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PERIODIC SOLUTIONS OF THE N-PREYS AND M-PREDATORS MODEL WITH VARIABLE RATES ON TIME SCALES

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In this paper, we establish the existence of periodic solution of a delayed predator-prey model with M-predators and N-preys over the time scales. We derive sufficient conditions for the existence of a periodic solution with the help of continuation theorem of coincidence degree theory. At the end, we give an example with numerical simulations to illustrate our analytical findings.

Key words : Time scale; prey-predator model; continuation theorem; coincidence degree; periodic solution.

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1. Introduction

In the past few decades, mathematical ecology has seen extensive progress, especially in population dynamics. It is the study of how a population changes over time and space or the relationship between the population and its environment. In the study of population dynamics, the interaction between a pair of predators and prey influences the population growth of both the species, i.e., when two or more species interact to each other their growth totally depends on each other species. For the understanding of dynamic behavior of the species, several population models are considered by many authors. For example, in 1989, the asymptotic behavior of two-dimensional Lotka-Volterra model has been studied by Ma and Wendi [18], and in 1995, two-predators and one-prey periodic Lotka-Volterra system was investigated by Zhonghua and Lansun, see [28]. In 1999, A Lotka-Volterra model with competition among predator species and among prey species or M-predators and N-preys was simultaneously considered by Yang and Rui [23], and then studied existence and uniqueness of the periodic solution of the system

$$\begin{cases} \dot{x}_i(t) = x_i(t)\left(b_i(t) - \sum\limits_{k=1}^{N} a_{ik}(t)x_k(t) - \sum\limits_{l=1}^{M} c_{il}(t)y_l(t)\right), \quad i = 1,2,\cdots,N, \\ \dot{y}_j(t) = y_j(t)\left(-r_j(t) - \sum\limits_{l=1}^{M} e_{jl}(t)y_l(t) + \sum\limits_{k=1}^{N} d_{jk}(t)x_k(t)\right), \quad j = 1,2,\cdots,M, \end{cases} \tag{1.1}$$

where $x_i(t)$ and $y_j(t)$ denoted the numbers of prey and predator species respectively and $b_i(t), r_j(t)$, $a_{ik}(t)$, $c_{il}(t), d_{jk}(t)$ and $e_{jl}(t)$ $(i,k = 1,2,\cdots,N$ and $l,j = 1,2,\cdots,M)$ denoted the coefficients, which are non-negative continuous periodic functions defined on $t \in (-\infty, \infty)$, where the coefficient b_i is the natural growth rate of prey species, r_j is the death rate of the predator species, a_{ik} measures the amount of the emulation between the prey species, e_{jl} measures the amount of the emulation between the predator species, and the constant $\bar{k}_{ij} \triangleq \frac{d_{ij}}{c_{ij}}$ denotes conversing prey species into new individual of predator species.

Many times it is necessary to consider the delayed models because many species start interacting after reaching a maturity period. It is important as sometimes, time delays may lead to oscillations, bisections, perturbations which may be detrimental to a system. So, Wen [21] has studied the positive periodic solution of a multi-species predator-prey system of (1.1) as follows:

$$\begin{cases} \dot{x}_i(t) = x_i(t)\left(b_i(t) - a_{ii}(t)x_i(t) - \sum\limits_{\substack{k=1 \\ k\neq i}}^{N} a_{ik}(t)x_k(t-\tau_{ik}) - \sum\limits_{l=1}^{M} c_{il}(t)y_l(t-\eta_{il})\right), \\ \dot{y}_j(t) = y_j(t)\left(-r_j(t) - e_{jj}(t)y_j(t) - \sum\limits_{\substack{l=1 \\ l\neq j}}^{M} e_{jl}(t)y_l(t-\xi_{jl}) + \sum\limits_{k=1}^{N} d_{jk}(t)x_k(t-\delta_{jk})\right), \end{cases} \tag{1.2}$$

where $b_i(t), r_j(t), a_{ik}(t), c_{il}(t), d_{jk}(t), e_{jl}(t)$ $(i,k = 1,2,\cdots,N$ and $j,l = 1,2,\cdots,M)$ are assumed to be continuous $\omega-$ periodic positive functions and delays $\tau_{ik}, \delta_{jk}, \eta_{il}, \xi_{jl}$ are assumed to be positive constants.

There may be some circumstances where the species do not overlap between the gradual generation, but according to [1, 2, 26], in such cases, it is better to use the model of difference equations. Indeed, as far as the discrete multispecies ecosystem is concerned, Chen [9] considered the following $N-$prey and $M-$predator Lotka-Volterra system of differential equations of (1.1) and investigated the dynamic behavior of it:

$$\begin{cases} x(k^{'}+1)=x(k^{'})exp\left(b_i(k^{'})-a_{ii}(k^{'})x_i(k^{'})-\sum\limits_{\substack{k=1\\k\neq i}}^{N}a_{ik}(k^{'})x_k(k^{'})-\sum\limits_{l=1}^{M}c_{il}(k^{'})y_l(k^{'})\right), \\ y(k^{'}+1)=y(k^{'})exp\left(-r_j(k^{'})-e_{jj}(k^{'})y_j(k^{'})-\sum\limits_{\substack{l=1\\l\neq j}}^{M}e_{jl}(k^{'})y_l(k^{'})+\sum\limits_{k=1}^{N}d_{jk}(k^{'})x_k(k^{'})\right), \end{cases} \tag{1.3}$$

Similarly, the differential equations system (1.2) can be carried over to their discrete analogues:

$$\begin{cases} x(k^{'}+1)=x(k^{'})exp\left(b_i(k^{'})-a_{ii}(k^{'})x_i(k^{'})-\sum\limits_{\substack{k=1\\k\neq i}}^{N}a_{ik}(k^{'})x_k(k^{'}-\tau_{ik})-\sum\limits_{l=1}^{M}c_{il}(k^{'})y_l(k^{'}-\eta_{il})\right), \\ y(k^{'}+1)=y(k^{'})exp\left(-r_j(k^{'})-e_{jj}(k^{'})y_j(k^{'})-\sum\limits_{\substack{l=1\\l\neq j}}^{M}e_{jl}(k^{'})y_l(k^{'}-\xi_{jl})+\sum\limits_{k=1}^{N}d_{jk}(k^{'})x_k(k^{'}-\delta_{jk})\right), \end{cases} \tag{1.4}$$

where $i=1,2,\cdots,N,\ j=1,2,\cdots,M.$

As seen above, some species grow both discretely and continuously in their real life e.g., 17 year cicada magicicada septendecim which lives as a larva for 17 years and as an adult for a short time or for at most a week and on the other hand the common mayfly stenonerna canadense lives as a larva for a year, but as an adult they live much shorter, may be at most 24 hours. It had been a challenge for mathematician to harmonize the discrete and the continuous calculus, to include them in one comprehensive mathematics and to remove ambiguity from both, see [5, 6].

To solve this dilemma, The theory of the time scales was introduced by the German mathematician Stefan Hilger in 1988 [14]. This theory has been developed by many authors and there are many books and monographs available on this topic [5, 7]. The motivation for the development of this theory is to unify the continuous and discrete analysis because it avoids the double analysis, once for the differential equations and other for the difference equations. The idea is to prove the result for the dynamic equations, where the domain of the unknown functions is called time scale. A time scale is an arbitrary non-empty closed subset of the real numbers i.e., $\mathbb{R}$, which is denoted by $\mathbb{T}$ and inherits the standard topology of $\mathbb{R}$. It may be in any of these forms e.g., $\mathbb{R},\mathbb{N},[a,b]\ \forall a,b\in\mathbb{R},\ \mathbb{Z},\mathbb{C},$ harmonic numbers, Cantor set etc. and $(a,b),[a,b),(a,b],\forall a,b\in\mathbb{R},\{\frac{1}{n}:n\in\mathbb{N}\},\mathbb{Q}$ etc. can't be time scales as these are not closed subsets of the real numbers $\mathbb{R}$ [5, 7].

It has many applications in different fields e.g., the mathematical models of phenomena and real processes such as biotechnology, population dynamics, economics, neural networks, quantum physics, and social science etc. [4, 10, 16]. In quantum physics, the time scale is taken as $q^{\mathbb{Z}} \bigcup \{0\}$, $q > 1$, for more details see [5, 6].

A large number of researchers have already focused on this topic for a long period of time, and we have plenty of essential papers, article etc. related to time scale. For example, in 2006, a solution to the first order dynamic equation on time scale which bounded and exponential stable is studied by Ai-lian [3]. A predator-prey dynamic system with Beddington-DeAngelis functional response was considered by Bohner *et al*. [8], and established the existence of the periodic solutions of dynamic equations on time scales. In 2010, a sufficient criteria for the existence of periodic solution of the predator-prey system on the time scale with Beddington-DeAngelis functional response and diffusion was investigated by Yange *et al*. [24]. In 2013, Wang [20] generalized the result of [11, 27] for the $n-$species competition, where $n-1$ competing preys and one predator on time scales with Holling-type II functional response are considered, the author established the existence of global solution. For more results concerning the periodic solutions, we refer the readers to see [2, 17].

In order to unify the continuous time model (1.2) and discrete time model (1.3), we consider following model on time scale:

$$\begin{aligned} u_i^{\Delta}(t) &= \left(b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{\substack{k=1 \\ k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \right), \\ v_j^{\Delta}(t) &= \left(-r_j(t) - e_{jj}(t)e^{v_j(t)} - \sum_{\substack{l=1 \\ l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})} + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} \right), \end{aligned} \tag{1.5}$$

where $b_i(t), r_j(t), a_{ik}(t), c_{il}(t), d_{jk}(t), e_{jl}(t) \quad (i,k = 1,2,\cdots,N; j,l = 1,2,\cdots,M)$ all are assumed to be rd-continuous positive $\omega-$ periodic functions.

Now the system (1.5) is supplemented with the initial conditions:

$$u_i(\theta) = \alpha_i(\theta), \quad v_j(\theta) = \beta_i(\theta), \quad \theta \in [-\tau, 0], \ \alpha_i(0) > 0, \ \beta_i(0) > 0,$$

where $\tau = \max\left[\max_{1\leq i,k\leq N} \tau_{ik}, \max_{1\leq i\leq N, 1\leq l\leq M} \eta_{il}, \max_{1\leq j,l\leq M} \xi_{jl}, \max_{1\leq j\leq M, 1\leq k\leq N} \delta_{jk}\right] > 0$. The solution of the system (1.5) remains positive for all $t \geq 0$ with the above initial conditions.

Remark 1.5 : In model (1.2), we set $x_i(t) = exp[u_i(t)]$ and $y_j(t) = exp[v_j(t)]$, $\forall i = 1,2,\cdots,N$ and $j = 1,2,\cdots,M$. If $\mathbb{T} = \mathbb{R}$, (1.5) turns into (1.2) and if $\mathbb{T} = \mathbb{Z}$, (1.5) is equivalent to the (1.4). The purpose of this paper is to study the simplest sufficient condition for $\omega-$ periodic solution of dynamic system on time scales (1.5) by employing a continuation theorem in coincidence degree theory.

The rest of the paper is arranged as follows: In Section 2, we present some basic definitions along with some examples, fundamental concepts, assumptions, inequalities and lemmas. In Section 3, we establish the existence of ω-periodic solution for the system (1.5). In the last section, we validate our theorem by giving an example with illustrations.

2. Preliminaries and Assumptions

We briefly recall some basic definitions, useful assumptions and lemmas that we will use in the sequel.

Definition 2.1 — [5, 7]. For $t \in \mathbb{T}$, we define a forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ where σ is monotonically increasing and ρ is monotonically decreasing operator on a time scale $\mathbb{T}$. In this definition, we put $\inf \phi = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t), $\sup \phi = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t) and a point $t \in \mathbb{T}$ is said to be

1. right-scattered if $t < \sigma(t)$;
2. left-scattered if $t > \rho(t)$;
3. right dense if $t <$sup$\mathbb{T}$ with $t = \sigma(t)$;
4. left dense if $t > \inf\mathbb{T}$ with $t = \rho(t)$;
5. isolate if $\rho(t) < t < \sigma(t)$;
6. dense if $\sigma(t) = t = \rho(t)$.

The graininess $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. See [5, 7] by Bohner and Peterson, Dynamic Equations on Time Scales.

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise: $\mathbb{T}^\kappa = \mathbb{T}$. e.g., if $\mathbb{T} = \{1, 2, 3, 4\}$, then $\mathbb{T}^\kappa = \{1, 2, 3\}$.

Definition 2.2 — [5, 7]. If a function $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$, then we define $f^\Delta(t)$, to be a number (provided it exists) with the property that given any $\epsilon > 0$, there exists a neighborhood $\mathcal{A} = (t - \delta, t + \delta) \bigcap \mathbb{T}$ for some $\delta > 0$ such that

$$\big|[f(\sigma(t)) - f(r)] - f^\Delta(t)[\sigma(t) - r]\big| \leq \epsilon|\sigma(t) - r| \quad \forall r \in \mathcal{A}.$$

Thus, we call $f^\Delta(t)$ the delta-or Hilger -derivative of f at t.

Definition 2.3 — [5, 7]. A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist(finite) at all right-dense points and left$-$sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set regulated functions is shown by $\mathbb{C}_r = \mathbb{C}_r(\mathbb{T}) = \mathbb{C}_r(\mathbb{T}, \mathbb{R})$.

Definition 2.4 — [5, 7]. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left$-$dense points in $\mathbb{T}$. The set rd$-$continuous functions is shown by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.5 — [5, 7]. A function $F : \mathbb{T} \to \mathbb{R}$ is called an anti$-$derivative of $f : \mathbb{T} \to \mathbb{R}$, provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then for all $a, b \in \mathbb{T}$, Cauchy integral is defined by

$$\int_a^b f(s)\Delta(s) = F(b) - F(a).$$

Definition 2.6 — [2]. A time scale $\mathbb{T}$ is periodic if there exists $L > 0$ such that $t \in \mathbb{T}$ implies $t \pm L \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive L is called the period of $\mathbb{T}$.
e.g., : $\mathbb{T} = \bigcup_{n \in \mathbb{Z}} [4n, 4n+1]$ with period $L = 4$.

Definition 2.7 — [2, 22]. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period $L > 0$. A function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period $\omega > 0$ if there exist a natural number n such that $\omega = nL$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$, where ω is a smallest positive number such that $f(t+\omega) = f(t)$ for all $t \in \mathbb{T}$.

Lemma 2.8 — [15]. If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then the following are true:

1. $\int_a^b [\alpha f_1(t) + \beta f_2(t)]\Delta t = \alpha \int_a^b f_1(t)\Delta t + \beta \int_a^b f_2(t)\Delta t$;
2. If $f_1(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f_1(t)\Delta t \geq 0$;
3. If $|f_1(t)| \leq f_2(t)$ for $t \in [a, b) \subset \mathbb{T}$, then $\left| \int_a^b f_1(t)\Delta t \right| \leq \int_a^b f_2(t)\Delta t$.

Lemma 2.9 — [25]. Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is $\omega-$ periodic, then the following inequalities hold:

$$f(t) \leq f(t_1) + \int_{I_\omega} |f^{\Delta}(s)|\Delta s$$

and

$$f(t) \geq f(t_2) - \int_{I_\omega} |f^{\Delta}(s)|\Delta s.$$

Lemma 2.10 — [5, 7]. Every delta-derivative function is continuous.

Lemma 2.11 — [5, 7]. Every rd-continuous function has a delta anti-derivative.

Definition 2.12 — [13]. Let X, Z be normed real Banach spaces, $L : Dom(L) \subset X \to Z$ be a linear mapping and $N' : X \to Z$ be a continuous mapping, then the mapping L is called a Fredholm mapping of index zero if $dimKer(L) = codimIm(L) < \infty$ and $Im(L)$ is closed in Z.

In order to explore the existence of positive periodic solutions of (1.5) and for convenience, we shall first summarize below a few concepts and results without proof.

Remark 2.13 : [13, 19]. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that Im $P = Ker(L), Ker(Q) = Im(L) = Im(I-Q)$. It follows that $L|_{dom(L)\bigcap Ker(P)} : (I-P)X \to Im(L)$ is invertible, we denote the inverse of this map by K_p.
If Ω is an open bounded subset of X, then mapping N' will be called $L-$compact on $\bar{\Omega}$ if $QN'(\bar{\Omega})$ is bounded and $K_p(I-Q)N' : \bar{\Omega} \to X$ is compact. Since $Im(Q)$ is isomorphic to $Ker(L)$, therefore there exists an isomorphism $J : Im(Q) \to Ker(L)$.

Definition 2.14 — [13, 22]. Let $\Omega \subset \mathbb{R}^n$ be open and bounded set. If $f \in C^1(\Omega, \mathbb{R}^n) \bigcap C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n \backslash f(\partial\Omega \cup N_f)$, i.e., y is a regular value of f and $N_f = [x \in \Omega : \mathbb{J}_f(x) = 0]$ is critical set of f and $\mathbb{J}_f(x)$ is the Jacobian of f at x, then the degree $\deg[f; \Omega; y]$ is defined by

$$\deg[f; \Omega; y] = \sum_{x \in f^{-1}(y)} sgn\, \mathbb{J}_f(x)$$

with agreement $\sum \phi = 0$.

Lemma 2.15 — (Continuation Theorem [13]). Let $\Omega \subset X$ be an open and bounded set and L be a Fredholm mapping of index zero and N' be $L-$compact on $\bar{\Omega}$. If the following are hold:

1. For each $\lambda \in (0, 1)$ and $x \in \partial\Omega \cap Dom(L), Lx \neq \lambda N'x$;

2. For each $x \in \partial\Omega \bigcap Ker(L), QN(x) \neq 0$ and $\deg[JQN'; \Omega \bigcap Ker(L); 0] \neq 0$

Then $Lx = N'x$ has at least one solution in $\overline{\Omega} \bigcap Dom(L)$.

In order to prove the periodic solution of model (1.5), we need to define the following notation and assumptions:

$$\begin{cases} \textbf{(A1)} : \bar{b}_i > \sum\limits_{\substack{k=1\\k\neq i}}^{N} \bar{a}_{ik} e^{M_k^1} + \sum\limits_{l=1}^{M} \bar{c}_{il} e^{K_l^1}; \\ \textbf{(A2)} : \bar{r}_j < \sum\limits_{k=1}^{N} \bar{d}_{jk} e^{M_k^*} - \sum\limits_{\substack{l=1\\l\neq j}}^{M} \bar{e}_{jl} e^{K_l^1}, \\ \textbf{(A3)} : |B| \neq 0 \text{ and the system } BV = \mathfrak{B} \text{ has a solution in } \mathbb{R}_+^M; \\ \textbf{(A4)} : b_i^l > \sum\limits_{\substack{k=1\\k\neq i}}^{N} a_{ik}^m e^{\mathfrak{M}_k^1} + \sum\limits_{l=1}^{M} c_{il}^m e^{\mathfrak{K}_l^1}; \\ \textbf{(A5)} : r_j^m < \sum\limits_{k=1}^{N} d_{jk}^l e^{\mathfrak{M}_k^*} - \sum\limits_{\substack{l=1\\l\neq j}}^{M} e_{jl}^m e^{\mathfrak{K}_l^1}. \end{cases} \tag{2.1}$$

Where

$$B = \begin{bmatrix} \bar{e}_{11} & \bar{e}_{12} & \cdots & \cdots & \bar{e}_{1M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{e}_{M1} & \bar{e}_{M2} & \cdots & \cdots & \bar{e}_{MM} \end{bmatrix}, \tag{2.2}$$

$$\log\left(\frac{b_i^m}{a_{ii}^l}\right) + 2b_i^m \omega = \mathfrak{M}_i^1, \quad \log\left(\frac{\sum\limits_{k=1}^{N} d_{jk}^m e^{\mathfrak{M}_i^1}}{e_{jj}^l}\right) + 2\sum_{k=1}^{N} d_{jk}^m e^{\mathfrak{M}_k^1}\omega = \mathfrak{K}_j^1, \tag{2.3}$$

$$\mathfrak{M}_i^* := \log\left(\frac{b_i^l - \sum\limits_{\substack{k=1\\k\neq i}}^{N} a_{ik}^m e^{\mathfrak{M}_k^1} - \sum\limits_{l=1}^{M} c_{il}^m e^{\mathfrak{K}_l^1}}{a_{ii}^m}\right) - 2b_i^m\omega \tag{2.4}$$

and

$$\mathfrak{K}_j^* := \log\left(\frac{\sum\limits_{k=1}^{N} d_{jk}^l e^{\mathfrak{M}_k^*} - r_j^m - \sum\limits_{\substack{l=1\\l\neq j}}^{M} e_{jl}^m e^{\mathfrak{K}_l^1}}{e_{jj}^m}\right) - 2\sum_{k=1}^{N} d_{jk}^m e^{\mathfrak{M}_k^1}\omega, \tag{2.5}$$

where $M_i^1, M_i^*, K_j^1, K_j^*$ are in (3.12), (3.21), (3.18) and (3.24) respectively, $\forall i = 1, 2, \cdots, N,\ j = 1, 2, \cdots, M$.

Notations : We introduce the following notations for more comfort and simplicity in this paper:

Let $k^* = \min\{\mathbb{R}^+ \bigcap \mathbb{T}\},\ I_\omega = [k^*, k^* + \omega] \bigcap \mathbb{T}$ and $\bar{f} = \frac{1}{\omega} \int_{I_\omega} f(t)\, \Delta t = \dfrac{1}{\omega} \int_{k^*}^{k^*+\omega} f(t)\, \Delta t$, for all periodic functions f, where $f \in C_{rd}$ is an $\omega-$ periodic function, i.e., $f(t+\omega) = f(t),\ \forall\, t \in \mathbb{T}$, det(B) = determinant of matrix $B = |B|$, $U(t) = (u_1(t), u_2(t), \cdots, u_N(t), v_1(t), \cdots, v_M(t))^T$, $V_e = (e^{v_1}, e^{v_2}, \cdots, e^{v_M})^T$, $V^* = (u_1^*, u_2^*, \cdots, u_N^*, v_1^*, \cdots, v_M^*)^T$,

$$V = (v_1, v_2, \cdots, v_M)^T \text{ and } \mathfrak{B} := \begin{pmatrix} \sum_{k=1}^{N} \bar{d}_{1k} \left(\frac{\bar{b}_k}{\bar{a}_{kk}}\right) - \bar{r}_1 \\ \sum_{k=1}^{N} \bar{d}_{2k} \left(\frac{\bar{b}_k}{\bar{a}_{kk}}\right) - \bar{r}_2 \\ \vdots \\ \sum_{k=1}^{N} \bar{d}_{Mk} \left(\frac{\bar{b}_k}{\bar{a}_{kk}}\right) - \bar{r}_M \end{pmatrix}, \bar{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{(N+M)\times 1}.$$

If $v = (v_1, v_2, \cdots, v_n)^T \in \mathbb{R}^n$, then the norm define as follows:

$$\|v\|_\infty = \max_{1 \le i \le n} |v_i|,$$

$$f^m := \sup_{t \in I_\omega} f(t) \text{ and } f^l := \inf_{t \in I_\omega} f(t).$$

3. Existence of Periodic Solution

In this section, we will use the Mawhin continuation theorem or Lemma 2.15 for the existence of the $\omega-$ periodic solution of model (1.5).

***Theorem* 3.1** — *Assume that $(A1) - (A3)$ hold, then model (1.5) has at least one $\omega-$ periodic solution.*

Proof : Let

$$X = Z = \left\{ \begin{array}{c} U(t) \in \mathbb{C}(\mathbb{T}, \mathbb{R}^{N+M}) : u_i(t+\omega) = u_i(t), v_j(t+\omega) = v_j(t), \\ \forall\, i = 1, 2, \cdots, N,\ j = 1, 2, \cdots, M \end{array} \right\}.$$

The norm is given by

$$\begin{aligned} \|U(t)\| &= \|(u_1(t), u_2(t), \cdots, u_N(t), v_1(t), \cdots, v_M(t))^T\| \\ &= \sum_{i=1}^{N} \max_{t \in I_\omega} |u_i(t)| + \sum_{j=1}^{M} \max_{t \in I_\omega} |v_j(t)|, \quad \forall\, U(t) \in X. \end{aligned}$$

Then both the spaces X and Z are Banach space endowed with the above norm $\|.\|_.$ We define a function $N' : X \to Z$ such that $N'[U(t)] =$

$$\begin{pmatrix} \vdots \\ p_i(t) \\ \vdots \\ q_j(t) \\ \vdots \end{pmatrix}_{(N+M)\times 1} = \begin{pmatrix} \vdots \\ b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{\substack{k=1 \\ k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \\ \vdots \\ -r_j(t) - e_{jj}(t)e^{v_j(t)} - \sum_{\substack{l=1 \\ l\neq j}}^{N} e_{jl}(t)e^{v_l(t-\xi_{jl})} + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} \\ \vdots \end{pmatrix}_{(N+M)\times 1} \in Z. \quad (3.1)$$

Let us define $L : Dom(L) \subset X \to Z$ by

$$\begin{aligned} &L\left[(u_1(t), u_2(t), \cdots, u_N(t), v_1(t), \cdots, v_M(t))^T\right] \\ &= \left(u_1^{\Delta}(t), u_2^{\Delta}(t), \ldots, u_N^{\Delta}(t), v_1^{\Delta}(t), \cdots, v_M^{\Delta}(t)\right)^T \in Z, \ \forall\, U(t) \in X, \end{aligned}$$

which is linear. Then, it is easy to find:

$$Ker(L) = \left\{ \ U(t) \in X : U(t) = \ (u_1, \cdots, u_N, v_1, \cdots, v_M)^T \in \mathbb{R}^{N+M}, \ \forall t \in \mathbb{T} \ \right\} = \mathbb{R}^{N+M},$$

and

$$Im(L) = \left\{ \ U(t) \in Z : \left(\int_{I_\omega} u_1(t)\,\Delta t, \cdots, \int_{I_\omega} u_N(t)\,\Delta t, \int_{I_\omega} v_1(t)\,\Delta t, \cdots, \int_{I_\omega} v_M(t)\,\Delta t\right)^T = \bar{0} \ \right\}.$$

From the above expressions, we obtain the following: $DimKer(L) = N + M, CodimIm(L) = Dim\left(\frac{Z}{Im(L)}\right) = N + M$ and $Im(L)$ is closed in Z. From Definition 2.8, it follows that the mapping L is Fredholm mapping of index zero and hence there exist two continuous projections, P and Q and $P : X \to X, Q : Z \to Z$ are defined by

$$\begin{aligned} P[U(t)] \quad = \quad & Q[U(t)] = \left(\frac{1}{\omega}\int_{I_\omega} u_1(t)\,\Delta t, \cdots, \frac{1}{\omega}\int_{I_\omega} u_N(t)\,\Delta t, \frac{1}{\omega}\int_{I_\omega} v_1(t)\,\Delta t, \cdots, \frac{1}{\omega}\int_{I_\omega} v_M(t)\,\Delta t\right)^T, \\ & \text{for all } U(t) \in X. \end{aligned} \quad (3.2)$$

From (3.2), we obtain

$$Im(P) = Ker(L) \text{ and } Im(L) = \ Ker(Q) \ = \ Im(I - Q).$$

From Remark (2.9), we conclude that $L_p = L|_{dom(L) \bigcap Ker(P)} : Dom(L) \bigcap Ker(P) \to Im(L)$

is invertible and its inverse, which is denoted by $K_p : Im(L) \to Dom(L) \bigcap Ker(P)$ is given by

$$K_p(U(t)) = \begin{pmatrix} \vdots \\ \displaystyle\int_{k^*}^{t} u_i(r)\,\Delta r - \frac{1}{\omega}\int_{k^*}^{k^*+\omega}\int_{k^*}^{t} u_i(r)\,\Delta r\,\Delta t \\ \vdots \\ \displaystyle\int_{k^*}^{t} v_j(r)\,\Delta r - \frac{1}{\omega}\int_{k^*}^{k^*+\omega}\int_{k^*}^{t} v_j(r)\,\Delta r\,\Delta t \\ \vdots \end{pmatrix}, \ \forall\, U(t) \in Im(L) \tag{3.3}$$

From (3.1) and (3.2), we have $QN' : X \to Z$ which is given by

$$QN'(U(t)) = \begin{pmatrix} \vdots \\ \frac{1}{\omega}\int_{I_\omega}\left(b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum\limits_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum\limits_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}\right)\Delta t \\ \vdots \\ \frac{1}{\omega}\int_{I_\omega}\left(-r_j(t) - e_{jj}(t)e^{v_j(t)} - \sum\limits_{\substack{l=1\\l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})} + \sum\limits_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\right)\Delta t \\ \vdots \end{pmatrix} \in Z. \tag{3.4}$$

From (3.3) and (3.4), we obtain a mapping $K_p(I-Q)N' : X \to X$ such that

$$K_p(I-Q)N'[U(t)] = \begin{pmatrix} \vdots \\ \int_{k^*}^{t} p_i(r)\Delta r - \frac{1}{\omega}\int_{I_\omega} \quad \int_{k^*}^{t} p_i(r)\Delta r \quad \Delta t - \quad t - k^* - \frac{1}{\omega}\int_{I_\omega}(t-k^*)\Delta t\Big)\,\bar{p}_i(t) \\ \vdots \\ \int_{k^*}^{t} q_j(r)\Delta r - \frac{1}{\omega}\int_{I_\omega} \quad \int_{k^*}^{t} q_j(r)\Delta r \quad \Delta t - \quad t - k^* - \frac{1}{\omega}\int_{I_\omega}(t-k^*)\Delta t\Big)\,\bar{q}_j(t) \\ \vdots \end{pmatrix}_{(N+M)\times 1}$$

$\forall i = 1, 2, \cdots, N, j = 1, 2, \cdots, M.$

Then, QN' and $K_p(I-Q)N'$ are continuous functions on any open bounded set $\Omega \subset X$. Therefore, by using the Arzela-Ascoli theorem, we obtain that $K_P(I-Q)N'(\overline{\Omega})$ is compact, Moreover $QN'(\overline{\Omega})$ is bounded. Thus, N' is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

Now, we are at the point to search for an appropriate open, bounded subset Ω for the application of the continuation theorem or Lemma 2.15. Corresponding to the operator equation $LU = \lambda N'U$, $\forall \lambda \in (0,1)$, we have

$$\begin{cases} u_i^{\Delta}(t) = \lambda\left(b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum\limits_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum\limits_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}\right), \quad i = 1, 2, \cdots, N, \\ v_j^{\Delta}(t) = \lambda\left(-r_j(t) - e_{jj}(t)e^{v_j(t)} - \sum\limits_{\substack{l=1\\l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})} + \sum\limits_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\right), \ j = 1, 2, \cdots, M, \end{cases} \tag{3.5}$$

where $j = 1, 2, \cdots, M, i = 1, 2, \cdots, N.$

From (3.5), we obtain

$$\int_{I_\omega} |u_i^\Delta(t)| \,\Delta t \le \lambda \int_{I_\omega} \left| b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \right| \Delta t$$

$$\le \bar{b}_i\omega + \int_{I_\omega} \left| a_{ii}(t)e^{u_i(t)} + \sum_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} + \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \right| \Delta t \tag{3.6}$$

and

$$\int_{I_\omega} \left(b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \right) \Delta t = \int_{I_\omega} u_i^\Delta(t)\,\Delta t$$

$$= 0,$$

which implies

$$\int_{I_\omega} \left(a_{ii}(t)e^{u_i(t)} + \sum_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} + \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \right) \Delta t = \bar{b}_i\omega. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\int_{I_\omega} |u_i^\Delta(t)|\,\Delta t \le 2\bar{b}_i\omega. \tag{3.8}$$

For any $U(t) \in X$, then there exist $\phi_i, \psi_i, q_j, p_j \in I_\omega,\ i = 1, 2, \cdots, N,\ \ j = 1, 2, \cdots, M$, such that

$$u_i(\phi_i) = \inf_{t\in I_\omega}[u_i(t)] \text{ and } u_i(\psi_i) = \sup_{t\in I_\omega}[u_i(t)], \tag{3.9}$$

$$v_j(q_j) = \inf_{t\in I_\omega}[v_j(t)] \text{ and } v_j(p_j) = \sup_{t\in I_\omega}[v_j(t)]. \tag{3.10}$$

From (3.7), we obtain

$$\frac{1}{\omega}\int_{I_\omega} a_{ii}(t)e^{u_i(t)}\,\Delta t \le \bar{b}_i$$

From the above relation and (3.9), we have

$$u_i(\phi_i) \le \log\left(\frac{\bar{b}_i}{\bar{a}_{ii}}\right) := L_i^1. \tag{3.11}$$

From (3.8), (3.11) and (2.1), we have

$$u_i(t) \le L_i^1 + 2\bar{b}_i\omega := M_i^1. \tag{3.12}$$

From (3.5), we obtain

$$\int_{I_\omega} |v_j^\Delta(t)|\Delta t \le \bar{r}_j\omega + \int_{I_\omega} \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\Delta t + \int_{I_\omega}\left(e_{jj}(t)e^{v_j(t)} + \sum_{\substack{l=1\\ l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}\right)\Delta t \tag{3.13}$$

as well as

$$\int_{I_\omega}\left(-r_j(t) - e_{jj}(t)e^{v_j(t)} - \sum_{\substack{l=1\\ l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})} + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\right)\Delta t = 0, \tag{3.14}$$

which implies

$$\int_{I_\omega}\left(e_{jj}(t)e^{v_j(t)} + \sum_{\substack{l=1\\ l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}\right)\Delta t = -\bar{r}_j\omega + \int_{I_\omega} \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\Delta t. \tag{3.15}$$

From (3.12), (3.13) and (3.15), we obtain

$$\int_{I_\omega} |v_j^\Delta(t)|\Delta t \le 2\omega \sum_{k=1}^{N} \bar{d}_{jk}e^{M_k^1}. \tag{3.16}$$

Further using (3.14), we have

$$\int_{I_\omega}\left(-e_{jj}(t)e^{v_j(t)} - \sum_{\substack{l=1\\ l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})} + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\right)\Delta t = \omega\bar{r}_j > 0,$$

$$\frac{1}{\omega}\int_{I_\omega}\left(e_{jj}(t)e^{v_j(t)} + \sum_{\substack{l=1\\ l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}\right)\Delta t < \frac{1}{\omega}\int_{I_\omega}\left(\sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\right)\Delta t.$$

Now from (3.10), (3.12) and using the above inequality, we obtain

$$\bar{e}_{jj}e^{v_j(q_j)} < \sum_{k=1}^{N} \bar{d}_{jk}e^{M_k^1},$$

which is equivalent to

$$v_j(q_j) < \log\left(\frac{\sum\limits_{k=1}^{N} \bar{d}_{jk}e^{M_k^1}}{\bar{e}_{jj}}\right) := S_j^1. \tag{3.17}$$

The following relation is obtained from (2.1) (3.16) and (3.17),

$$v_j(t) < S_j^1 + 2\omega\sum_{k=1}^{N} \bar{d}_{jk}e^{M_k^1} := K_j^1 \quad \forall\, t \in I_\omega. \tag{3.18}$$

From (3.5) and (3.9), we obtain

$$\frac{1}{\omega}\int_{I_\omega}\left(b_i(t) - \sum_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}\right)\Delta t = \frac{1}{\omega}\int_{I_\omega} a_{ii}(t)e^{u_i(t)} \le \bar{a}_{ii}e^{u_i(\psi_i)}.$$

From (3.12) and (3.18), we have

$$\bar{b}_i - \sum_{\substack{k=1\\k\neq i}}^{N} \bar{a}_{ik}e^{M_k^1} - \sum_{l=1}^{M} \bar{c}_{il}e^{K_l^1} \le \bar{a}_{ii}e^{u_i(\psi_i)}. \tag{3.19}$$

Using condition $(A1)$ in (2.2) and (3.19), we get

$$L_i^* := \log\left(\frac{\bar{b}_i - \sum\limits_{\substack{k=1\\k\neq i}}^{N} \bar{a}_{ik}e^{M_k^1} - \sum\limits_{l=1}^{M} \bar{c}_{il}e^{K_l^1}}{\bar{a}_{ii}}\right) \le u_i(\psi_i). \tag{3.20}$$

From (2.1), (3.8) and (3.20), we have

$$M_i^* := L_i^* - 2\bar{b}_i\omega \le u_i(t) \;\; \forall\, t \in I_\omega. \tag{3.21}$$

We obtain the following from (3.9), (3.14), (3.14) and (3.16):

$$\sum_{k=1}^{N} \bar{d}_{jk}e^{M_k^*} - \bar{r}_j - \sum_{l=1}^{M} \bar{e}_{jl}e^{K_l^1} \le \bar{e}_{jj}e^{v_j(p_j)} \quad \forall\, t \in I_\omega. \tag{3.22}$$

From condition $(A2)$ in (2.1) and (3.19), we deduce

$$S_j^* := \log\left(\frac{\sum_{k=1}^{N}\bar{d}_{jk}e^{M_k^*} - \bar{r}_j - \sum_{\substack{l=1\\ l\neq j}}^{M}\bar{e}_{jl}e^{K_l^1}}{\bar{e}_{jj}}\right) \leq v_j(p_j). \tag{3.23}$$

From (2.1), (3.8) and (3.20), we obtain

$$K_j^* := S_j^* - 2\omega\sum_{k=1}^{N}\bar{d}_{jk}e^{M_k^1} \leq v_j(t). \tag{3.24}$$

From the above inequalities, for any $t \in \mathbb{I}_\omega$, we have

$$M_i^* \leq u_i(t) \leq M_i^1 \quad \forall i, \tag{3.25}$$

$$K_j^* \leq v_j(t) \leq K_j^1 \quad \forall j, \tag{3.26}$$

Therefore, we establish the bounds on u_j, v_j by using (3.25) and (3.26), which are as follows,

$$|u_i(t)| \leq \max_{t\in I_\omega}\left[\,|M_i^1|\,,\ |M_i^*|\,\right] := H_i^1, \tag{3.27}$$

$$|v_j(t)| \leq \max_{t\in I_\omega}\left[\,|K_j^1|\,,\ |K_j^*|\,\right] := H_j^2, \tag{3.28}$$

where H_i^1, H_j^2 $i = 1, 2, \cdots, N$ $j = 1, 2, \cdots, M$ in $(3.27), (3.28)$ are independent of λ.

Now let us consider the following algebraic equations

$$\begin{pmatrix} \vdots \\ \bar{b}_i - \bar{a}_{ii}e^{u_i} - \sum_{\substack{k=1\\ k\neq i}}^{N}\mu\bar{a}_{ik}e^{u_k} - \sum_{l=1}^{M}\mu\bar{c}_{il}e^{v_l} \\ \vdots \\ -\bar{r}_j - \bar{e}_{jj}e^{v_j} - \sum_{\substack{l=1\\ l\neq j}}^{M}\bar{e}_{jl}e^{v_l} + \sum_{k=1}^{N}\bar{d}_{jk}e^{u_k} \\ \vdots \end{pmatrix}_{(N+M)\times 1} = \bar{0}, \tag{3.29}$$

for $(u_1, u_2, \cdots, u_N, v_1, \cdots, v_M)^T \in \mathbb{R}^{N+M}$, where $\mu \in [0, 1]$ is a parameter. The equation is obtained by replace the coefficients $a_{ik}(t)$ and $c_{il}(t)$ by $\mu a_{ik}(t)$ and $\mu c_{il}(t)$ $\forall t \in \mathbb{I}_\omega$ respectively in (3.5). By carrying out similar arguments as above, it is not difficult to show that any solution $(u_1', u_2', \cdots, u_N', v_1', \cdots, v_M')^T \in \mathbb{R}^{N+M}$ of (3.29) with $\mu \in [0,1]$ satisfies

$$|u_i'| \le \max_{t\in I_\omega} \left[\, |M_i^1|\,,\ |M_i^*|\,\right] := H_i^1 \ \text{ and } \ |v_j'| \le \max_{t\in I_\omega} \left[\, |K_j^1|\,,\ |K_j^*|\,\right] := H_j^2.$$

Now, we define an open bounded subset Ω of X:

$$\Omega = \left\{\ U(t) \in X\ :\ \|U(t)\| = \|(u_1(t), u_2(t), \cdots, u_N(t), v_1(t), \cdots, v_M(t))^T\| < W\ \right\} \quad (3.30)$$

with $W = \sum_{i=1}^{N} H_i^1 + \sum_{j=1}^{M} H_j^2 + H$, where H is taken sufficient large such that $H > \sum_{i=1}^{N} |L_i^1| + \sum_{i=1}^{N} |L_i^*| + \sum_{j=1}^{M} |S_j^1| + \sum_{j=1}^{M} |S_j^*|$. It is clear that Ω satisfies the condition (1) of Lemma 2.15. If $U(t) \in \partial\Omega \bigcap$ KerL=$\partial\Omega \bigcap \mathbb{R}^{N+M}$ i.e., $U(t) = (u_1, u_2, \cdots, v_M)^T \in \mathbb{R}^{N+M}$(a constant vectors) with $\|U(t)\| = \|(u_1, u_2, \cdots, u_N, v_1, \cdots, v_M)^T\| = W$, then from (3.29) and definition of W, we obtain

$$\begin{pmatrix} \vdots \\ \bar{b}_i - \bar{a}_{ii}e^{u_i} - \sum\limits_{\substack{k=1\\k\neq i}}^{N} \mu\bar{a}_{ik}e^{u_k} - \sum\limits_{l=1}^{M} \mu\bar{c}_{il}e^{v_l} \\ \vdots \\ -\bar{r}_j - \bar{e}_{jj}e^{v_j} - \sum\limits_{\substack{l=1\\l\neq j}}^{M} \bar{e}_{jl}e^{v_l} + \sum\limits_{k=1}^{N} \bar{d}_{jk}e^{u_k} \\ \vdots \end{pmatrix}_{(N+M)\times 1} \neq \ \bar{0}, \quad (3.31)$$

where $\forall \mu \in [0,1], \bar{0} \in \mathbb{R}^{N+M}$ and at $\mu = 1,\ QN(U(t)) \neq 0$ on $\partial\Omega \bigcap KerL$. It is suffices to show that the Brouwer degree i.e., $\deg[JQN; \Omega \bigcap KerL; \bar{0}] \neq 0$. For $\mu \in [0,1]$, define a Homotopy H_μ such that $H_\mu : DomL \to \mathbb{R}^{N+M}$ by

$$H_\mu(U(t)) = \begin{pmatrix} \vdots \\ \bar{b}_i - \frac{1}{\omega}\int_{I_\omega} a_{ii}(t)e^{u_i(t)}\Delta t \\ \vdots \\ -\bar{r}_j - \frac{1}{\omega}\int_{I_\omega} e_{jj}(t)e^{v_j(t)}\Delta t - \frac{1}{\omega}\int_{I_\omega} \sum\limits_{\substack{l=1\\l\neq j}}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}\Delta t + \frac{1}{\omega}\int_{I_\omega} \sum\limits_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}\Delta t \\ \vdots \end{pmatrix}$$

$$+\mu\begin{pmatrix} \vdots \\ -\frac{1}{\omega}\int_{I_\omega}\sum_{\substack{k=1\\k\neq i}}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})}\Delta t - \frac{1}{\omega}\int_{I_\omega}\sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}\Delta t \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{N+M}. \quad (3.32)$$

From (3.31) and (3.32), we obtain

$H_\mu(U(t)) \neq \bar{0}$ on $\partial\Omega \bigcap KerL$ i.e., $\bar{0} \notin H_\mu(\partial\Omega \bigcap KerL)$ $\forall \mu \in [0,1]$ with $\|U(t)\| = W$.

For $U(t) \in \Omega \bigcap KerL$. From (3.32) at $\mu = 0$ and (3.29), we obtain

$$H_0(U(t)) = \begin{pmatrix} \vdots \\ \bar{b}_i - \bar{a}_{ii}e^{u_i} \\ \vdots \\ -\bar{r}_j - \bar{e}_{jj}e^{v_j} - \sum_{\substack{l=1\\l\neq j}}^{M} \bar{e}_{jl}e^{v_l} + \sum_{k=1}^{N} \bar{d}_{jk}e^{u_k} \\ \vdots \end{pmatrix}_{(N+M)\times 1} = \bar{0}, \quad (3.33)$$

which further implies

$$\bar{b}_i - \bar{a}_{ii}e^{u_i} = 0 \quad \forall i = 1, 2, \cdots, N,$$

which is equivalent to

$$u_i^* = \log\left(\frac{\bar{b}_i}{\bar{a}_{ii}}\right), \ \forall\ i = 1, 2, \cdots, N.$$

Now for $j = 1, 2, \cdots, M$, and using (3.33), we have

$$\begin{pmatrix} \vdots \\ -\bar{r}_j - \bar{e}_{jj}e^{v_j} - \sum_{\substack{l=1\\l\neq j}}^{M} \bar{e}_{jl}e^{v_l} + \sum_{k=1}^{N} \bar{d}_{jk}e^{u_k} \\ \vdots \end{pmatrix}_{M\times 1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{M\times 1}.$$

The above relation can also be written as

$$BV_e = \mathfrak{B}, \tag{3.34}$$

where the matrix B is defined in (2.2). Now from condition $(A3)$ in (2.1) and (3.34), we easily obtain that the system (3.34) has a unique solution. Let it call $V^* \in \Omega \bigcap KerL$. Choose the isomorphism J to be the identity mapping, by a direct computation and the invariance property of homotopy, we have

$$\begin{aligned}\deg[JQN'; \Omega \bigcap \text{KerL}; \bar{0}] &= \deg[QN'; \Omega \bigcap KerL; \bar{0}] \\ &= \deg[H_1; \Omega \bigcap KerL; \bar{0}] \\ &= \deg[H_0; \Omega \bigcap KerL; \bar{0}] \\ &\neq 0.\end{aligned}$$

Since $\mathbb{J}_{H_0}(V^*) = (-1)^{N+M}.\bar{a}_{11}.\bar{a}_{22} \ldots \bar{a}_{NN}.e^{u_1^*+u_2^* \cdots + u_N^* + v_1^* \cdots + v_M^*}.|B| \neq 0$ for $V^* \in \Omega \bigcap$ KerL and sign $\mathbb{J}_{H_0}(V^*)$ is either positive or negative. Therefore, $\deg[H_0 \ ; \ \Omega \bigcap \text{KerL} \ ; \ \bar{0}] \neq 0$. Thus, we have shown that Ω satisfies all the conditions of Lemma 2.15. Hence our system (1.5) has at least one ω-periodic solution in $Dom(L) \bigcap \bar{\Omega}$. □

We have the following corollary of the above theorem.

Corollary 3.2 — Assume that (A3)-(A5) hold. Then, system (1.5) has at least one ω-periodic solution.

PROOF : By observation the proof of above Theorem 3.1 and from (2.3), (2.4) and (2.5), we can easily find the appropriate bounds such that

$$\mathfrak{M}_i^* \leq u_i(t) \leq \mathfrak{M}_i^1 \text{ and } \mathfrak{R}_j^* \leq v_j(t) \leq \mathfrak{R}_j^1 \tag{3.35}$$

where $i = 1, 2, \cdots, N, \ \ j = 1, 2, \cdots, M$, and observe the equations from (3.25) to (3.30), we are able to find a set $\Omega \subset X$ which satisfies all the conditions of Lemma 2.5. Thus, the system (1.5) has at least one ω-periodic solution. □

4. EXAMPLE

In this section, we construct an example and perform the numerical simulation to illustrate the practicability and effectiveness of our Theorem 3.1.

Let us consider the following system of one-prey and two predators on an arbitrary time scale $\mathbb{T}$.

Example 4.1 :

$$u_1^{\Delta}(t) = 0.040 - 0.0027e^{u_1(t)} - (0.0000012)e^{v_1(t-1)} - 0.00000045e^{v_2(t-1)}$$

$$v_1^{\Delta}(t) = -0.0069 + 0.0033e^{u_1(t-1)} - 0.000027e^{v_1(t)} - 0.00000033e^{v_2(t-1)} \tag{4.1}$$

$$v_2^{\Delta}(t) = -0.0069 + 0.0033e^{u_1(t-1)} - 0.0000009e^{v_1(t-1)} - 0.0000098e^{v_2(t)}.$$

Solution : In view of Theorem 3.1, and $\omega = 2\pi$, we choose the parameters such that:

$$\bar{b}_1 = \frac{1}{\omega}\int_{I_\omega} b_1(t)\,\Delta t = 0.040, \bar{r}_1 = \bar{r}_2 = 0.0069,\ \bar{a}_{11} = 0.0027,\ \bar{d}_{11} = \bar{d}_{21} = 0.0033, \bar{c}_{11}$$
$$= 0.0000012, \bar{c}_{12} = 0.00000045\ \bar{e}_{11} = 0.000027,\ \bar{e}_{22} = 0.0000098,\ \bar{e}_{12} = 0.00000033,$$
$$\bar{e}_{21} = 0.0000009,$$

$$|B| = \begin{vmatrix} \bar{e}_{11} & \bar{e}_{12} \\ \bar{e}_{21} & \bar{e}_{22} \end{vmatrix} \neq 0 \text{ and } BV = \mathfrak{B} \text{ i.e., } \begin{pmatrix} 0.000027 & 0.00000033 \\ 0.0000009 & 0.0000098 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0.04198889 \\ 0.04198889 \end{pmatrix} \Rightarrow V = B^{-1}\mathfrak{B} \text{ has a unique positive solution. Thus, we conclude that}$$

$$e^{M_1^1} := \left(\frac{\bar{b}_1 e^{2\bar{b}_1\omega}}{\bar{a}_{11}}\right) \approx 24.491, e^{K_1^1} := \left(\frac{e^{M_1^1}\bar{d}_{11}e^{\ 2\omega\bar{d}_{11}e^{M_1^1}}}{\bar{e}_{11}}\right) \approx 8,264.8291,$$
$$e^{K_2^1} := \left(\frac{e^{M_1^1}\bar{d}_{21}e^{\ 2\omega\bar{d}_{21}e^{M_1^1}}}{\bar{e}_{22}}\right) \approx 22,770.4497, e^{M_1^*} := \left(\frac{e^{-2\bar{b}_1\omega}\left(\bar{b}_1 - \bar{c}_{11}e^{k_1^1} - \bar{c}_{12}e^{k_2^1}\right)}{\bar{a}_{11}}\right)$$
$$\approx 4.44405286,$$

and

(1) $\bar{c}_{11}e^{K_1^1} + \bar{c}_{12}e^{K_2^1} \approx 0.02016 < 0.040 = \bar{b}_1;$

(2) $\bar{r}_1 \approx 0.0069 < 0.007151 \approx \bar{d}_{11}e^{M_1^*} - \bar{e}_{12}e^{K_2^1};$

(3) $\bar{r}_2 \approx 0.0069 < 0.007227 \approx \bar{d}_{21}e^{M_1^*} - \bar{e}_{21}e^{K_1^1}.$

Therefore, the system (4.1) satisfy the conditions $(A1) - (A3)$ of the Theorem 3.1. Thus it has ω-periodic solution. In particular, we may choose the time scale $\mathbb{T} = \mathbb{R}$, which is given below.

Example 4.2 : We consider the following system of differential equation

$$\begin{aligned} u_1'(t) &= \left(0.040 + \tfrac{sin(t)}{26}\right) - \left(0.0027 - \tfrac{cos(t)}{371}\right) e^{u_1(t)} - (0.0000012)e^{v_1(t-1)} - (0.00000045)e^{v_2(t-1)} \\ v_1'(t) &= -\left(0.0069 + \tfrac{sin(t)}{146}\right) + \left(0.0033 + \tfrac{cos(t)}{305}\right) e^{u_1(t-1)} - (0.000027)e^{v_1(t)} - (0.00000033)e^{v_2(t-1)} \\ v_2'(t) &= -\left(0.0069 + \tfrac{cos(t)}{146}\right) + \left(0.0033 + \tfrac{cos(t)}{305}\right) e^{u_1(t-1)} - \left(0.0000009 - \tfrac{sin(t)}{(1111112)}\right) e^{v_1(t-1)} \\ &\quad -(0.0000098)e^{v_2(t)}. \end{aligned} \tag{4.2}$$

Solution : We modify the above parameters for the time scale $\mathbb{T} = \mathbb{R}$ as follow

$$\begin{aligned} b_1(t) &= \left(0.040 + \frac{sin(t)}{26}\right), a_{11}(t) = \left(0.0027 - \frac{cos(t)}{371}\right), c_{11}(t) = 0.0000012, c_{12}(t) \\ &= 0.00000045; \\ r_1(t) &= \left(0.0069 + \frac{sin(t)}{146}\right), d_{11}(t) = \left(0.0033 + \frac{cos(t)}{305}\right),\ e_{11}(t) = 0.000027, e_{12}(t) = 0.00000033 \\ r_2(t) &= \left(0.0069 + \frac{cos(t)}{146}\right), d_{21}(t) = \left(0.0033 + \frac{cos(t)}{305}\right),\ e_{21}(t) = \left(0.0000009 - \frac{sin(t)}{(1111112)}\right), \\ e_{22}(t) &= 0.0000098. \end{aligned}$$

Thus, the average value of all the coefficients in Example 4.1 are same as the average value of the above coefficients for time scale $\mathbb{R}$. therefore we obtain that (4.2) satisfies $(A1)-(A3)$. Therefore, it has $2\pi-$ periodic solution. The numerical results are illustrated in Figs. 1, 2 and 3, which clearly shows the periodicity of the solution of the system (4.2) with initial values $u_1(0) = 2.6660$, $v_1(0) = 7.2350$, $v_2(0) = 8.2420$ respectively as follows:

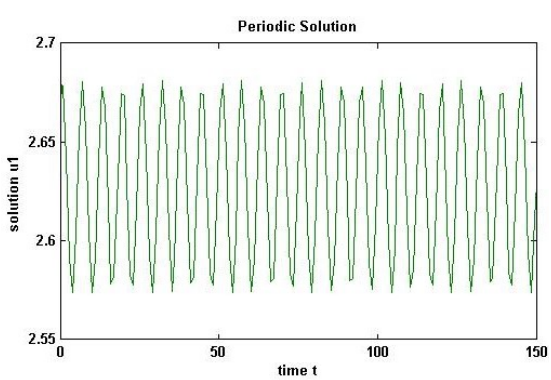

Figure 1: State variable u_1 of system (4.2).

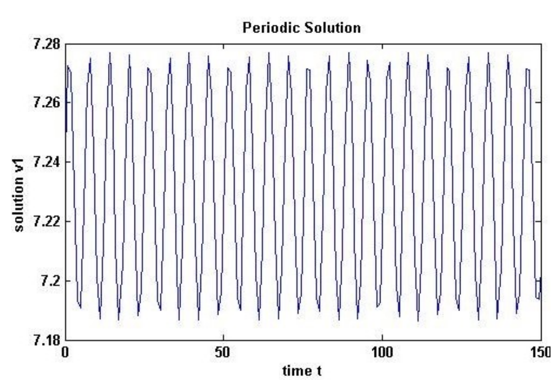

Figure 2: State variable v_1 of system (4.2).

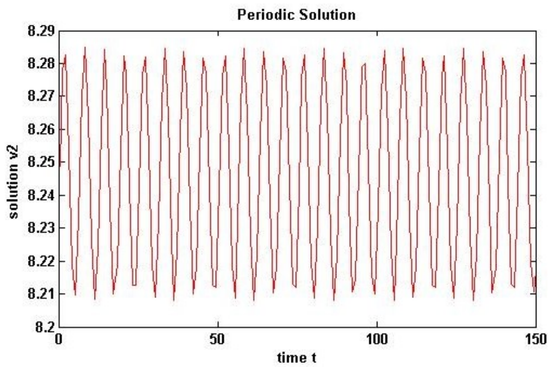

Figure 3: State variable v_2 of system (4.2).

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