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FINITE GROUPS WITH SYSTEMS OF Σ-$\mathfrak{F}$-EMBEDDED SUBGROUPS[1]

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Let $\mathfrak{F}$ denote a class of groups. A maximal subgroup M of G is called $\mathfrak{F}$-abnormal provided $G/M_G \notin \mathfrak{F}$. We say that (K, H) is an $\mathfrak{F}$-abnormal pair of G provided K is a maximal $\mathfrak{F}$-abnormal subgroup of H. Let $\Sigma = \{G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n\}$ be a subgroup series of G. A subgroup H of G is said to be Σ-$\mathfrak{F}$-embedded in G if H either covers or avoids every $\mathfrak{F}$-abnormal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$ for some $i \in \{0, 1, \cdots, n\}$. In this paper, some new characterizations of p-supersoluble and p-soluble are given by discussing the properties of Σ-$\mathfrak{F}$-embedded of subgroups.

Key words : Finite group; $\mathfrak{F}$-abnormal pair; Σ-$\mathfrak{F}$-embedded; p-supersoluble groups; p-soluble groups.

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1. INTRODUCTION

Throughout this paper, all groups considered are finite, G always denotes a group and p denotes a prime. Let G_p denote the Sylow p-subgroup of G and $\pi(G)$ denote the set of all prime divisors of $|G|$.

A class of groups $\mathfrak{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is called saturated (resp. solubly saturated) if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ (resp. $G/\Phi(N) \in \mathfrak{F}$ for a soluble normal subgroup N of G). We use $\mathfrak{U}$ and $\mathfrak{S}_p$ to

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denote the classes of all supersoluble groups and p-soluble groups, respectively. It is well known that $\mathfrak{U}$ and $\mathfrak{S}_p$ are all s-closed saturated formations. All unexplained notation and terminology are standard, as in [1-3] if necessary.

Let A be a subgroup of G, $K \leq H \leq G$ and $\Sigma = \{G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n\}$ some subgroup series of G. Then we say: (i) A covers the pair (K, H) if $AK = AH$; (ii) A aviods the pair (K, H) if $A \cap K = A \cap H$; (iii) A is Σ-embedded in G [4] if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$ for some i; (iv) A is Σ_p-embedded G [4] if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$ for some i and p divides $|G_i : G_{i-1}|$, where p is a prime. Let $\mathfrak{F}$ denote a class of groups. A maximal subgroup M of G is called $\mathfrak{F}$-abnormal provided $G/M_G \notin \mathfrak{F}$. We say that (K, H) is an $\mathfrak{F}$-abnormal pair of G provided K is a maximal $\mathfrak{F}$-abnormal subgroup of H. This observation leads us to the following generalization of the Σ-embedded (resp. Σ_p-embedded) subgroups.

Definition 1.1 — Let $\mathfrak{F}$ be a formation and $\Sigma = \{G_0 \leq G_1 \leq G_2 \leq \cdots G_n\}$ some subgroup series of G. A subgroup H of G is said to be Σ-$\mathfrak{F}$-embedded in G if H either covers or avoids every $\mathfrak{F}$-abnormal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$ for some i.

Definition 1.2 — Let $\mathfrak{F}$ be a formation, p a prime and $\Sigma = \{G_0 \leq G_1 \leq G_2 \leq \cdots G_n\}$ some subgroup series of G. A subgroup H of G is said to be Σ_p-$\mathfrak{F}$-embedded in G if H either covers or avoids every $\mathfrak{F}$-abnormal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$ for some i and p divides $|G_i : G_{i-1}|$.

In this paper, we use the properties of Σ-$\mathfrak{F}$-embedded (resp. Σ_p-$\mathfrak{F}$-embedded) of subgroups to characterize the structure of finite group.

2. Preliminaries

Lemma 2.1 — Let E be $\{A \leq G\}$-$\mathfrak{F}$-embedded in G, $A \leq V \leq G$ and $N \trianglelefteq G$.

(1) Then $E \cap V$ is $\{A \cap V \leq V\}$-$\mathfrak{F}$-embedded in V.

(2) Then EN/N is $\{AN/N \leq G/N\}$-$\mathfrak{F}$-embedded in G/N.

Proof : (1) Let (K, H) be an $\mathfrak{F}$-abnormal pair such that $A \cap V \leq K < H \leq V$. Then clearly, (K, H) is an $\mathfrak{F}$-abnormal pair such that $A \leq K < H \leq G$. Thus E either covers or avoids (H, K). If E covers (K, H), then $(E \cap V)K = EK \cap V = EH \cap V = (E \cap V)H$. If E avoids (K, H), then clearly, $(E \cap V) \cap K = (E \cap V) \cap H$. It follows that $E \cap V$ either covers or avoids (K, H). Hence $E \cap V$ is $\{A \cap V \leq V\}$-$\mathfrak{F}$-embedded in V.

(2) Let $(K/N, H/N)$ be an $\mathfrak{F}$-abnormal pair such that $AN/N \leq K/N < H/N \leq G/N$. Then $H/K_H \cong (H/N)/(K_H/N) = (H/N)/(K/N)_{(H/N)} \notin \mathfrak{F}$. It follows that (K, H) is an $\mathfrak{F}$-abnormal pair such that $A \leq AN \leq K < H \leq G$. Hence E either covers or avoids (K, H). If E covers (K, H), then $KEN = HEN$ and so $(K/N)(EN/N) = (H/N)(EN/N)$. Now assume that E avoids (K, H), then $(K/N) \cap (EN/N) = (K \cap E)N/N = (H \cap E)N/N = (H/N) \cap (EN/N)$. Thus EN/N either covers or avoids $(K/N, H/N)$. Consequently, EN/N is $\{AN/N \leq G/N\}$-$\mathfrak{F}$-embedded in G/N.

Lemma 2.2 — [4, Lemma 2.4]. Let A and B be proper subgroups of G such that $G = AB$. Then $G = AB^x$ and $G \neq AA^x$ for all $x \in G$.

Let $\mathfrak{F}$ be a formation. A subgroup H of G is said to be $\mathfrak{F}$-subnormal (in the sense of Kegel [5]) or K-$\mathfrak{F}$-subnormal in G [6, p. 236] if either $H = G$ or there is a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all $i = 0, 1, 2, \cdots, n$.

Lemma 2.3 — Let $H \leq G$. If every Sylow subgroup of G is $\{H \leq G\}$-$\mathfrak{F}$-embedded in G. Then H is K-$\mathfrak{F}$-subnormal in G.

PROOF : If $H = G$, it is clear. Assume that $H \neq G$. Let M be a maximal subgroup of G containing H and p any prime divisor of $|M|$. Then for every Sylow p-subgroup M_p of M, there exists a Sylow p-subgroup G_p of G such that $M_p = M \cap G_p$. By Lemma 2.1(1), M_p is $\{H \leq M\}$-$\mathfrak{F}$-embedded in M. By induction, H is K-$\mathfrak{F}$-subnormal in M.

If $G/M_G \in \mathfrak{F}$, then H is K-$\mathfrak{F}$-subnormal in G. Now, assume that $G/M_G \notin \mathfrak{F}$. Then (M, G) is an $\mathfrak{F}$-abnormal pair. Let p dividing $|G : M|$ and G_p be a Sylow p-subgroup of G. Then $G_p \nleq M$. By hypothesis, $G = MG_p$. Let q be any prime divisor of $|G|$ such that $q \neq p$ and G_q be any Sylow q-subgroup of G. It is easy to see that (M^x, G) is an $\mathfrak{F}$-abnormal pair for any $x \in G$. If $G_q \nleq M^x$ for some $x \in G$, then $G = G_qM^x$ and so $G_p \leq M^{xy}$, where $y \in G$. Since $G = G_pM$, by Lemma 2.2, it is impossible. Hence $G_q \leq M_G$ and so $O^p(G) \leq M_G$. It follows that G/M_G is a p-group. Hence M is normal in G. Consequently, H is K-$\mathfrak{F}$-subnormal in G.

Lemma 2.4 — [7, Theorem 2.1.6]. If G is a p-supersoluble group and $O_{p'}(G) = 1$, then p is the largest prime dividing $|G|$ and G is supersoluble.

Lemma 2.5 — [7, Theorem 3.2.28]. Let the group $G = AB$ be the product of an abelian subgroup A and a nilpotent subgroup B. Then $AF(G)$ is a normal subgroup of G.

Lemma 2.6 — Every $\{1 \leq G\}$-$\mathfrak{F}$-embedded subgroup of G is K-$\mathfrak{F}$-subnormal in G.

PROOF : Suppose that the result is false and let G be a counterexample of minimal order. Let E be a $\{1 \leq G\}$-$\mathfrak{F}$-embedded subgroup of G. If $E = G$, it is evident. Assume that $E \neq G$. Then there exists a maximal subgroup M of G such that $E \leq M$. Let (K, H) be any $\mathfrak{F}$-abnormal pair such that $1 \leq K < H \leq M$. Clearly, (K, H) is an $\mathfrak{F}$-abnormal pair such that $1 \leq K < H \leq G$. Hence E either covers or avoids (K, H). By induction, E is K-$\mathfrak{F}$-subnormal in M. If $G/M_G \in \mathfrak{F}$, then E is K-$\mathfrak{F}$-subnormal in G. Now assume that $G/M_G \notin \mathfrak{F}$. It is easy to see that (M^x, G) is an $\mathfrak{F}$-abnormal pair of G for any $x \in G$. Suppose that $E \nleq M^x$ for some $x \in G$, then $G = EM^x$, which is impossible by Lemma 2.2. Hence $E \leq M_G$. By a similar discussion as above, we have that E is K-$\mathfrak{F}$-subnormal in M_G by induction. Consequently, E is K-$\mathfrak{F}$-subnormal in G. □

Lemma 2.7 — [8, Lemma 2.14]. Let $\mathfrak{F}$ be a saturated (solubly saturated) formation and F the canonical local (the canonical composition, respectively) satellite of $\mathfrak{F}$. Let E be a normal p-subgroup of G. Then $E \leq Z_{\mathfrak{F}}(G)$ if and only if $G/C_G(E) \in F(p)$.

Lemma 2.8 — [9, Chapter VI, Theorem 4.8]. Let $G = G_1G_2$, where G_1 and G_2 are two nilpotent subgroups of G. If G is soluble and H_1 and H_2 are π-Hall subgroups of G_1 and G_2, respectively, then $H = H_1H_2$ is a π-Hall subgroup of G.

Lemma 2.9 — [7, Chapter 1, Theorem 1.1.9]. Let A and B be two subgroups of G such that $AB^g = B^gA$ for every $g \in G$. If X and Y are two subsets of G, then $[\langle A^X\rangle, \langle B^Y\rangle]$ is a subnormal subgroup of G.

Lemma 2.10 — [1, Lemma 3.8.8]. If G is a soluble group and $|G : N_G(G_p)| = q^\beta > 1$, then G has a q-subgroup $Q \neq 1$ such that G_pQ is a normal subgroup of G.

Lemma 2.11 — [1, Theorem 1.8.6]. The following conditions on a group G are equivalent:

(1) G is p-nilpotent.

(2) G is p-soluble and every maximal subgroup with index a power of p is normal in G.

Lemma 2.12 — Let $E \leq G$ and E be Σ_p-$\mathfrak{F}$-embedded in G for some composition series Σ of G.

(1) If $V \leq G$, then $E \cap V$ is $\{\Sigma \cap V\}_p$-$\mathfrak{F}$-embedded in V.

(2) If $N \leq E$ and N is normal in G, then E/N is $(\Sigma N/N)_p$-$\mathfrak{F}$-embedded in G/N.

PROOF : (1) Let $\Sigma = \{1 = G_0 < G_1 < \cdots < G_n\}$ be a composition series of G such that E is Σ_p-$\mathfrak{F}$-embedded in G. Then $\Sigma \cap V = \{1 = G_0 \cap V \leq G_1 \cap V \leq \cdots \leq G_n \cap V\}$ is a subgroup series of V. Let (K, H) be an $\mathfrak{F}$-abnormal pair such that $G_{i-1} \cap V \leq K < H \leq G_i \cap V$ for some i, where p divides $|G_i \cap V/G_{i-1} \cap V|$. We show that $E \cap V$ either covers or avoids (K, H).

Since $G_i \cap V/G_{i-1} \cap V \cong G_{i-1}(G_i \cap V)/G_{i-1} \leq G_i/G_{i-1}$, p divides $|G_i/G_{i-1}|$. If $G_{i-1} \cap H \nleq K$, then $H = K(H \cap G_{i-1}) \leq K(V \cap G_{i-1}) \leq K$, a contradiction. Hence $H \cap G_{i-1} = K \cap G_{i-1}$. Now, we consider that the subgroup series $G_{i-1} \leq KG_{i-1} \leq HG_{i-1} \leq G_i$. Assume that $KG_{i-1} \leq T \leq HG_{i-1}$. Then $T = G_{i-1}(T \cap H)$ and $K \leq T \cap H \leq H$. If $K = T \cap H$, then $T = KG_{i-1}$. Assume that $H = T \cap H$, then $T = HG_{i-1}$. Hence (KG_{i-1}, HG_{i-1}) is a maximal pair. Obviously, $K \cap G_{i-1} \leq K_H$ and $K_H G_{i-1} \leq (KG_{i-1})_{HG_{i-1}}$. Since $(KG_{i-1})_{HG_{i-1}}/G_{i-1} \leq (KG_{i-1}/G_{i-1})_{HG_{i-1}/G_{i-1}} \cong (K/K \cap G_{i-1})_{H/H\cap G_{i-1}} = (K/K \cap G_{i-1})_{H/K\cap G_{i-1}} = K_H/(K_H \cap G_{i-1}) = K_H G_{i-1}/G_{i-1}$, $(KG_{i-1})_{HG_{i-1}} \leq K_H G_{i-1}$. So $(KG_{i-1})_{HG_{i-1}} = K_H G_{i-1}$. It follows that $HG_{i-1}/(KG_{i-1})_{HG_{i-1}} = HG_{i-1}/K_H G_{i-1} \cong H/K_H(H \cap G_{i-1}) = H/K_H \notin \mathfrak{F}$. Hence (KG_{i-1}, HG_{i-1}) is an $\mathfrak{F}$-abnormal pair. By hypothesis, E either covers or avoids (KG_{i-1}, HG_{i-1}). If E covers (KG_{i-1}, HG_{i-1}), then $(E \cap V)K = (E \cap V)(V \cap KG_{i-1}) = V \cap EKG_{i-1} = V \cap EHG_{i-1} = V \cap (E \cap V)(HG_{i-1}) = (E \cap V)H$. Assume that E avoids (KG_{i-1}, HG_{i-1}). Then $(E \cap V) \cap K = (E \cap V) \cap KG_{i-1} \cap V = E \cap KG_{i-1} \cap V = E \cap HG_{i-1} \cap V = (E \cap V) \cap H$. It follows that $E \cap V$ either covers or avoids (K, H). Therefore $E \cap V$ is $\{\Sigma \cap V\}_p$-$\mathfrak{F}$-embedded in V.

(2) Let $\Sigma = \{1 = G_0 < G_1 < \cdots < G_n\}$ be a composition series of G such that E is Σ_p-$\mathfrak{F}$-embedded in G. Then $\Sigma N/N = \{G_0 N/N \leq G_1 N/N \leq \cdots \leq G_n N/N\}$ is a subgroup series of G/N. Let $(K/N, H/N)$ be an $\mathfrak{F}$-abnormal pair such that $G_{i-1}N/N \leq K/N < H/N \leq G_i N/N$ for some i, where p divides $|G_i N/G_{i-1}N|$. Clearly, $1 \neq G_i N/G_{i-1}N \cong G_i/G_{i-1}$. Thus p divides $|G_i/G_{i-1}|$. Now, we consider that the subgroup series $G_{i-1} \leq K \cap G_i < H \cap G_i \leq G_i$. We show that $(K \cap G_i, H \cap G_i)$ is an $\mathfrak{F}$-abnormal pair.

It is easy to see that $K = N(K \cap G_i) = K_H(K \cap G_i)$, $H = N(H \cap G_i) = K_H(H \cap G_i)$ and $K_H \cap G_i \leq (K \cap G_i)_{H\cap G_i}$. Hence $(K \cap G_i)_{H\cap G_i}/K_H \cap G_i \cong (K \cap G_i/K_H \cap G_i)_{H\cap G_i/K_H\cap G_i} \cong ((K \cap G_i)K_H/K_H)_{(H\cap G_i)K_H/K_H} = (K/K_H)_{H/K_H} = 1$ and so $(K \cap G_i)_{H\cap G_i} = K_H \cap G_i$. Then $H \cap G_i/(K \cap G_i)_{H\cap G_i} = H \cap G_i/(K_H \cap G_i) \cong (H \cap G_i)K_H/K_H = H/K_H \notin \mathfrak{F}$. Let $K \cap G_i \leq L \leq H \cap G_i$. Then $N(K \cap G_i) \leq NL \leq N(H \cap G_i)$, that is, $K \leq NL \leq H$. If $K = NL$, then $K \cap G_i = NL \cap G_i = L(N \cap G_i) = L$. Assume that $H = NL$, we have that $H \cap G_i = NL \cap G_i = L(N \cap G_i) = L$. Hence $(K \cap G_i, H \cap G_i)$ is an $\mathfrak{F}$-abnormal pair. By hypothesis, E either covers or avoids $(K \cap G_i, H \cap G_i)$. If E covers $(K \cap G_i, H \cap G_i)$, then E covers $(N(K \cap G_i), N(H \cap G_i)) = (K, H)$. Assume that E avoids $(K \cap G_i, H \cap G_i)$. Since $N \leq E$, $E \cap H = E \cap N(H \cap G_i) = N(E \cap H \cap G_i) = N(E \cap K \cap G_i) = E \cap N(K \cap G_i) = E \cap K$. Thus E avoids (K, H). It follows that E/N either covers or avoids $(K/N, H/N)$. Therefore E/N is $(\Sigma N/N)_p$-$\mathfrak{F}$-embedded in G/N.

3. Main Results

***Theorem* 3.1** — *Let p be a prime. Suppose that $G = AB$, where A and B are p-nilpotent and every Sylow subgroup of G is $\{A \cap B \leq G\}$-$\mathfrak{U}$-embedded in G. Then G is p-supersoluble.*

Proof : Suppose that the result is false and let G be a counterexample of minimal order. Clearly, a $\{A \cap B \leq G\}$-$\mathfrak{U}$-embedded subgroup is also a $\{A \leq G\}$-$\mathfrak{U}$-embedded subgroup and a $\{B \leq G\}$-$\mathfrak{U}$-embedded subgroup. By Lemma 2.3, we have that A and B are K-$\mathfrak{U}$-subnormal in G. Clearly, $G \neq A$ and $G \neq B$. Hence G has a proper subgroup H_1 such that $A \leq H_1$ and H_1 is normal in G or $G/(H_1)_G \in \mathfrak{U}$. Meanwhile, G has a proper subgroup H_2 different from H_1 such that $B \leq H_2$ and H_2 is normal in G or $G/(H_2)_G \in \mathfrak{U}$. Let M_1 and M_2 be two distinct maximal subgroups of G such that $H_1 \leq M_1$ and $H_2 \leq M_2$. It is obvious that $1 \neq (H_1)_G \leq (M_1)_G$ and $1 \neq (H_2)_G \leq (M_2)_G$. We proceed the proof via the following steps:

(1) $O_{p'}(G) = 1$, G has a unique minimal normal subgroup N and G/N is p-supersoluble.

Let L be any normal subgroup of G. By Lemma 2.1(2), G/L satisfies the hypothesis of the theorem. The choice of G implies that G/L is p-supersoluble. It implies that $O_{p'}(G) = 1$ and G has a unique minimal normal subgroup N.

(2) M_1 and M_2 are p-supersoluble.

Let P be a Sylow p-subgroup of M_1, where p is any prime divisor of $|M_1|$. Then there exists a Sylow p-subgroup G_p of G such that $P = M_1 \cap G_p$. By Lemma 2.1(1), P is $\{A \cap B \leq M_1\}$-$\mathfrak{U}$-embedded in M_1. The choice of G implies that M_1 is p-supersoluble. Similarly, M_2 is p-supersoluble.

(3) G is p-soluble and G has a maximal subgroup M such that $G = N \rtimes M$ and $N = O_p(G) = C_G(N) = F(G)$.

If H_1 is normal in G , then $G/H_1 = H_1B/H_1$ is p-soluble and so G is p-soluble by (2). Assume that $G/(H_1)_G \in \mathfrak{U}$. Clearly by (2), G is p-soluble too. Hence by (1), we know that N is an abelian p-group and $N \nleq \Phi(G)$. Therefore G has a maximal subgroup M such that $G = N \rtimes M$ and $N = O_p(G) = C_G(N)$.

(4) $O_{p'}(M_1) = O_{p'}(M_2) = 1$, M_1 and M_2 are supersoluble and p is the largest prime divisor of $|M_1|$ and $|M_2|$.

Since $N \leq M_1$, $[N, O_{p'}(M_1)] = 1$ and so $O_{p'}(M_1) = 1$ by (3). By (2) and Lemma 2.4, we have that p is the largest prime dividing $|M_1|$ and M_1 is supersoluble. Similarly, $O_{p'}(M_2) = 1$, p is the largest prime dividing $|M_2|$ and M_2 is supersoluble.

(5) $A \cap B \nleq M$.

Assume that $A \cap B \leq M$. Obviously, (M, G) is a $\mathfrak{U}$-abnormal pair. By hypothesis, every Sylow subgroup of G either covers or avoids (M, G). Suppose that for any prime divisor $q \neq p$ of $|G|$, there exists a Sylow q-subgroup G_q of G such that $G_q \nleq M$, then $G = G_qM$, which contradicts $|G : M|$ is a power of p by (3). Hence M contains all Sylow q-subgroups of G. It follows from (1) that $N \leq O^p(G) \leq M$, which is impossible. Hence $A \cap B \nleq M$.

Final contradiction

It is obvious from (4) that N is the Sylow p-subgroup of $(M_1)_G$ and $(M_2)_G$. We will draw contradictions according to the following cases:

(i) $G/(H_1)_G \in \mathfrak{U}$ and $G/(H_2)_G \in \mathfrak{U}$.

Obviously, $G/(M_1)_G \in \mathfrak{U}$ and $G/(M_2)_G \in \mathfrak{U}$. Hence $|G : M_1|$ and $|G : M_2|$ are prime numbers. Firstly, assume that $|G : M_1| = |G : M_2| = p$. By considering the permutation representation of $G/(M_1)_G$ on the right coset of $M_1/(M_1)_G$, we can see that $G/(M_1)_G$ is isomorphic to some subgroup of the symmetric group S_p of degree p. Hence the order of Sylow p-subgroup of $G/(M_1)_G$ is p. Then $M_1/(M_1)_G$ is a p'-group. It implies that N is the Sylow p-subgroup of M_1. Similarly, N is also the Sylow p-subgroup of M_2. Since $G/N = (AN/N)(BN/N) = (M_1/N)(M_2/N)$, N is the Sylow p-subgroup of G, which contradicts $|G : M_1| = |G : M_2| = p$.

Secondly, assume that $|G : M_1| = p$ and $|G : M_2| = q$ or $|G : M_1| = q$ and $|G : M_2| = p$, where $q \neq p$. Without loss of generality, we may assume that $|G : M_1| = p$ and $|G : M_2| = q$. Then M_2 contains a Sylow p-subgroup G_p of G. With a same discussion as above, we have that N is the Sylow p-subgroup of M_1. Since M_1 is supersoluble, M_1' is nilpotent, which implies by (4) that $M_1' \leq F(M_1) = N$. Hence M_1/N is abelian. It is obvious that $G = M_1G_p$ and so $G/N = (M_1/N)(G_p/N)$. By Lemma 2.5, we have that $(M_1/N)F(G/N)$ is normal in G/N. If $(M_1/N)F(G/N) = M_1/N$, then M_1 is normal in G. This implies that $M_1 \cap M_2$ is normal in G. Hence $|G/(M_1 \cap M_2)| = pq$. By (4), we have that q is the smallest prime divisor of $|G/(M_1 \cap M_2)|$. Since $|G/(M_1 \cap M_2) : M_2/(M_1 \cap M_2)| = q$, $M_2/(M_1 \cap M_2)$ is normal in $G/(M_1 \cap M_2)$ and so M_2 is normal in G. By (4), we have that $G_p = N$ is normal in G, which is impossible for $|G : M_1| = p$. Therefore assume that $(M_1/N)F(G/N) = G/N$. Then $G_p/N \leq F(G/N)$ because M_1/N is a p'-group, and thereby $G_p = N$ is normal in G, a contradiction too.

Finally, we suppose that $|G : M_1| = q$ and $|G : M_2| = r$, where $q, r \neq p$. Then there exists Sylow p-subgroups G_p and G_p^x of G such that $G_p \leq M_1$ and $G_p^x \leq M_2$ for some $x \in G$. By (4),

$M_1 \leq N_G(G_p)$ and $M_2 \leq N_G(G_p^x)$. Assume that G_p is normal in G. Then $G_p = N$ is the Sylow p-subgroups of M_1 and M_2. By (4), we know that M_1/N and M_2/N are abelian. Hence $M_1 \cap M_2$ is normal in G.

We show that A and B are nilpotent. If $A \cap N = 1$, then $A \cong AN/N \leq M_1/N$ is abelian and A is a p'-group. Hence $B \cap N \neq 1$. Since B is p-nilpotent and $B \cap N$ is a normal p-subgroup of B, $B/C_B(B \cap N)$ is a p-group by Frobenius Theorem [10, Chapter 7, Theorem 4.5]. It follows from Lemma 2.7 that $B \cap N \leq Z_\infty(B)$. Because $B/B \cap N \leq M_2/N$ is abelian, so B is nilpotent. Assume that $A \cap N \neq 1$. By a similar argument as above, we know that A is nilpotent. In this case, whether $B \cap N = 1$ or $B \cap N \neq 1$, which is easy to derive that B is nilpotent.

If A is a p-group, then M_2 contains a Sylow r-subgroup of G, which contradicts $|G : M_2| = r$. Hence A is not a p-group. With a same argument, B is not a p-group. Therefore, let $A_{p'}$ and $B_{p'}$ be p'-Hall subgroups of A and B, respectively. Since $G = AB^g$ for any $g \in G$ by Lemma 2.2, $A_{p'}B_{p'}^g$ is a p'-Hall subgroup of G by Lemma 2.8. It follows from Lemma 2.9 that $[A_{p'}, B_{p'}^g]$ is subnormal in G. Thus by (1), $[A_{p'}, B_{p'}^g] \leq O_{p'}(G) = 1$. This implies that $B_{p'}^G \leq C_G(A_{p'})$. Similarly, $A_{p'}^G \leq C_G(B_{p'})$. Hence $N \leq C_G(A_{p'}) \cap C_G(B_{p'})$. Clearly, $N \leq C_G(A_p) \cap C_G(B_p)$, where A_p and B_p are Sylow p-subgroups of A and B, respectively. Therefore $N \leq Z(G)$. Since AN/N and BN/N are nilpotent, AN and BN are nilpotent. If $q = r$, then $G/(M_1 \cap M_2)$ is an abelian group of order of q^2. Hence M_1 and M_2 are normal in G, and so AN and BN are subnormal in G. It derived that $A \leq N$ and $B \leq N$, a contradiction. Thereby $q \neq r$, without loss of generality, we assume that q is the smallest prime dividing $|G/(M_1 \cap M_2)|$. Since $|G/(M_1 \cap M_2)| = qr$ and $|G/(M_1 \cap M_2) : M_1/(M_1 \cap M_2)| = q$, M_1 is normal in G. It follows that AN is subnormal in G. Then $A \leq N$, which is impossible. Therefore G_p is not normal in G. Consequently, $M_2 = N_G(G_p^x) = (N_G(G_p))^x = (M_1)^x$, which is a contradiction by Lemma 2.2.

(ii) $G/(H_1)_G \in \mathfrak{U}$ and H_2 is normal in G or H_1 is normal in G and $G/(H_2)_G \in \mathfrak{U}$.

Without loss of generality, we assume that $G/(H_1)_G \in \mathfrak{U}$ and H_2 is normal in G. Clearly, $G/(M_1)_G \in \mathfrak{U}$ and $H_2 \leq (M_2)_G$. Since H_2 is normal in G, $G/H_2 \cong (G/N)/(H_2/N) = (AN/N)(H_2/N)/(H_2/N)$ is supersoluble by (4), which is the same case as (i).

(iii) H_1 and H_2 are normal in G.

By (i) and (ii), it is easy to derive a contradiction. The theorem is thus proved.

***Theorem* 3.2** — *Let p be a prime. Suppose that $G = AB$, where A is p-nilpotent, B is p-supersoluble and B has a maximal series all members of which are $\{1 \leq G\}$-$\mathfrak{U}$-embedded in G.*

Then G is p-supersoluble.

PROOF : Suppose that the result is false and let G be a counterexample of minimal order. Let $1 = B_0 < B_1 < B_2 < \cdots < B_{t-1} < B_t = B$ be a maximal subgroup series of B and all members of which are $\{1 \le G\}$-$\mathfrak{U}$-embedded in G. Then by Lemma 2.6, B is K-$\mathfrak{U}$-subnormal in G. Since $B \neq G$, there exists a proper subgroup H of G such that $B \le H$ and H is normal in G or $G/H_G \in \mathfrak{U}$. Let M_1 and M_2 be two maximal subgroups of G such that $H \le M_1$ and $A \le M_2$. We proceed the proof via the following steps:

(1) $O_{p'}(G) = 1$, G has a unique minimal normal subgroup N and G/N is p-supersoluble.

Let L be any normal subgroup of G. Clearly, $1 = B_0L/L \le B_1L/L \le B_2L/L \le \cdots \le B_{t-1}L/L \le B_tL/L = BL/L$ is a subgroup series of BL/L. Suppose that $B_iL/L \le M/L \le B_{i+1}L/L$ $(i = 0, 1, 2 \cdots t)$. Then $B_i \le M \cap B_{i+1} \le B_{i+1}$. If $B_i = M \cap B_{i+1}$, then $M/L = M/L \cap B_{i+1}L/L = (M \cap B_{i+1})L/L = B_iL/L$. Assume that $B_{i+1} \le M$. Then $B_{i+1}L/L = M/L$. Thus $1 = B_0L/L < B_1L/L < B_2L/L < \cdots < B_{t-1}L/L < B_tL/L = BL/L$ is a maximal subgroup series of BL/L. Since B_i is $\{1 \le G\}$-$\mathfrak{U}$-embedded in G, by Lemma 2.1(2), B_iL/L is $\{1 \le G/L\}$-$\mathfrak{U}$-embedded in G/L for every i. The choice of G implies that G/L is p-supersoluble. It implies that $O_{p'}(G) = 1$ and G has a unique minimal normal subgroup N.

(2) M_1 is p-supersoluble, G is p-soluble and G has a maximal subgroup M such that $G = N \rtimes M$ and $N = O_p(G) = C_G(N)$.

Clearly, $M_1 = B(A \cap M_1)$. By Lemma 2.1(1), M_1 satisfies the hypothesis of the theorem. The choice of G implies that M_1 is p-supersoluble and so H is p-supersoluble. If $G/H_G \in \mathfrak{U}$, then G is p-soluble. Assume that H is normal in G. Since $G/H = AH/H$ is p-nilpotent, G is p-soluble. Hence by (1), G has a maximal subgroup M such that $G = N \rtimes M$ and $N = O_p(G) = C_G(N)$.

(3) $O_{p'}(M_1) = 1$, p is the largest prime divisor of $|M_1|$ and M_1 is supersoluble.

Because either H is normal in G or $G/H_G \in \mathfrak{U}$, we have $N \le H \le M_1$. Since $[N, O_{p'}(M_1)] = 1$, $O_{p'}(M_1) \le C_G(N) = O_p(G)$. It follows that $O_{p'}(M_1) = 1$. By (2) and Lemma 2.4, we know that p is the largest prime divisor of $|M_1|$ and M_1 is supersoluble.

(4) $B \le M$.

Assume that $B \not\le M$. Clearly, by (1) and (2), $G/M_G \notin \mathfrak{U}$. Hence (M, G) is a $\mathfrak{U}$-abnormal pair. Let i be the smallest integer such that $B_i \not\le M$, where $i \in \{1, 2, \cdots, t\}$. Then $B_{i-1} \le M \cap B_i \le B_i$ and so $B_{i-1} = M \cap B_i$. By hypothesis, $G = B_iM$. It follows by (2) that $|N| = |G : M| = |B_i : B_{i-1}| = p$ because B is supersoluble. By (1), we have that G is p-supersoluble, a contradiction.

Hence $B \leq M$.

(5) If $G/(M_2)_G \notin \mathfrak{U}$ or M_2 is normal in G, then M_2 is p-supersoluble.

We show that the subgroup series

$$1 = B_0 \cap M_2 < B_1 \cap M_2 < B_2 \cap M_2 < \cdots < B_{t-1} \cap M_2 < B_t \cap M_2 = B \cap M_2 \qquad (*)$$

is a maximal subgroup series of $B \cap M_2$. Clearly, $B \nleq M_2$. Let i be the smallest integer such that $B_i \nleq M_2$, where $i \in \{1, 2, \cdots, t\}$. Then $B_j \leq M_2$ for $j = 0, 1, \cdots, i-1$. If $G/(M_2)_G \notin \mathfrak{U}$, then (M_2, G) is a $\mathfrak{U}$-abnormal pair, by hypothesis, $G = B_i M_2$. Assume that M_2 is normal in G. Obviously, $G = B_i M_2$. Hence $B_{i+1} = B_i(B_{i+1} \cap M_2)$ and so $|B_{i+1} \cap M_2 : B_i \cap M_2| = |B_{i+1} : B_i|$ is a prime. This implies that $B_i \cap M_2$ is a maximal subgroup of $B_{i+1} \cap M_2$ for $i = 0, 1, \cdots, t$. Therefore the $(*)$ series is a maximal subgroup series of $B \cap M_2$. Let (K, H) be a $\mathfrak{U}$-abnormal pair such that $1 \leq K < H \leq M_2$. By hypothesis, B_i either covers or avoids (K, H) for every i. If $B_i K = B_i H$, then $(B_i \cap M_2)K = B_i K \cap M_2 = B_i H \cap M_2 = (B_i \cap M_2)H$. For the latter case, $(B_i \cap M_2) \cap K = (B_i \cap M_2) \cap H$. Hence M_2 satisfies the hypothesis of the theorem. The choice of G implies that M_2 is p-supersoluble.

(6) If $N \leq A$, then A is a p-group.

Let $A_{p'}$ be a normal p-complement of A. Then $[N, A_{p'}] = 1$. Hence by (2), $A_{p'} \leq C_G(N) = O_p(G)$. It follows that $A_{p'} = 1$ and so A is a p-group.

(7) $G/H_G \notin \mathfrak{U}$.

Assume that $G/H_G \in \mathfrak{U}$. Clearly, G is soluble and $H_G \neq 1$. Then $H_G \leq (M_1)_G$ and so $G/(M_1)_G \in \mathfrak{U}$. Hence $|G : M_1|$ is a prime.

Firstly, we suppose that $|G : M_1| = p$. By considering the permutation representation of $G/(M_1)_G$ on the right coset of $M_1/(M_1)_G$, we can see that $G/(M_1)_G$ is isomorphic to some subgroup of the symmetric group S_p of degree p. Hence the order of Sylow p-subgroup of $G/(M_1)_G$ is p. It implies that $M_1/(M_1)_G$ is a p'-group. Since $(M_1)_G$ is supersoluble, N is the Sylow p-subgroup of $(M_1)_G$, and thereby N is the Sylow p-subgroup of M_1. This follows from (4) that B is a p'-group. Hence A contains a Sylow p-subgroup G_p of G. Then $N \leq A$ and so $N \leq (M_2)_G$. By (6), A is a p-group and so $A = G_p$. If $G/(M_2)_G \notin \mathfrak{U}$, then by (5), M_2 is p-supersoluble. Clearly, $B \nleq M_2$. Therefore we may suppose that i is the smallest integer such that $B_i \nleq M_2$, where $i \in \{1, 2, \cdots, t\}$. Then $B_{i-1} \leq M_2 \cap B_i \leq B_i$ and so $B_{i-1} = M_2 \cap B_i$. Since (M_2, G) is a $\mathfrak{U}$-abnormal pair, by hypothesis, $G = B_i M_2$. This implies that $|G : M_2| = |B_i : B_{i-1}|$ is a prime. Assume that $G/(M_2)_G \in \mathfrak{U}$. Then $|G : M_2|$ is a prime. Since $G_p \leq M_2$, we may assume that $|G : M_2| = q$, where $p \neq q$. By (3), we have $F(M_1) = N$. Since M_1 is supersoluble, M_1' is nilpotent and so $M_1' \leq N$. It follows that M_1/N is abelian. Since $G/N = (G_p/N)(M_1/N)$, by Lemma 2.5, $(M_1/N)F(G/N)$ is normal in G/N. If $(M_1/N)F(G/N) = M_1/N$, then M_1 is normal in G. Because $G/N = (M_1/N)(M_2/N)$, so $M_1 \cap M_2$ is normal in G. It is easy to see that $|G/(M_1 \cap M_2)| = pq$ and q is the smallest prime divisor of $G/(M_1 \cap M_2)$. Since $|G/(M_1 \cap M_2) : M_2/(M_1 \cap M_2)| = q$, M_2 is normal in G. By (5), M_2 is p-supersoluble. Appling using a similar argument as in (3), we have that p is the largest prime dividing $|M_2|$ and M_2 is supersoluble. Hence G_p is normal in M_2 and so $G_p = N$ is normal in G. This contradicts with $|G : M_1| = p$. Therefore $(M_1/N)F(G/N) = G/N$. Then $G_p/N \leq F(G/N)$. Hence $G_p = N$ is normal in G, which is impossible too for $|G : M_1| = p$.

Secondly, suppose that $|G : M_1| = q$, where $q \neq p$ is a prime divisor of $|G|$. Let G_p be a Sylow p-subgroup of G contained in M_1. By (3), G_p is normal in M_1 and so $M_1 \leq N_G(G_p)$. If $N_G(G_p) = G$, then G_p is normal in G. So $N = G_p$. This follows from by (4) that B is a p'-group. Thus $N \leq A$. By (6), $A = N$, and thereby $M = B \leq M_1$ by (2) and (4), which is impossible. Hence $N_G(G_p) = M_1$. By Lemma 2.10, G has a q-subgroup $Q \neq 1$ such that G_pQ is a normal subgroup of G. Let G_q be a Sylow q-subgroup of G containing Q. Then $G_pQG_q = G_pG_q$ is a subgroup of G. By (3), we know that p is the largest prime divisor of M_1. If p is the largest prime divisor of G, then it is easy to see that $|G_pG_q : G_pG_q \cap M_1| = |G_pG_q : G_p(G_q \cap M_1)| = |G_q : G_q \cap M_1| = q$. Since q is the smallest prime divisor of $|G_pG_q|$, $G_pG_q \cap M_1$ is normal in G_pG_q. Because $G_pG_q \cap M_1$ is supersoluble, we have that G_p is normal in $G_pG_q \cap M_1$ and so G_p is normal in G_pG_q. Then G_p is normal in G_pQ. Thus G_p is normal in G, this is impossible as above. Therefore p is not the largest prime divisor of G, then q is the largest prime divisor of G. Since $|G : M_1| = q$, $|G_q| = q$, where G_q is the Sylow q-subgroup of G. Thus $G_pQ = G_pG_q$ is a normal subgroup of G. It follows from (1) that $(G_qN/N)(G_p/N)$ is supersoluble. Hence $G_qN/N \trianglelefteq (G_qN/N)(G_p/N)$ and thereby G_qN is normal in G. By (2), without loss of generality, we may assume that $G_q \leq M$. Then $G_q = M \cap G_qN$ is normal in M. Hence $B \leq M \leq N_G(G_q)$. Obviously, there exists some $x \in A$ such that $G_q^x \leq A$ because $q \notin \pi(B)$. It implies that $G_q^G = G_q^A \leq A_G$. This shows that $A_G \neq 1$ and so $N \leq A_G$ by (1). By (6), we have that A is a p-group, which is impossible. Consequently, G/H_G is not supersoluble.

Final contradiction

By (7), we know that H is normal in G. Let P be a Sylow p-subgroup of H. Since H is supersoluble by (3), P is normal in G and so $P = N$. By (4), we have $B \cap N = 1$. It implies that B is a p'-group. Hence A contains a Sylow p-subgroup G_p of G and so $N \leq A$. By (6), $A = G_p$. Hence $G/H = AH/H \cong A/A \cap H$ is p-nilpotent. It follows that G/N is p-nilpotent because $(G/N)/(H/N) \cong G/H$. Since $|G/N : M_1/N| = |(G_p/N)(M_1/N) : M_1/N|$ is a power of p, by

Lemma 2.11(1) is normal in G. Hence $|G/M_1| = p$. By using a similar discussion as in (7), we can derive a final contradiction. The theorem is thus proved.

***Theorem* 3.3** — *Let p be a prime and P a Sylow p-subgroup of G. The following assertions holds:*

(a) *If every maximal subgroup of G is Σ_p-$\mathfrak{S}_p$-embedded in G for some composition series Σ of G, then G is p-soluble.*

(b) *If p is a prime divisor of $|G|$ and P is Σ_p-$\mathfrak{S}_p$-embedded in G for some composition series Σ of G, then G is p-soluble.*

(c) *If every non-supersoluble Schmidt subgroup of G is Σ_p-$\mathfrak{S}_p$-embedded in G for some composition series Σ of G, then G is p-soluble.*

PROOF : (a) Suppose that the result is false and let G be a counterexample of minimal order.

Let H be any normal subgroup of G. Clearly, by Lemma 2.12(2), G/H satisfies the hypothesis of theorem. The choice of G implies that G/H is p-soluble. Hence $O_{p'}(G) = 1$, G has the unique non-abelian minimal normal subgroup N and p divides $|N|$. Then $N = N_1 \times N_2 \times \cdots \times N_t$, where N_i $(i = 1, 2, \cdots, t)$ are isomorphic non-abelian simple groups. Let $P \leq G_p$, where P and G_p are the Sylow p-subgroups of N and G, respectively. Obviously, $N_G(P) \neq G$. Therefore there exists a maximal subgroup M of G such that $N_G(P) \leq M$. By hypothesis, M is Σ_p-$\mathfrak{S}_p$-embedded in G for some composition series Σ of G.

Let L be a minimal subnormal subgroup of G. If $N \cap L = 1$, then $L \cong NL/N$ is p-soluble. It follows that $O_{p'}(L) \neq 1$ or $O_p(L) \neq 1$, which is impossible. Hence $L \leq N$. Without loss of generality, we may assume that M is $\{1 \leq N_i\}$-$\mathfrak{S}_p$-embedded in G for some i. By Lemma 2.1(1), $M_i = M \cap N_i$ is $\{1 \leq N_i\}$-$\mathfrak{S}_p$-embedded in N_i. Hence by Lemma 2.6, M_i is K-$\mathfrak{S}_p$-subnormal in N_i. It follows that $M_i = 1$ or $M_i = N_i$. By Frattini argument, $G = NN_G(P) = NM$. If $M_i = N_i$, then $N_i^G = N_i^M \leq M_G$. Thus $N \leq M$, a contradiction. Hence assume that $M_i = 1$. Clearly, $G_p \leq N_G(P)$. Let $P_i = N_i \cap P$. Then $P_i \leq G_p^x \leq M^n$, for some $x \in G$, $n \in N$. Hence $(P_i)^{n^{-1}} \leq M$. Since N_i is normal in N, $(P_i)^{n^{-1}} \leq N_i$. This implies that $(P_i)^{n^{-1}} \leq M \cap N_i = M_i$, a contradiction. Therefore (a) holds.

(b) Suppose that the result is false and let G be a counterexample of minimal order. Let

$$\Gamma_p = \{1 = G_0 < G_1 < G_2 < \cdots < G_{n-1} < G_n = G\} \qquad (*)$$

be a composition series of G such that P is Σ_p-$\mathfrak{S}_p$-embedded in G.

Since G is not p-soluble, there exists a composition factor G_i/G_{i-1} such that G_i/G_{i-1} is a non-abelian simple group and p divides $|G_i/G_{i-1}|$. Hence $G_i \neq G_{i-1}(G_i \cap P) \neq G_{i-1}$. Then P is $\{G_{i-1} \leq G_i\}$-$\mathfrak{S}_p$-embedded in G. By Lemma 2.1(1), $G_i \cap P$ is $\{G_{i-1} \leq G_i\}$-$\mathfrak{S}_{\mathfrak{p}}$-embedded in G_i. Therefore by Lemma 2.1(2), $(G_i \cap P)G_{i-1}/G_{i-1}$ is $\{1 \leq G_i/G_{i-1}\}$-$\mathfrak{S}_p$-embedded in G_i/G_{i-1}. It follows from Lemma 2.6 that $(G_i \cap P)G_{i-1}/G_{i-1}$ is K-$\mathfrak{S}_p$-subnormal in G_i/G_{i-1}. It is impossible because G_i/G_{i-1} is a non-abelian simple group. Hence (b) holds.

(c) Suppose that the result is false and let G be a counterexample of minimal order. Let K be any non-supersoluble Schmidt subgroup of G and M a maximal subgroup of G. By Lemma 2.12(1), $M \cap K$ is $\{\Sigma \cap M\}_p$-$\mathfrak{S}_p$-embedded in M. The choice of G implies that M is p-soluble. Hence every subgroup of G is p-soluble. Let q be a smallest prime dividing $|G|$. It is obvious that G is not q-nilpotent. By [9, Chapter IV, Theorem 5.4], G has a q-closed Schmidt subgroup H. Clearly, H is not supersoluble and $H \neq G$. By hypothesis, H is Σ_p-$\mathfrak{S}_p$-embedded in G. If G is a simple group, then $\{1 \leq G\}$ is a unique composition series of G. By Lemma 2.6, H is K-$\mathfrak{S}_p$-subnormal in G, a contradiction. Hence G is not simple.

Let L be any proper subnormal subgroup of G and N a normal subgroup of G such that G/N is a non-abelian simple group and p divides $|G/N|$. If $L \nleq N$, then $G = NL$ is p-soluble, a contradiction. Hence $L \leq N$. If $N \nleq \Phi(G)$, then there exists a maximal subgroup A of G such that $G = AN$ is p-soluble, which is impossible. Thus $N \leq \Phi(G)$. Clearly, $\Phi(G) \leq N$ and so $N = \Phi(G)$. Since $H \neq G$, G has a maximal subgroup M such that $H \leq M$. It is easy to see that $M_G = \Phi(G)$. Hence $G/M_G \notin \mathfrak{S}_p$. Therefore (M, G) is an $\mathfrak{S}_p$-abnormal pair. Since N contains every proper subnormal subgroups of G, $N = M_G = \Phi(G)$ and p divides $|G/N|$, H covers or avoids (M, G). If $H \nleq M^x$ for some $x \in G$, then $G = HM^x = MM^x$, which contradicts Lemma 2.2. Hence $H \leq M_G = \Phi(G) \leq F(G)$, a contradiction. □

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