1$. However, at least $n - a + 2a = n + a$ different elements of e_i are exists. Since $(n, m) = 1$, we have $N = km$ and by substitution the equality $m - \frac{1}{k} + a \leq m$ satisfies. Therefore, $k = 1$ and $a = 1$.

Conversely, assume $\frac{Nn}{m} = a$ where $a \neq N-1$. Without loss of generality, assume again $(n, m) = 1$. And $(N, a) = 1$ because if for example we have $(N, a) = k \neq 1$ then $|A_1 \cap A_{k+1}| \neq |A_1 \cap A_2|$. By this assumption, we have $|A_1 \cap A_N| = n - 1$, while $|A_1 \cap A_N| < n - 1$.

Example 15 : Let p be a prime number. By Theorem 14, an optimal Grassmannian fusion frame can be constructed for $\mathbb{F}^p$ with p subspaces of dimension $p - 1$. Then, by Theorem 8 we have an optimal Grassmannian fusion frame for $\mathbb{F}^{kp}$ with p subspaces of dimension $k(p - 1)$ for a given integer k.

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