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LAPLACIAN SPECTRAL CHARACTERIZATION OF (BROKEN) DANDELION GRAPHS[1]

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Let $H(p, tK^*_{1,m})$ be a connected unicyclic graph with $p + t(m + 1)$ vertices obtained from the cycle C_p and t copies of the star $K_{1,m}$ by joining the center of $K_{1,m}$ to each one of t consecutive vertices of the cycle C_p through an edge, respectively. When $t = p$, the graph is called a dandelion graph and when $t \neq p$, the graph is called a broken dandelion graph. In this paper, we prove that the dandelion graph $H(p, pK^*_{1,m})$ and the broken dandelion graph $H(p, tK^*_{1,m})$ $(0 < t < p)$ are determined by their Laplacian spectra when $m \neq 2$ and p is even.

Key words : Laplacian spectrum; graph determined by its Laplacian spectrum; unicyclic graph; bipartite graph.

2010 Mathematics Subject Classification : 05C50.

1. INTRODUCTION

Let G be a simple and undirected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. If $v_i, v_j \in V(G)$ are adjacent, then we denote by v_iv_j the edge joining v_i and v_j. Let $d_i = d_G(v_i)$ denote the degree of a vertex v_i in G and $deg(G) = (d_1, d_2, ..., d_n)$ the degree sequence of G. For convenience, the degree sequence of G can be written as $deg(G) = (1^{n_1}, ..., k^{n_k}, ..., \Delta^{n_\Delta})$, where $\Delta = \Delta(G)$ denotes the maximum degree of all vertices of G and n_k is the number of vertices of

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degree k in the graph G. Let $A(G) = (a_{ij})_{n\times n}$ denote the adjacency matrix of G, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. Let $D(G) = diag(d_1, d_2, ..., d_n)$ be the diagonal matrix of vertex degrees of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and the signless Laplacian matrix of G, respectively. The polynomials $\phi(G) = \phi(G, x) = det(xI_n - L(G)) = l_0x^n + l_1x^{n-1} + ... + l_n$ and $\psi(G) = \psi(G, x) = det(xI_n - Q(G)) = q_0x^n + q_1x^{n-1} + ... + q_n$ are defined as the Laplacian characteristic polynomial and the signless Laplacian characteristic polynomial of G, respectively. Since $A(G), L(G)$ and $Q(G)$ are real and symmetric, their eigenvalues are all real numbers. Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ are the adjacency eigenvalues, the Laplacian eigenvalues and the signless Laplacian eigenvalues of G, respectively.

Two graphs are said to be A-cospectral (L-cospectral or Q-cospectral) if they have the same A-spectrum (L-spectrum or Q-spectrum). A graph is said to be determined by the A-spectrum (L-spectrum or Q-spectrum) if there is no other non-isomorphic graph with the same A-spectrum (L-spectrum or Q-spectrum). We can also denote these graphs as DAS (DLS or DQS) graphs.

Günthard and Primas [11] first put forward the question in 1956: Which graphs are determined by their spectra? It is a difficult and interesting problem in the theory of graph spectra. Only some graphs with special structure are shown to be determined by their spectra. Another application comes from Fisher in 1966, who considered a question of Kac [13]: ”Can one hear the shape of a drum?” He modeled the shape of the drum by a graph. Then the frequency of the sound of that drum is characterized by the eigenvalues of the graph. This problem, in fact, is a problem of spectral determination of graphs which we are studying. It has been revisited by Dam and Haemers in 2003 [9] and has drawn much attention from researchers recently. For additional remarks on this topic we refer the reader to see two excellent surveys [9, 10]. Recently, Wang *et al*. [26] proved that $H(n; q, n_1, n_2, n_3, n_4)$ is determined by its Laplacian spectrum. In 2016, Sorgun and Topcu [24] proved that the Kite graph $Kite_{p,2}$ is determined by its adjacency spectrum. As well, in 2017, Liu *et al*. [17] proved that all the butterfly-like graphs $W_{2^{(s)};\mathbf{a}(\mathbf{k})}$ are determined by their Laplacian spectra and signless Laplacian spectra, respectively. In 2018, He and Dam [12] proved that except for two specific examples, the rose graphs are determined by their Laplacian spectra. There are many results on DAS, DLS and DQS graphs. Some of these results can be referred in [1, 2, 9, 10, 24], [1, 5, 6, 9, 12, 16-19, 25-29] and in [3, 4, 5, 9, 17-21, 23, 30] respectively.

In previous work, $H(p, tK_{1,m})$ denotes an unicyclic graph with $p+tm$ vertices obtained from C_p by attaching the center of star $K_{1,m}$ to each one of t consecutive vertices of the cycle C_p respectively. When $m = 1$, $t = p$ or $t = p - q > 0$, $p > 2$ is even, Boulet [1] proved that $H(p, tK_{1,m}) =$

$H(p, pK_{1,1})$ and $H(p, tK_{1,m}) = H(p, (p-q)K_{1,1})$ ($p > 2$ is even) are determined by their Laplacian spectra, respectively. When $m = 2$, $t = p$ or $t = p - 1$, Bu *et al*. [6] and Wang [27] proved that $H(p, pK_{1,2})$ and $H(p, (p - 1)K_{1,2})$ are determined by their Laplacian spectra, respectively. When $m = 2$, p is even, Wang [27] proved that $H(p, 2K_{1,2})$, $H(p, 3K_{1,2})$, $H(p, (p - 3)K_{1,2})$ and $H(p, (p - 2)K_{1,2})$ are determined by their Laplacian spectra, respectively. When $m = 3$ or 4, $t = p$ or $t = p - 1$, Mei *et al*. [29] proved that $H(p, pK_{1,4})$, $H(p, pK_{1,3})$ and $H(p, (p - 1)K_{1,3})$ are determined by their Laplacian spectra, respectively. When $m = 3$, p is even, Mei *et al*. [29] proved that $H(p, 2K_{1,3})$, $H(p, (p-2)K_{1,3})$ and $H(p, (p-3)K_{1,3})$ are determined by their Laplacian spectra, respectively. When $m = 4$ or 5, $t = p$ or $t = p - 1$, Sun *et al*. [25] proved that $H(p, pK_{1,5})$ and $H(p, (p - 1)K_{1,4})$ are determined by their Laplacian spectra, respectively. When $m = 4$, p is even, Sun *et al*. [25] proved that $H(p, 2K_{1,4})$, $H(p, (p - 2)K_{1,4})$ and $H(p, (p - 3)K_{1,4})$ are determined by their Laplacian spectra, respectively.

In this paper, let $H(p, tK^*_{1,m})$ be a connected unicyclic graph with $p + t(m + 1)$ vertices obtained from the cycle C_p and t copies of the star $K_{1,m}$ by joining the center of $K_{1,m}$ to each one of t consecutive vertices of the cycle C_p through an edge, respectively. When $t = p$, the graph is called a dandelion graph and when $t \neq p$, the graph is called a broken dandelion graph (shown in Fig. 1). The rest of this paper is organized as follows. First, we prove that the dandelion graph $H(p, pK^*_{1,m})$ is determined by its Laplacian spectrum when $m \neq 2$ and p is even. Then we prove that the broken dandelion graph $H(p, tK^*_{1,m})$ $(0 < t < p)$ is determined by its Laplacian spectrum when $m \neq 2$ and p is even.

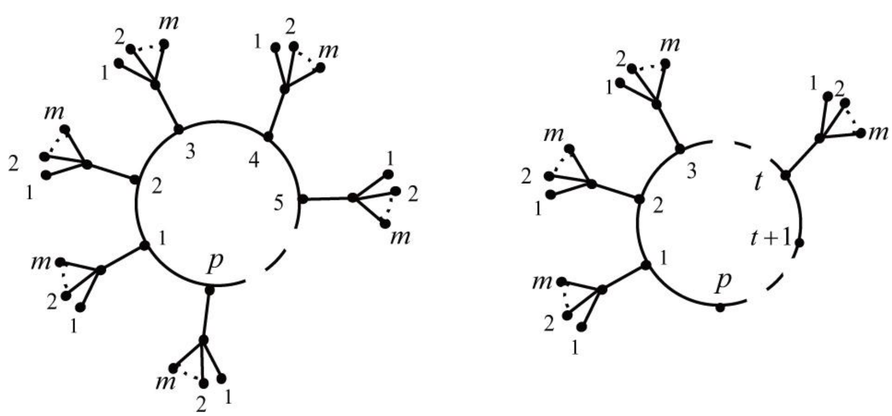


Figure 1 : The dandelion graph $H(p, pK^*_{1,m})$ and the broken dandelion graph $H(p, tK^*_{1,m})$.

2. PRELIMINARIES

In this section, we provide some known results that are used in this paper.

Lemma 2.1 — [9, 22]. If G and G' are L-cospectral graphs, then

(i) G and G' have the same number of vertices.

(ii) G and G' have the same number of edges.

(iii) G and G' have the same number of components.

(iv) G and G' have the same number of spanning trees.

(v) G and G' have the same sum of the squares of degrees of vertices.

If G and G' are A-cospectral graphs, then

(vi) G and G' have the same number of closed walks of any length.

Lemma 2.2 — [14, 15]. Let G be a graph with $V(G) \neq \varnothing$ and $E(G) \neq \varnothing$. Then

$$\Delta + 1 \leq \mu_1 \leq \max\{\frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j}, v_iv_j \in E(G)\},$$

where Δ denotes the maximum vertex degree of G and m_i denotes the average of degrees of the vertices adjacent to vertex v_i in G.

Lemma 2.3 — [18]. Let G be a connected unicylic graph with n vertices and its cycle C_p. If G' is L-cospectral to G, then G' must be a connected unicyclic graph with n vertices and one cycle C_p. Moreover,

$$\sum_{i=1}^{n} d_i(G)^3 = \sum_{i=1}^{n} d_i(G')^3.$$

Lemma 2.4 — [7]. In a simple graph, the number of closed walks of length 4 is equal to the sum of 2 times of the number of edges, 4 times of induced paths of length 2 and 8 times of the number of 4-cycles C_4.

Lemma 2.5 — [8, 21]. If G and G' are L-cospectral graphs and they are both bipartite graphs, then their line graphs $l(G)$ and $l(G')$ are A-cospectral.

3. (Broken) Dandelion Graphs are Determined by Their Laplacian Spectra

In this section, we mainly investigate Laplacian spectral characterization of (broken) dandelion graphs. For Theorem 3.1, we firstly prove that when $m \geq 5$ and p is even, the dandelion graph $G = H(p, pK^*_{1,m})$ is determined by its L-spectrum, then we prove that when $m = 4, m = 3$ or $m = 1$, the graph $G = H(p, pK^*_{1,m})$ is also determined by its L-spectrum. Moreover, we prove that when $t < p, m \neq 2$ and p is even, the broken dandelion graph $H(p, tK^*_{1,m})$ is determined by its L-spectrum.

***Theorem* 3.1** — *Let p and m be positive integers. If p is even and $m \neq 2$, then the dandelion graph $G = H(p, pK^*_{1,m})$ is determined by its L-spectrum.*

PROOF : The first step of the proof is to determine the degree distribution of the vertices of G. Here, we assume that G' is L-cospectral to G. By Lemma 2.3, we know that G' is a connected unicyclic graph with the same number of vertices, the same number of edges and cycle C_p with G. By Lemma 2.2 we have

$$\Delta + 1 \leq \mu_1 \leq \max\{\frac{m+16}{3}, \frac{m^2+4m+6}{m+2}, \frac{m^2+4m+20}{m+4}\}. \tag{1}$$

where Δ denotes the maximum degree of all vertices of the graph G.

Then we will prove this theorem separately to distinguish four cases that $m \geq 5, m = 4, m = 3, m = 1$.

Case 1 : $m \geq 5$.

It is easy to see from Fig. 1, the graph G has $(m+2)p$ vertices, $(m+2)p$ edges and cycle C_p. By Eq. (1) we have

$$\Delta + 1 \leq \mu_1 \leq \max\{\frac{m+16}{3}, \frac{m^2+4m+6}{m+2}, \frac{m^2+4m+20}{m+4}\} = \frac{m^2+4m+6}{m+2} \quad (m \geq 5).$$

After careful calculation, we obtain $\Delta \leq m+1$. Let n_i denote the number of vertices of degree i in G'. According to Lemma 2.1 (i),(ii),(v) and Lemma 2.3, we have

$$\begin{cases} n_1 + n_2 + n_3 + ... + n_{m+1} = (m+2)p \\ n_1 + 2n_2 + 3n_3 + ... + (m+1)n_{m+1} = 2(m+2)p \\ n_1 + 4n_2 + 9n_3 + ... + (m+1)^2 n_{m+1} = (m^2+3m+10)p \\ n_1 + 8n_2 + 27n_3 + ... + (m+1)^3 n_{m+1} = (m^3+3m^2+4m+28)p. \end{cases} \tag{2}$$

The above system of linear equations in variables n_1, n_2, n_3, n_4 has the equivalent form as follows

Ax=b,

where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}, x = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix},$$

$$b=((m+2)p-\sum_{i=5}^{m+1}n_i,2(m+2)p-\sum_{i=5}^{m+1}in_i,(m^2+3m+10)p-\sum_{i=5}^{m+1}i^2n_i,$$
$$(m^3+3m^2+4m+28)p-\sum_{i=5}^{m+1}i^3n_i)^T,$$

here, the transpose of a vector C is denoted by C^T.

By using Mathematica 8.0, we can obtain the solution

$$\begin{cases} n_1=\frac{1}{6}((-m^3+6m^2-5m+6)p+\sum_{i=5}^{m+1}(i^3-9i^2+26i-24)n_i) \\ n_2=\frac{1}{2}((m^3-5m^2+6m)p+\sum_{i=5}^{m+1}(-i^3+8i^2-19i+12)n_i) \\ n_3=\frac{1}{2}((-m^3+4m^2-3m+2)p+\sum_{i=5}^{m+1}(i^3-7i^2+14i-8)n_i) \\ n_4=\frac{1}{6}((m^3-3m^2+2m)p+\sum_{i=5}^{m+1}(-i^3+6i^2-11i+6)n_i). \end{cases} \tag{3}$$

Since $n_i\geq 0\ \ (i=1,2,3,4)$, we get the inequalities as follows

$$\sum_{i=5}^{m+1}(i^3-9i^2+26i-24)n_i\geq(m^3-6m^2+5m-6)p, \tag{4}$$

$$\sum_{i=5}^{m+1}(i^3-8i^2+19i-12)n_i\leq(m^3-5m^2+6m)p, \tag{5}$$

$$\sum_{i=5}^{m+1}(i^3-7i^2+14i-8)n_i\geq(m^3-4m^2+3m-2)p, \tag{6}$$

$$\sum_{i=5}^{m+1}(i^3-6i^2+11i-6)n_i\leq(m^3-3m^2+2m)p. \tag{7}$$

Next, we consider the following two cases that $n_{m+1}=p$ and $n_{m+1}\neq p$.

Case 1.1 : $n_{m+1}=p$.

When $n_{m+1}=p$, by Eq. (5), we have

$$\sum_{i=5}^{m}(i^3-8i^2+19i-12)n_i\leq 0.$$

Here, we assume that $f(i)=i^3-8i^2+19i-12$, thus $f'(i)=3i^2-16i+19>0\ (i>5)$, and $f(5)>0$. Then we have $f(i)=i^3-8i^2+19i-12>0\ (i>5)$. However, $n_i\geq 0$ $(i=1,2,3,...,m+1)$. When $m\geq 5$, we obtain one of the solutions

$$\begin{cases} n_i=0\ (i=5,6,7,...,m), \\ \qquad n_{m+1}=p. \end{cases} \tag{8}$$

We substitute the formula (8) into the inequations (4)-(7), and find that the solution satisfies the conditions of Theorem 3.1.

Then, we can obtain the solution n_1, n_2, n_3, n_4 from Eq. (3)

$$\begin{cases} n_1 = mp, \\ n_i = 0 \ (i = 2, 4), \\ n_3 = p. \end{cases} \tag{9}$$

Case 1.2 : $n_{m+1} \neq p$.

Apparently, according to Eq. (5), we have $n_{m+1} \leq p$. When $n_{m+1} \neq p$, we have $n_{m+1} = p-1, p-2, p-3, ...,$ or 0. Without loss of generality, we assume that $n_{m+1} = p-c \ (1 \leq c \leq p-1)$. Eq. (5) can be written as

$$\sum_{i=5}^{m}(i^3 - 8i^2 + 19i - 12)n_i \leq (m^3 - 5m^2 + 6m)c.$$

However, here c is a constant and it is irrelevant to p while the left side of the above inequation is always relevant to p (By Eq. (2), we know that at least one of n_i is relevant to p. Besides, for $i > 5$, $f(i) > 0$, the left side of the above inequation is always relevant to p). When p is arbitrary, it can't be guaranteed that above inequality always holds up. Thus, we obtain that when $n_{m+1} \neq p$, $c = p$ is the only possibility. Under such circumstance, n_{m+1} can only be 0.

To simplify the expression, we assume that $a_i = i^3 - 7i^2 + 14i - 8 \ (i \geq 5)$ and $a_0 = (m^3 - 4m^2 + 3m - 2)p$, thus Eq. (6) can be expressed as

$$\sum_{i=5}^{m} a_i n_i \geq a_0.$$

The same goes for b_i. We assume that $b_i = i^3 - 6i^2 + 11i - 6 \ (i \geq 5)$ and $b_0 = (m^3 - 3m^2 + 2m)p$, thus Eq. (7) can be expressed as

$$\sum_{i=5}^{m} b_i n_i \leq b_0.$$

Clearly

$$\frac{a_i}{b_i} = 1 + \frac{1}{3-i} \ (5 \leq i \leq m)$$

which is monotonically increasing on i. Thus we assume that $M = 1 + \frac{1}{3-m}$, we have

$$\frac{a_i}{b_i} = 1 + \frac{1}{3-i} \leq 1 + \frac{1}{3-m} = M \ (5 \leq i \leq m). \tag{10}$$

Meanwhile

$$\frac{a_0}{b_0} = 1 + \frac{-m^2 + m - 2}{m^3 - 3m^2 + 2m} > M. \tag{11}$$

Let $\overrightarrow{\mathbf{B}}$ denote a row vector: $\overrightarrow{\mathbf{B}} = (b_5 n_5, b_6 n_6, b_7 n_7, ..., b_m n_m)$ and let $\overrightarrow{\mathbf{X}}$ denote a column vector: $\overrightarrow{\mathbf{X}}=[\frac{a_5}{b_5}, \frac{a_6}{b_6}, \frac{a_7}{b_7}, ... \frac{a_m}{b_m}]^T$, and $\overrightarrow{\mathbf{M}} = M[1, 1, 1, ...1]^T$.

Thus, we can obtain the following inequality

$$a_0 \leq \overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{X}} \leq \overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{M}} \leq b_0 M. \tag{12}$$

Hence,

$$\frac{a_0}{b_0} \leq M (b_0 > 0)$$

which contradicts to (11). Thus Case 1.2 doesn't exist.

From Cases 1.1 and 1.2, we obtain the unique solution that $n_1 = mp, n_2 = 0, n_3 = p, n_4 = 0, n_i = 0 (i = 5, 6, 7, ..., m), n_{m+1} = p$. Hence $deg(G') = (1^{mp}, 3^p, (m + 1)^p)$.

For the graph $G = H(p, tK^*_{1,m})$, we introduce three operations. By Operation I, Operation II and Mixed Operation, we can obtain the graphs G^* and G^{**} and G^{***} whose degree sequences are the same as that of the graph G.

Operation I

To clearly illustrate, we introduce three new terms: normal branch, leafless branch and multi-leaf branch (shown in Fig. 2). In fact, they are induced subgraphs of G^* respectively.

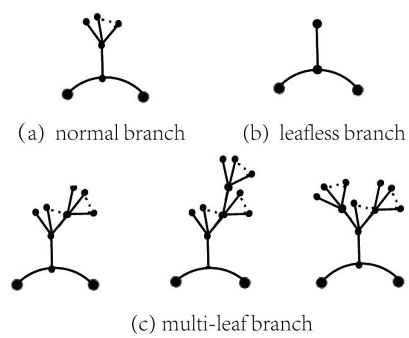


Figure 2 : The different kinds of branches in the graph G^*

The graph G^* is obtained from the graph G by Operation I (shown in Fig. 3). We suppose that there are r normal branches, k leafless branches and s multi-leaf branches in graph G^*. Here we assume that the missing *leaves* of leafless branch of G have been transferred to multi-leaf branch by the overlap of one-degree vertex randomly in each shift.

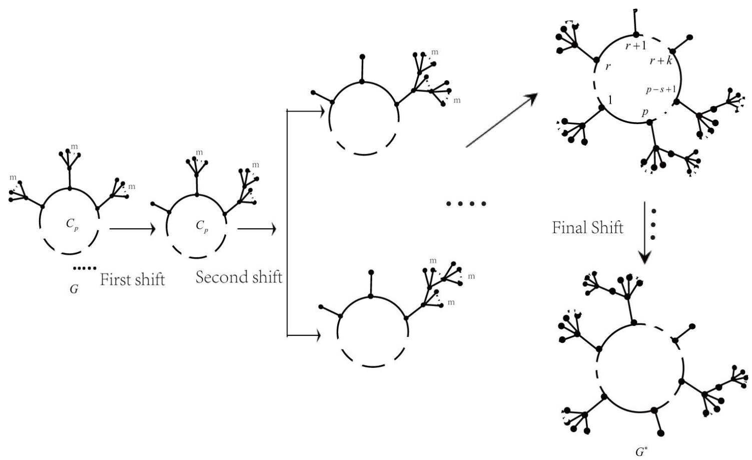


Figure 3 : Operation I $G \to G^*$

Note that p is even, the graphs G and G^* are both bipartite graphs. From Lemma 2.5, we know that their line graphs $l(G)$ and $l(G^*)$ are A-cospectral. Then by Lemma 2.1, we know that the number of closed walks of any length in their line graphs should be equal. We obtain the degree sequences of line graphs of G and G^* respectively as follows. $deg(l(G)) = ((m+2)^p, m^{mp}, 4^p)$, $deg(l(G^*)) = ((2m)^k, (m+2)^{p-k}, m^{mp-k}, 4^p, 2^k)$.

Clearly, the graphs G and G^* have same number of edges and closed walks of length 4. By Lemma 2.4, we know that the induced path of length 2 should also be equal. Then we have

$$p\binom{m+2}{2}+mp\binom{m}{2}+p\binom{4}{2}=k\binom{2m}{2}+(p-k)\binom{m+2}{2}+(mp-k)\binom{m}{2}+p\binom{4}{2}+k\binom{2}{2}. \tag{13}$$

By (13), we deduce $(2m^2-4m)k=0$. This is contradictory to the fact that $m \geq 5$ and $k > 0$. In addition, we can also find that the right side of the above equation is larger than its left side. This means that the number of induced path of G^* which is obtained from G by using Operation I will increase. Hence the above situation is not right. We can conclude that the graph G^* is not L-cospectral to the graph G.

Operation II

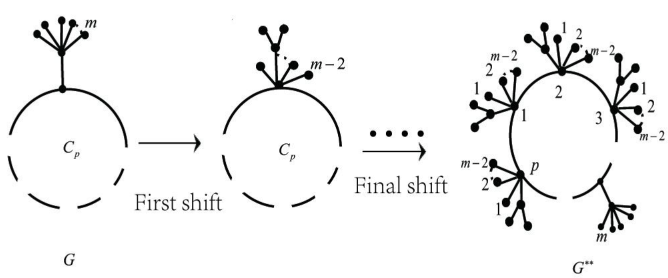


Figure 4 : Operation II $G \to G^{**}$

Graph G^{**} is obtained from graph G by using Operation II. In the first shift of graph G in Fig. 4, we can see that $(m-2)$ pendent edges at $(m+1)$-degree vertex *move* to 3-degree vertex on the C_p. And similarly, we can prove that graph G^{**} is not L-cospectral to graph G. The graph G^{**} is obtained through three different kinds of shifts as follows (Fig. 5).

As shown in Fig. 5, there are three important kinds of important shifts in Operation II. In Operation I, we know that the induced path of length 2 should also be equal. Next, we will prove that, by these shifts, their number of induced path of length 2 will increase, thus it is unlikely to be equal with the degree sequence of the graph G.

In Operation II.(1) in Fig. 5, let Δh_1 denote the changed number of induced path of length 2 after shift (1), thus

$$\Delta h_1 = 2C_{m+2}^2 - 2C_m^2 - 2C_4^2 + 2C_2^2 = 4m - 8 > 0, (m \geq 5), \tag{14}$$

where $C_n^r = \begin{pmatrix} n \\ r \end{pmatrix}$ denotes the number of r-subsets of an n-element set.

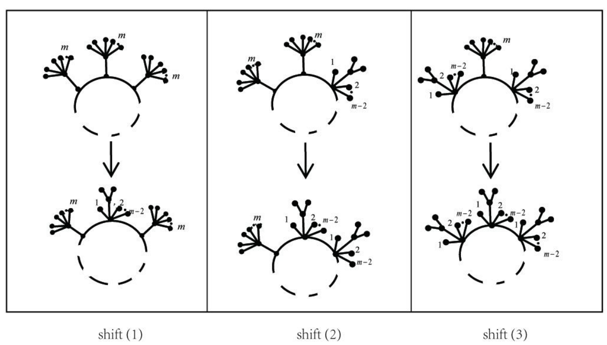


Figure 5 : Three kinds of shifts in Operation II

In Operation II.(2) in Fig. 5, let Δh_2 denote the changed number of induced path of length 2 after shift (2), thus

$$\Delta h_2 = C_{2m}^2 - 2C_m^2 - C_4^2 + 2C_2^2 = m^2 - 4 > 0, (m \geq 5). \tag{15}$$

In Operation II.(3) in Fig. 5, let Δh_3 denote the changed number of induced path of length 2 after shift (3), thus

$$\Delta h_3 = 2C_{2m}^2 - 2C_{m+2}^2 - 2C_m^2 + 2C_2^2 = 2m^2 - 4m > 0, (m \geq 5). \tag{16}$$

By Eqs. (14)-(16), we know that, through Operation II, the number of induced path of length 2 changes from G to G^{**}, which contradicts to Lemma 2.4, thus the graph G^{**} is not L-cospectral to the graph G.

Mixed Operation

Graph G^{***} is obtained by Mixed operation shown in Fig. 6. Mixed operation combines Operation I and Operation II. Nevertheless, its degree sequence of line graph can not be obtained easily. Instead, we use above relevant outcomes to prove our results.

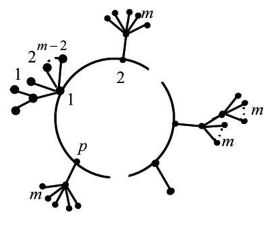


Figure 6 : The graph G^{***} is obtained by Mixed operation

Based on Eq. (13) and Eqs. (14)-(16), we know that through Operation I and Operation II, after each shift, the induced path of length 2 in line graph of G will increase. It means that it can't be equal with that of graph G which contradicts to Lemma 2.4. Hence, their degree sequences are not equal. We can draw the conclusion that they are not L-cospectral, either.

After totally discussed, the dandelion graph G is unique and $G' \cong G$.

Next, we will prove that when $m < 5$, the conclusion remains right.

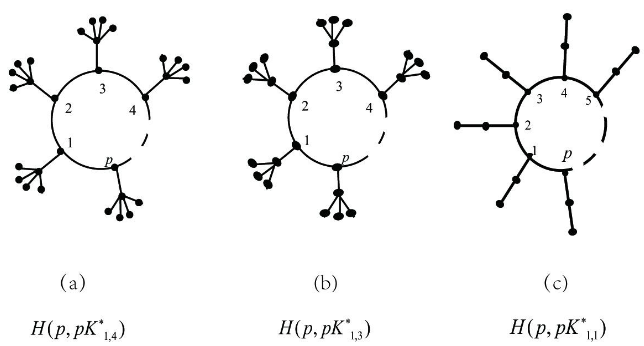


Figure 7 : The dandelion graphs $H(p, pK^*_{1,m})(m = 4, 3, 1)$

Case 2 : $m = 4$.

Clearly seen in Fig. 7(a), the graph $H(p, pK^*_{1,4})$ has $6p$ edges and $6p$ vertices. Thus $\Delta + 1 \leq \mu_1 \leq \max\{\frac{20}{3}, \frac{13}{2}, \frac{19}{3}\}$, so $\Delta \leq 5$. According to Lemma 2.1 and Lemma 2.3, we have

$$\sum_{i=1}^{5} n_i = 6p, \ \sum_{i=1}^{5} in_i = 12p, \ \sum_{i=1}^{5} i^2 n_i = 38p, \ \sum_{i=1}^{5} i^3 n_i = 156p.$$

By solving above equations, we obtain that

$$n_1 = 3p + n_5,\ n_2 = 4p - 4n_5,\ n_3 = -5p + 6n_5,\ n_4 = 4p - 4n_5.$$

Due to that $n_3 \geq 0, n_4 \geq 0$, we get $\frac{5}{6}p \leq n_5 \leq p$, in addition, $n_5 \in Z$ (where Z denotes the set of integers) and p is arbitrary, hence we have $n_5 = p$. It is easy to get $n_1 = 4p, n_2 = 0, n_3 = p, n_4 = 0$ and $deg(G') = (1^{4p}, 3^p, 5^p)$. Till now, we notice that the solution satisfies Eqs. (8)-(9) when $m = 4$. Thus similarly, the following proof is the same to Case 1. Thus, we obtain the conclusion that when $m = 4$, Dandelion graph $G = H(p, pK^*_{1,m})$ is determined by L-spectrum when p is even.

Case 3 : $m = 3$.

Clearly seen in Fig. 7(b), the graph $H(p, pK^*_{1,3})$ has $5p$ edges and $5p$ vertices. Thus $\Delta + 1 \leq \mu_1 \leq \max\{\frac{27}{5}, \frac{41}{7}, \frac{19}{3}\}$, so $\Delta \leq 5$. According to Lemma 2.1 and Lemma 2.3, we have

$$\sum_{i=1}^{5} n_i = 5p,\ \sum_{i=1}^{5} in_i = 10p,\ \sum_{i=1}^{5} i^2 n_i = 28p,\ \sum_{i=1}^{5} i^3 n_i = 94p.$$

By solving above equations, we obtain that

$$n_1 = 3p + n_5,\ n_2 = -4n_5,\ n_3 = p + 6n_5,\ n_4 = p - 4n_5.$$

Since that $n_i \geq 0$, we have $n_5 = 0$, it is natural to get $n_1 = 3p, n_2 = 0, n_3 = p, n_4 = p$. Thus, $deg(G') = (1^{3p}, 3^p, 4^p)$. Till now, we notice that the solution satisfies Eqs. (8)-(9) when $m = 3$. Thus similarly, the following proof is the same to Case 1. Thus, we obtain the conclusion that when $m = 3$, Dandelion graph $G = H(p, pK^*_{1,m})$ is determined by L-spectrum when p is even.

Case 4 : $m = 1$.

Similarly, we take use of the above method. Clearly seen in Fig. 7(c), the graph $H(p, pK^*_{1,1})$ has $3p$ edges and $3p$ vertices. Thus $\Delta + 1 \leq \mu_1 \leq \max\{\frac{17}{3}, \frac{11}{3}, 5\}$, so $\Delta \leq 4$. According to Lemma 2.1 and Lemma 2.3, we have

$$\sum_{i=1}^{4} n_i = 3p,\ \sum_{i=1}^{4} in_i = 6p,\ \sum_{i=1}^{4} i^2 n_i = 14p,\ \sum_{i=1}^{4} i^3 n_i = 36p.$$

By solving above equations, we obtain that

$$n_1 = p,\ n_2 = p,\ n_3 = p,\ n_4 = 0.$$

Thus, $deg(G') = (1^p, 2^p, 3^p)$.

However, through Operation I, there is graph G^* with the same degree sequence of graph G (shown in Fig. 8). We assume that there are r normal branches, k leafless branches and s multi-leaf branches in graph G^*.

Note that p is even, the graphs G and G^* are both bipartite graphs. From Lemma 2.5, we know that their line graphs $l(G)$ and $l(G^*)$ are A-cospectral. Then by Lemma 2.1, we know that the number of closed walks of any length in their line graphs should be equal. We obtain that the degree sequence of line graphs of G and G^* are $deg(l(G)) = (4^p, 3^p, 1^p)$, $deg(l(G^*)) = (4^p, 3^{p-k}, 2^{2k}, 1^{p-k})$, respectively.

Clearly, they have same number of edges and closed walks of length 4. By Lemma 2.4, we know that the induced path of length 2 should also be equal. Then we have

$$p\binom{4}{2} + p\binom{3}{2} = p\binom{4}{2} + (p-k)\binom{3}{2} + 2k\binom{2}{2}. \tag{17}$$

However, by Eq. (17), we deduce $-2k = 0$ which is contradictory to the fact that $k > 0$. Besides, the right side of the above equation is smaller than left side which means that after Operation I, the number of induced path will decrease. Thus, above situation is not right. We can conclude that graph G^* is not L-cospectral to graph G. So the dandelion graph $H(p, pK^*_{1,1})$ is unique and $G' \cong G$.

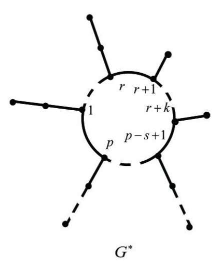


Figure 8 : The graphs with the same degree sequence ($m = 1$)

Till now, we completely prove that dandelion graph $H(p, pK^*_{1,m})$ is determined by Laplacian spectrum when p is even and $m \neq 2$. □

***Theorem* 3.2** — *Let p, t and m be positive integers. If p is even, $m \neq 2$ and $t < p$, then the broken dandelion graph $H(p, tK^*_{1,m})$ is determined by its L-spectrum.*

PROOF : As we define earlier, $H(p, tK^*_{1,m})$ is a connected unicyclic graph with $p + t(m+1)$ vertices obtained from the cycle C_p and t copies of the star $K_{1,m}$ by joining the center of $K_{1,m}$ to each one of t consecutive vertices of the cycle C_p through an edge, respectively. Here, we assume that there is a graph G' with the same L-spectrum and C_p of the graph $G = H(p, tK^*_{1,m})$. Let n_i

denote the number of vertices of degree i of G'. Clearly seen in Fig. 1, the graph $H(p, tK^*_{1,m})$ has $tm+t+p$ edges and $tm+t+p$ vertices. We apply Lemma 2.2 and we get the following results

Table 1: Discussion on maximum degree Δ of the graph G $(m \geq 5)$.

t	$\frac{d_i(d_i+m_i)+d_j(d_j+m_j)}{d_i+d_j}$	maximum	Δ
$t=p-1$	$\{\frac{m+16}{3}, \frac{m^2+4m+6}{m+2}, \frac{m^2+4m+20}{m+4}, \frac{m^2+4m+19}{m+4}, \frac{2m+31}{6}, \frac{m+25}{5}\}$	$\frac{m^2+4m+6}{m+2}$	$m+1$
$t=p-2$	$\{\frac{m+16}{3}, \frac{m^2+4m+6}{m+2}, \frac{m^2+4m+20}{m+4}, \frac{m^2+4m+19}{m+4}, \frac{2m+31}{6}, \frac{m+24}{5}, \frac{9}{2}\}$	$\frac{m^2+4m+6}{m+2}$	$m+1$
$t=p-3$	$\{\frac{m+16}{3}, \frac{m^2+4m+6}{m+2}, \frac{m^2+4m+20}{m+4}, \frac{m^2+4m+19}{m+4}, \frac{2m+31}{6}, \frac{m+24}{5}, \frac{17}{4}\}$	$\frac{m^2+4m+6}{m+2}$	$m+1$
$1 \leq t \leq p-4$	$\{\frac{m+16}{3}, \frac{m^2+4m+6}{m+2}, \frac{m^2+4m+20}{m+4}, \frac{m^2+4m+19}{m+4}, \frac{2m+31}{6}, \frac{m+24}{5}, \frac{17}{4}, 4\}$	$\frac{m^2+4m+6}{m+2}$	$m+1$

Thus, when $m \geq 5$, Δ is irrelevant to t and we can conclude that $\Delta = m+1$.

According to Lemma 2.1*(i)*,*(ii)*, *(v)* and Lemma 2.3, we have

$$\begin{cases} n_1 + n_2 + n_3 + ... + n_{m+1} = tm + t + p \\ n_1 + 2n_2 + 3n_3 + ... + (m+1)n_{m+1} = 2(tm+t+p) \\ n_1 + 4n_2 + 9n_3 + ... + (m+1)^2 n_{m+1} = (m^2+3m+6)t + 4p \\ n_1 + 8n_2 + 27n_3 + ... + (m+1)^3 n_{m+1} = (m^3+3m^2+4m+20)t + 8p. \end{cases} \tag{18}$$

Similar to Eq. (2), we obtain following solution by using Mathematica 8.0.

$$\begin{cases} n_1 = \frac{1}{6}((-m^3+6m^2-5m+6)t + \sum_{i=5}^{m+1}(i^3 - 9i^2 + 26i - 24)n_i) \\ n_2 = \frac{1}{2}((m^3-5m^2+6m)t + 2p - 2t + \sum_{i=5}^{m+1}(-i^3 + 8i^2 - 19i + 12)n_i) \\ n_3 = \frac{1}{2}((-m^3+4m^2-3m+2)t + \sum_{i=5}^{m+1}(i^3 - 7i^2 + 14i - 8)n_i) \\ n_4 = \frac{1}{6}((m^3-3m^2+2m)t + \sum_{i=5}^{m+1}(-i^3 + 6i^2 - 11i + 6)n_i). \end{cases} \tag{19}$$

It can be clearly seen that, $n_i \geq 0$ $(i = 1, 2, 3, 4)$, thus, we get the inequalities as follows

$$\sum_{i=5}^{m+1}(i^3 - 9i^2 + 26i - 24)n_i \geq (m^3 - 6m^2 + 5m - 6)t, \tag{20}$$

$$\sum_{i=5}^{m+1}(i^3 - 8i^2 + 19i - 12)n_i \leq (m^3 - 5m^2 + 6m)t + 2p - 2t, \tag{21}$$

$$\sum_{i=5}^{m+1}(i^3 - 7i^2 + 14i - 8)n_i \geq (m^3 - 4m^2 + 3m - 2)t, \tag{22}$$

$$\sum_{i=5}^{m+1}(i^3 - 6i^2 + 11i - 6)n_i \leq (m^3 - 3m^2 + 2m)t. \tag{23}$$

Next, similarly, we consider the following two cases $n_{m+1} = t$ and $n_{m+1} \neq t$.

Case 2.1 : $n_{m+1} = t$.

When $n_{m+1} = t$, by Eq. (23), we have

$$\sum_{i=5}^{m}(i^3 - 6i^2 + 11i - 6)n_i \leq 0.$$

However, $n_i \geq 0 (i \geq 0)$ and $(i^3 - 6i^2 + 11i - 6) > 0$ when $m \geq 5$, thus, we obtain one of the solutions

$$\begin{cases} n_i = 0 \ (i = 5, 6, 7, ..., m), \\ n_{m+1} = t. \end{cases} \tag{24}$$

Substituting Eq. (24) into Eqs. (20)-(23), we find that the solution satisfies the conditions of Theorem 3.2.

Then, by Eq. (19), we can easily obtain that

$$\begin{cases} n_1 = mt, \\ n_2 = p - t, \\ n_3 = t, \\ n_4 = 0. \end{cases} \tag{25}$$

Case 2.2 : $n_{m+1} \neq t$.

Apparently, according to Ineq. (23), we have $n_{m+1} \leq t$. When $n_{m+1} \neq t$, we have $n_{m+1} = t - 1, t - 2, t - 3, ..., 0$. When $n_{m+1} = t - c \ (1 \leq c \leq t - 1)$. Eq. (23) can be written as

$$\sum_{i=5}^{m}(i^3 - 6i^2 + 11i - 6)n_i \leq (m^3 - 3m^2 + 2m)c.$$

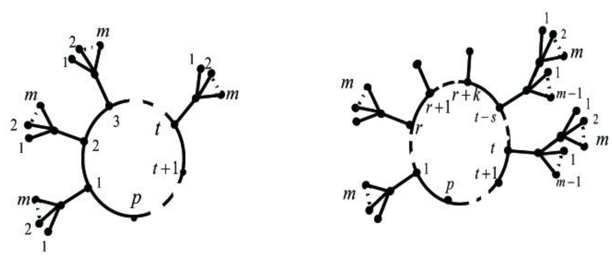
However, here c is a constant and it is irrelevant to p while the left side of above inequation is always relevant to p (according to Eqs. (19)). When p is arbitrary, it can't be guaranteed that above inequality always holds up. Thus, we obtain that when $n_{m+1} \neq t$, $c = p$ is the last possibility. Under such circumstance, n_{m+1} can only be 0.

For the proof of Case 2.2 of Theorem 3.2, the result is proved by the method which is used in Case 1.2 of Theorem 3.1. By the same way, we can draw the conclusion that Case 2.2 doesn't exist and the unique solution of above inequalities is shown in Eqs. (24) and (25). Hence $deg(G) = (1^{mt}, 2^{p-t}, 3^t, (m+1)^t)$.

Till now we have already known $deg(G)$. Next, we are supposed to prove that the graph G is unique and can be determined by its L-spectrum.

However, the biggest difference of the proof between Theorem 3.1 and Theorem 3.2 is the determination of Δ. By Table 1, we know that the number of Δ is certain, which is always $m+1$ when $m \geq 5$.

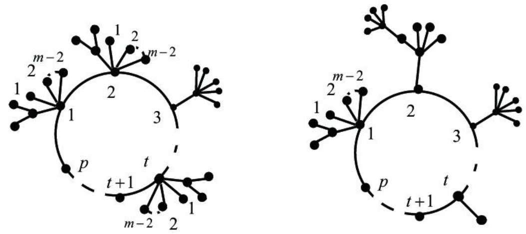
Similar to Theorem 3.1, by using three operations (Operation I, Operation II and Mixed Operation), we can also obtain the graphs G^*, G^{**} and G^{***} whose degree sequences are the same with graph G's. Next, we have to prove that the graphs G^* , G^{**} and G^{***} are not L-cospectral to graph G. And only $G' \cong G$. We find that the rest proving method is almost the same but with different number to consider. Thus, we only provide figures and tables to show our results. Note that when $m=1$, the graph G^{**} doesn't exist.



G G^*

Figure 9 : The broken dandelion graphs of G and G^*

Fig. 9 shows the broken dandelion graphs of G and G^* which is obtained by Operation I. We have: $deg(l(G)) = ((m+2)^t, m^m t, 4^{t-1}, 3^2, 2^{p-t-1})$, $deg(l(G^*) = ((2m)^k, (m+2)^{t-k}, m^{mt-k}, 4^{t-1}, 3^2, 2^{p-t-1+k})$. After careful calculation, we have $deg(l(G)) \neq deg(l(G^*))$. Thus, G^* is not L-cospectral to the graph G. Next, we will prove that G^{**} and G^{***} (shown in Fig. 9) are not L-cospectral to graph G, neither.



G^{**} G^{***}

Figure 10 : The graphs G^{**} and G^{***} are obtained by Operation II and Mixed Operation respectively

For the graph G^{**}, similar to the proof in Theorem 3.1, by Fig. 5 and Eqs. (14)-(15), we notice that the number of induced path of length 2 also changes from G to G^{**}, which contradicts to Lemma 2.4, thus the graph G^{**} is not L-cospectral to graph G. As for the graph G^{***} which is obtained by Mixed Operation, it combines the Operation I and Operation II, the number of induced path of length 2 should also change.

Thus, even though these graphs have same degree sequences, they are not L-conspectral. So we can conclude that graph $H(p, tK_{1,m}^*)$ can be determined by L-spectrum. □

4. Remarks

In this paper, we prove that dandelion graph $H(p, pK_{1,m}^*)$ is determined by its Laplacian spectrum when $m \neq 2$ and p is even. And we conclude that broken dandelion graph $H(p, tK_{1,m}^*)$ $(0 < t < p)$ is determined by Laplacian spectrum when $m \neq 2$ and p is even as well.

However, when $m = 2$, after our research, we find that $deg(l(G)) \equiv deg(l(G^*))$ in Eq. (13). It makes it hard for us to continue the work. Besides, when p is odd, whether dandelion graph can be determined by L-spectrum is still the open problem. Under such circumstances, the graph G is not the bipartite graph and we can not apply the corresponding theories of A-spectrum to our result.

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