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THE GEOMETRIC PROPERTIES OF A CLASS OF NONSYMMETRIC CONES

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Geometric methods are important for researching the differential properties of metric projectors, sensitivity analysis, and the augmented Lagrangian algorithm. Sun [3] researched the relationship among the strong second-order sufficient condition, constraint nondegeneracy, B-subdifferential nonsingularity of the KKT system, and the strong regularity of KKT points in investigating nonlinear semidefinite programming problems. Geometric properties of cones are necessary in studying second-order sufficient condition and constraint nondegeneracy. In this paper, we study the geometric properties of a class of nonsymmetric cones, which is widely applied in optimization problems subjected to the epigraph of vector k-norm functions and low-rank-matrix approximations. We compute the polar, the tangent cone, the linear space of the tangent cone, the critical cone, and the affine hull of this critical cone. This paper will support future research into the sensitivity and algorithms of related optimization problems.

Key words : Critical cone; geometric properties; nonsymmetric cone; tangent cone.

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1. Introduction

The geometric properties of convex closed cones play an important role in theories and algorithms for optimization problems. In particular, the tangent and critical cones are widely used in determining the differential properties of metric projectors and in sensitivity analysis. Moreover, they are necessary for the design of the augmented Lagrangian algorithm. In this paper, we study the geometric properties of a class of cones: the intersection of a closed half-space and a variable box, denoted

$$\mathcal{C} := \{(y,\tau) \in \mathcal{R}^n \times \mathcal{R} : 0 \leq y \leq \tau e, e^T y \leq \kappa\tau\},$$

where $\kappa > 0$ and e denotes the vector of which all elements are 1's. $\mathcal{C}$ appears widely in optimization problems related to the vector k-norm functions and the matrix Ky Fan k-norm functions [1, 2]. For each $(y, \tau) \in \mathcal{C}$, it is clear that (y, τ) satisfies $0 \leq y \leq \tau e$, then the constraint $e^T y \leq \kappa\tau$ is not binding if $\kappa \geq n$. In this case, $\mathcal{C}$ degenerates into $\{(y, \tau) \in \mathcal{R}^n \times \mathcal{R} : 0 \leq y \leq \tau e\}$. In this paper, we restrict $0 < \kappa < n$.

There are two motivations of our work. One is the fact that geometric properties are associated with the sensitivity analysis of optimization problems. For nonlinear semidefinite programming problems, Sun researched the relationship among the strong second-order sufficient condition, constraint nondegeneracy, B-subdifferential nonsingularity of the KKT system, and the strong regularity of KKT points [3]. Hence, geometric properties, including the tangent cone, its linear space, the critical cone, and its affine hull, are necessary for sensitivity analysis. In our future work, we will discuss sensitivity analysis of the problem

$$min\{f(z) : G(z) \in \mathcal{C}\} \tag{1}$$

where $f : \mathcal{R}^m \to \mathcal{R}$ and $G : \mathcal{R}^m \to \mathcal{R}^n \times \mathcal{R}$ are twice continuously differentiable. Since $\mathcal{C}$ is a nonsymmetric cone, it is not easy to deal with the optimization problem related to $\mathcal{C}$. In order to support further research into differential properties of the metric projector and sensitivity analysis of problem (1.1), we focus upon the geometric properties of $\mathcal{C}$ in this paper. The tangent cone and the critical cone will be beneficial to the B-subdifferential of the KKT system. The linear space of the tangent cone and the affine hull of the critical cone will promote research in the strong second-order sufficient condition and constraint nondegeneracy.

Our other motivation is the numerous applications of the problem

$$\min\left\{\frac{1}{2}(\|y - x\|^2 + (\tau - t)^2) : (y, \tau) \in \mathcal{C}\right\}, \tag{2}$$

where the point $(x, t) \in \mathcal{R}^n \times \mathcal{R}$ is given. This is a convex optimization problem. We know that the unique optimal solution is actually the projection over $\mathcal{C}$, denoted by $\Pi_{\mathcal{C}}(x, t)$. Problem of this type may appear in optimizations subject to the epigraph of the vector k-norm functions and the matrix Ky Fan k-norm functions or when finding the fastest mixing Markov chain on a graph [1, 2] or the structured low-rank-matrix approximation [4]. For more applications, one can see [5]. There have been many results when computing the metric projector over $\mathcal{C}$ (such as [6-9]); Liu, Wang and Sun [9] obtained the closed form of this projector. The study in this paper is carried out based on [9].

The rest of this paper is organised as follows. Section 2 introduces some preliminaries necessary for our study. Sections 3-5 focus on the geometric properties related to $\mathcal{C}$. Specifically, Section 4 is devoted to the dual and polar of the cone, Section 6 focuses on the tangent cone and its linear space, and Section 5 is dedicated to the critical cone associated with Problem (1.2) and its affine hull. We present the final conclusions of this paper in Section 6.

2. Symbols, Concepts and Preliminaries

In this section, we shall introduce some relevant concepts.

2.1 *Metric projector of the cone*

The metric projector has been computed by Proposition 2.2 and Algorithm 3.1 in [9]. In this section, we briefly introduce the main results. For a point $z \in \mathcal{R}^n$, we use $z^{\downarrow}$ to denote the rearrangement of z satisfying $z^{\downarrow}_1 \geq z^{\downarrow}_2 \geq \cdots \geq z^{\downarrow}_n$. For given $(x,t) \in \mathcal{R}^n \times \mathcal{R}$, let $\sigma^* \geq 0$ be the parameter relative to (x,t). And let m be the cardinal number of the set $\{x_i : x_i > \sigma^*\}$ and λ be the smallest integer $i \in \{0,1,\ldots,m-1\}$ such that $x^{\downarrow}_{i+1} - \sigma^* < \dfrac{t + \sum_{k=1}^{i} x^{\downarrow}_k + (\kappa - i)\sigma^*}{1+i} \leq x^{\downarrow}_i - \sigma^*$. If this inequality is not satisfied, let $\lambda = m$. Denote

$$\theta(x,t,\sigma^*) = \frac{t + \sum_{i=1}^{\lambda} x^{\downarrow}_i + (\kappa - \lambda)\sigma^*}{1+\lambda}.$$

Then the metric projector $\Pi_{\mathcal{C}}$ at (x,t) can be computed as $(\bar{x},\bar{t})$:

$$\bar{t} = \max\{0, \theta(x,t,\sigma^*)\}, \tag{3}$$

and for $i = 1,\cdots,n$,

$$\bar{x}_i = \begin{cases} 0, & x_i \leq \sigma^*, \\ x_i - \sigma^*, & \sigma^* < x_i < \bar{t} + \sigma^*, \\ \bar{t}, & x_i \geq \bar{t} + \sigma^*. \end{cases} \tag{4}$$

For the sake of simplicity of description, we denote

$$\alpha := \{i : x_i \leq \sigma^*\}, \gamma := \{i : x_i \geq \bar{t} + \sigma^*\}, \beta := \{1,2,\cdots,n\} \backslash (\alpha \cup \beta) \tag{5}$$

and

$$\alpha^{=} := \{i \in \alpha : x_i = \sigma^*\}, \ \ \alpha^{\neq} := \alpha \backslash \alpha^{=}, \ \ \gamma^{=} := \{i \in \gamma : x_i = \bar{t} + \sigma^*\}, \ \ \gamma^{\neq} := \gamma \backslash \gamma^{=}. \tag{6}$$

2.2 *Some concepts*

In this paper, we perform our work on the finite dimensional Euclidean space $\mathcal{R}^n \times \mathcal{R}$. Let $\mathcal{S}$ be a given closed convex cone in $\mathcal{R}^n \times \mathcal{R}$. We denote the interior of $\mathcal{S}$ by $int(\mathcal{S})$ and the boundary of $\mathcal{S}$ by $bd(\mathcal{S})$. The linearity space and the affine hull can be defined, respectively, by

$$lin(\mathcal{S}) := \mathcal{S} \cap (-\mathcal{S}), \quad aff(\mathcal{S}) := \mathcal{S} - \mathcal{S}. \tag{7}$$

The dual $\mathcal{S}^*$ is defined by

$$\mathcal{S}^* := \{(s, s_0) \in \mathcal{R}^n \times \mathcal{R} : \langle (s, s_0), (\xi, \xi_0) \rangle \geq 0, \forall (\xi, \xi_0) \in \mathcal{S}\},$$

and the polar $\mathcal{S}^o$ is defined by

$$\mathcal{S}^o := \{(s, s_0) \in \mathcal{R}^n \times \mathcal{R} : \langle (s, s_0), (\xi, \xi_0) \rangle \leq 0, \forall (\xi, \xi_0) \in \mathcal{S}\}.$$

We know that $\mathcal{S}^* = -\mathcal{S}^o$. Hence, it is enough to determine one of them.

Now, let us characterize the tangent cone of $\mathcal{S}$. We write the tangent cone of $\mathcal{S}$ at $(s, s_0) \in \mathcal{S}$ as $T_{\mathcal{S}}(s, s_0)$, i.e.,

$$T_{\mathcal{S}}(s, s_0) = \{(h, h_0) \in \mathcal{R}^n \times \mathcal{R} : dist\big((s, s_0) + l(h, h_0), \mathcal{S}\big) = o(l), l \geq 0\},$$

where "$dist$" denotes the distance of a point and a closed convex set. If $\mathcal{S}$ is the epigraph of a positively homogeneous convex function $\mathcal{F} : \mathcal{R}^n \to \mathcal{R}$, that is, there is a positively homogeneous convex function $\mathcal{F} : \mathcal{R}^n \to \mathcal{R}$ such that $\mathcal{S}$ can be written as

$$\mathcal{S} = \{(s, s_0) \in \mathcal{R}^n \times \mathcal{R} : \mathcal{F}(s) \leq s_0\}, \tag{8}$$

following Theorem 2.4.9 in [10], for $(s, s_0) \in bd(\mathcal{S})$, the tangent cone of $\mathcal{S}$ at (s, s_0) can be computed as

$$T_{\mathcal{S}}(s, s_0) := \{(h, h_0) \in \mathcal{R}^n \times \mathcal{R} : \mathcal{F}'(s; h) \leq h_0\}, \tag{9}$$

where $\mathcal{F}'(s; h)$ is the directional derivative of $\mathcal{F}$ at point s along direction h, i.e.

$$\mathcal{F}'(s; h) = \lim_{l \downarrow 0} \frac{\mathcal{F}(s + lh) - \mathcal{F}(s)}{l}.$$

Many sets can be written as (2.8). For example, the epigraph of k-norm function. The set $\mathcal{C}$ is also can be written as (2.8) (see Theorem 4.1).

The critical cone of the general optimization problem (1.1) at the optimal solution $\bar{z}$ is defined by

$$\bar{\mathcal{C}} := \{\eta \in \mathcal{R}^n \times \mathcal{R} : G'(\bar{z})\eta \in T_{\mathcal{C}}(G(\bar{z})), f'(\bar{z}; \eta) \leq 0\}.$$

It is easy to see that, for given point $(x, t) \in \mathcal{R}^n \times \mathcal{R}$, the critical cone at the optimal solution $(\bar{x}, \bar{t})$ of Problem (1.2) (that is, $(\bar{x}, \bar{t}) = \Pi_{\mathcal{C}}(x, t)$) can be computed by

$$\bar{\mathcal{C}} = T_{\mathcal{C}}(\bar{x}, \bar{t}) \cap \big((x, t) - (\bar{x}, \bar{t})\big)^{\perp}. \tag{10}$$

2.3 *Symbols*

For $x \in \mathcal{R}^n$ and $Q \subseteq \{1, 2, \cdots, n\}$, we use $|Q|$ to denote the cardinality of Q and use x_Q to denote the sub-vector of x formed by removing all components of x not in Q. e denotes the vector with all entries being ”1”, and its dimension is dependent upon the specific situation. For a given point $r \in \mathcal{R}$, we denote $r_- := \min\{r, 0\}$ and $r_+ := \max\{r, 0\}$.

3. The Dual and Polar of the Cone

3.1 *Symbold*

For convenience of description, let $\lfloor \kappa \rfloor$ be the largest integer number which is less than κ, and define

$$\Omega = \{x \in \mathcal{R}^n : 0 \leq x \leq e, e^T x \leq \kappa\}.$$

For a given point $x \in \mathcal{R}^n$, let $x^\uparrow$ and $x^\downarrow$ denote the rearrangement of x satisfying $x_1^\uparrow \leq x_2^\uparrow \leq \cdots \leq x_n^\uparrow$ and $x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_n^\downarrow$, respectively. For notational convenience, let

$$\begin{aligned}
I_= &= \{i : x_i = (x^\uparrow_{\lfloor \kappa \rfloor + 1})_-\}, \quad J_= = \{i : x_i = (x^\downarrow_{\lfloor \kappa \rfloor + 1})_+\},\\
I_< &= \{i : x_i < (x^\uparrow_{\lfloor \kappa \rfloor + 1})_-\}, \quad J_< = \{i : x_i < (x^\downarrow_{\lfloor \kappa \rfloor + 1})_+\},\\
I_> &= \{i : x_i > (x^\uparrow_{\lfloor \kappa \rfloor + 1})_-\}, \quad J_> = \{i : x_i > (x^\downarrow_{\lfloor \kappa \rfloor + 1})_+\}.
\end{aligned}$$

3.2 *Dual of the cone*

Lemma 3.1 — For a given point $x \in \mathcal{R}^n$, denote the optimal solution set of the optimization problem

$$\min\{x^T z : z \in \Omega\} \tag{11}$$

by $SOL_\Omega(x)$. Then for $z^* \in SOL_\Omega(x)$, we have

(a) $i \in I_<$, then $z_i^* = 1$;

(b) $i \in I_>$, then $z_i^* = 0$;

(c) $i \in I_=$, then $0 \leq z_i^* \leq 1$ and satisfies: $\sum_{i \in I_=} z_i^* = \kappa - |I_<|$ if $x^\uparrow_{\lfloor \kappa \rfloor + 1} < 0$; $\sum_{i \in I_=} z_i^* \leq \kappa - |I_<|$ if $x^\uparrow_{\lfloor \kappa \rfloor + 1} \geq 0$.

That is,

$$SOL_\Omega(x) = \begin{cases} \{z^* : z^*_{I_<} = e; z^*_{I_>} = 0; 0 \le z^*_{I_=} \le e; \sum_{i \in I_=} z^*_i \le \kappa - |I_<|\}, & x^\uparrow_{\lfloor\kappa\rfloor+1} \ge 0 \\ \{z^* : z^*_{I_<} = e; z^*_{I_>} = 0; 0 \le z^*_{I_=} \le e; \sum_{i \in I_=} z^*_i = \kappa - |I_<|\}, & x^\uparrow_{\lfloor\kappa\rfloor+1} < 0. \end{cases} \quad (12)$$

PROOF : We need to describe the optimal solutions of $min\{x^T z : z \in \Omega\}$. If $x^\uparrow_{\lfloor\kappa\rfloor+1} \ge 0$, we have $(x^\uparrow_{\lfloor\kappa\rfloor+1})_- = 0$. In this case, we have

$$I_< = \{i : x^\uparrow_i < 0\}, I_= = \{i : x^\uparrow_i = 0\}, I_> = \{i : x^\uparrow_i > 0\},$$

and $|I_<| \le \lfloor\kappa\rfloor$. Let z^* be a solution of the problem $min\{x^T z : z \in \Omega\}$. Then z^* obviously satisfies

$$z^*_{I_<} = e, z^*_{I_>} = 0, 0 \le z^*_{I_=} \le e, \sum_{i \in I_=} z^*_i \le \kappa - |I_<|.$$

Similarly, if $x^\uparrow_{\lfloor\kappa\rfloor+1} < 0$, we have $|I_<| \le \lfloor\kappa\rfloor \le |\{i \in [1 : n] : x_i < 0\}|$ and $\left(x^\uparrow_{\lfloor\kappa\rfloor+1}\right)_- = x^\uparrow_{\lfloor\kappa\rfloor+1}$. Then for any $z^* \in SOL_\Omega(x)$, z^* satisfies

$$z^*_{I_<} = e, z^*_{I_>} = 0, 0 \le z^*_{I_=} \le e, \sum_{i \in I_=} z^*_i = \kappa - |I_<|.$$

which means that (3.3) holds. □

***Theorem* 3.1** — *Let $SOL_\Omega(x)$ be the optimal solution set of (3.12). The dual of cone $\mathcal{C}$ can be computed as*

$$\begin{aligned} \mathcal{C}^* &= \{(x,t) \in \mathcal{R}^n \times \mathcal{R} : t \ge 0, x^T z^* + t \ge 0, z^* \in SOL_\Omega(x)\} \\ &= \{(x,t) \in \mathcal{R}^n \times \mathcal{R} : t \ge 0, \textstyle\sum_{i \in I_<} x_i + (\kappa - |I_<|)(x^\uparrow_{\lfloor\kappa\rfloor+1})_- + t \ge 0\}; \end{aligned} \quad (13)$$

PROOF : Let $\mathcal{B}$ denote the right-hand side of the first equality in (3.14), i.e.,

$$\mathcal{B} = \{(x,t) \in \mathcal{R}^n \times \mathcal{R} : t \ge 0, x^T z^* + t \ge 0, z^* \in SOL_\Omega(x)\}. \quad (14)$$

We need to show that:

(i) $\mathcal{B} = \mathcal{C}^*$;

(ii) $\mathcal{B} = \{(x,t) \in \mathcal{R}^n \times \mathcal{R} : t \ge 0, \sum_{i \in I_<} x_i + (\kappa - |I_<|)(x^\uparrow_{\lfloor\kappa\rfloor+1})_- + t \ge 0\}$.

PROOF OF (I) : First, we will prove $\mathcal{C}^* \subseteq \mathcal{B}$. Let $(x,t) \in \mathcal{C}^*$ be given. We obtain from the definition of dual cone that

$$0 \leq \langle (x,t),(y,\tau)\rangle = \langle x,y\rangle + t\tau \tag{15}$$

for any $(y,\tau) \in \mathcal{C}$. Then, it is obvious that $t \geq 0$ since $(0,1) \in \mathcal{C}$. Since $(y,\tau) \in \mathcal{C}$, we have $\tau \geq 0$. If $\tau = 0$, then $y = 0$ by the definition of $\mathcal{C}$, and (3.16) holds for all $(x,t) \in \mathcal{R}^n \times \mathcal{R}$. We now consider the condition $\tau > 0$. From the inequality (3.16), it holds that $-t \leq (\frac{1}{\tau})x^T y$, which implies

$$-t \leq \min\{x^T z : z = (\frac{1}{\tau})y, \tau > 0, (y,\tau) \in \mathcal{C}\} = \min\{x^T z : z \in \Omega\}.$$

This implies that $-t \leq x^T z^*$ where $z^* \in argmin\{x^T z : z \in \Omega\}$. That is, $x^T z^* + t \geq 0, z^* \in SOL_\Omega(x)$. Hence we have $\mathcal{C}^* \subseteq \mathcal{B}$.

Now, we will show that $\mathcal{B} \subseteq \mathcal{C}^*$. For each $(x,t) \in \mathcal{B}$ and any $(y,\tau) \in \mathcal{C}$, we will show that the inequality (3.16) holds. In fact, if $\tau = 0$, then $y = 0$, and $x^T y + t\tau = 0$, which implies (3.16). If $\tau > 0$, then $x^T y + t\tau = \tau(x^T(\frac{y}{\tau}) + t) \geq \tau(\min\{x^T z : z \in \Omega\} + t) \geq 0$, this implies that (3.16) holds, i.e., $(x,t) \in \mathcal{C}^*$, which means $\mathcal{B} \subseteq \mathcal{C}^*$. Thus, we have proved that $\mathcal{C}^* = \mathcal{B}$.

PROOF OF (II) : Combining (3.13) and (3.15), we investigate the inequality $x^T z^* + t \geq 0$. If $x^{\uparrow}_{\lfloor\kappa\rfloor+1} \geq 0$, we have $(x^{\uparrow}_{\lfloor\kappa\rfloor+1})_- = 0$. Then, $0 \leq x^T z^* + t = \sum_{i\in I_<} x_i z_i^* + \sum_{i\in I_>} x_i z_i^* + \sum_{i\in I_=} x_i z_i^* + t = \sum_{i\in I_<} x_i + 0 + (x^{\uparrow}_{\lfloor\kappa\rfloor+1})_- \sum_{i\in I_=} z_i^* + t = \sum_{i\in I_<} x_i + t = \sum_{i\in I_<} x_i + 0(\kappa - |I|) + t = \sum_{i\in I_<} x_i + (x^{\uparrow}_{\lfloor\kappa\rfloor+1})_-(\kappa - |I|) + t$. If $x^{\uparrow}_{\lfloor\kappa\rfloor+1} < 0$, the deduction is similar. Then, we obtain the second equality of (3.14). The proof is completed. □

3.3 *Polar of the cone*

By a similar deduction to that used in Lemma 3.1, we can obtain the following result.

Lemma 3.2 — Let the optimal solution set of $\max_{z\in\Omega} x^T z$ be denoted by $sol_\Omega(x)$. We have, for $z^* \in sol_\Omega(x)$,

(a) $i \in J_>$, then $z_i^* = 1$;

(b) $i \in J_<$, then $z_i^* = 0$;

(c) $i \in J_=$, then $0 \leq z_i^* \leq 1$ and satisfies: $\sum_{i\in J_=} z_i^* = \kappa - |J_>|$ if $x^{\downarrow}_{\lfloor\kappa\rfloor+1} > 0$; $\sum_{i\in J_=} z_i^* \leq \kappa - |J_>|$ if $x^{\downarrow}_{\lfloor\kappa\rfloor+1} \leq 0$.

That is,

$$sol_\Omega(x) = \begin{cases} \{z^* : z^*_{J_>} = e; z^*_{J_<} = 0; 0 \leq z^*_{J_=} \leq e; \sum_{i\in J_=} z_i^* \leq \kappa - |J_>|\}, & x^{\downarrow}_{\lfloor\kappa\rfloor+1} \leq 0 \\ \{z^* : z^*_{J_>} = e; z^*_{J_<} = 0; 0 \leq z^*_{J_=} \leq e; \sum_{i\in J_=} z_i^* = \kappa - |J_>|\}, & x^{\uparrow}_{\lfloor\kappa\rfloor+1} > 0. \end{cases} \tag{16}$$

***Theorem* 3.2** — *Let $sol_{\Omega}(x)$ be defined as in Lemma 3.2. The polar of cone $\mathcal{C}$ can be computed as*

$$\begin{aligned} \mathcal{C}^o &= \{(x,t) \in \mathcal{R}^n \times \mathcal{R} : t \leq 0, x^T z^* + t \leq 0, z^* \in sol_{\Omega}(x)\} \\ &= \{(x,t) \in \mathcal{R}^n \times \mathcal{R} : t \leq 0, \textstyle\sum_{i \in J_>} x_i + (\kappa - |J_>|)(x^{\downarrow}_{\lfloor \kappa \rfloor + 1})_+ + t \leq 0\}. \end{aligned} \tag{17}$$

PROOF : Because $\mathcal{C}^o = -\mathcal{C}^*$, by examining (3.14) and (3.17), we can obtain the expression of $\mathcal{C}^o$ as (3.18). The proof is completed. □

4. TANGENT CONE AND ITS LINEARITY SPACE

***Theorem* 4.1** — *Let $(x,t) \in \mathcal{R}^n \times \mathcal{R}$ be a given point and $\Pi_{\mathcal{C}}(x,t) = (\bar{x}, \bar{t})$. α, β, γ are the index sets defined in (2.5). The tangent cone $T_{\mathcal{C}}(\cdot,\cdot) \subseteq \mathcal{R}^n \times \mathcal{R}$ of $\mathcal{C}$ at $(\bar{x}, \bar{t})$ can be computed as follows:*

(a) *if $(x,t) \in int(\mathcal{C})$, then $T_{\mathcal{C}}(\bar{x}, \bar{t}) = \mathcal{R}^n \times \mathcal{R}$;*

(b) *if $(x,t) \in \mathcal{C}^o$, then $T_{\mathcal{C}}(\bar{x}, \bar{t}) = \mathcal{C}$;*

(c) *Otherwise*

$$T_{\mathcal{C}}(\bar{x}, \bar{t}) = \begin{cases} \{(d, d_0) : -d_\alpha \leq 0, d_\gamma \leq d_0 e\}, & e^T \bar{x} \neq \kappa \bar{t}, \\ \{(d, d_0) : -d_\alpha \leq 0, d_\gamma \leq d_0 e, e^T d - \kappa d_0 \leq 0\}, & e^T \bar{x} = \kappa \bar{t}. \end{cases} \tag{18}$$

PROOF : It is clear that (a) and (b) hold. Let us consider the proof of (c). Let

$$f(y, \tau) = max\{-y, y - \tau e, e^T y - \kappa \tau\},$$

then $\mathcal{C}$ can be rewritten as

$$\mathcal{C} = \{(y,\tau) : f(y,\tau) \leq 0\}.$$

Following (9), we know that

$$T_{\mathcal{C}}(\bar{x}, \bar{t}) = \{(d, d_0) : f'\big((\bar{x}, \bar{t}); (d, d_0)\big) \leq 0\}.$$

We consider the situation $e^T \bar{x} \neq \kappa \bar{t}$. By (2.4), we know $-\bar{x} \leq 0$, $\bar{x} - \bar{t} e \leq 0$, $e^T \bar{x} - \kappa \bar{t} < 0$, $-\bar{x}_\alpha = 0$ and $\bar{x}_\gamma - \bar{t} e = 0$. Then we know $f(\bar{x}, \bar{t}) = 0$. For $(d, d_0) \in T_{\mathcal{C}}(\bar{x}, \bar{t})$, if l is sufficiently small, we have

$$\begin{aligned} 0 &\geq f'\big((\bar{x},\bar{t});(d,d_0)\big) \\ &= \lim_{l\downarrow 0}\frac{f(\bar{x}+ld,\bar{t}+ld_0)-f(\bar{x},\bar{t})}{l} \\ &= \lim_{l\downarrow 0}\frac{\max\{-\bar{x}-ld,\bar{x}+ld-\bar{t}e-ld_0e,e^T(\bar{x}+ld)-\kappa(\bar{t}+ld_0)\}}{l} \\ &= \lim_{l\downarrow 0}\frac{\max\{-ld_\alpha,ld_\gamma-ld_0e\}}{l} \\ &= \max\{-d_\alpha,d_\gamma-d_0e\}. \end{aligned}$$

That is, the first equality of (4.19) holds. If $e^T\bar{x}=\kappa\bar{t}$, the proof is similar, therefore, it is omitted. □

Theorem 4.2 — *Let $(x,t)\in\mathcal{R}^n\times\mathcal{R}$ be a given point and $\Pi_{\mathcal{C}}(x,t)=(\bar{x},\bar{t})$. α,β,γ are defined in (2.5). Then the linearity space $lin(T_{\mathcal{C}}(\bar{x},\bar{t}))\subseteq\mathcal{R}^n\times\mathcal{R}$ is given by*

(a) *if $(x,t)\in int(\mathcal{C})$, then $lin(T_{\mathcal{C}}(\bar{x},\bar{t}))=\mathcal{R}^n\times\mathcal{R}$;*

(b) *if $(x,t)\in\mathcal{C}^o$, then $lin(T_{\mathcal{C}}(\bar{x},\bar{t}))=\{(0,0)\}$;*

(c) *Otherwise*

$$lin(T_{\mathcal{C}}(\bar{x},\bar{t}))=\begin{cases}\{(d,d_0):d_\alpha=0,d_\gamma=d_0e\}, & e^T\bar{x}\neq\kappa\bar{t},\\ \{(d,d_0):d_\alpha=0,d_\gamma=d_0e,e^Td-\kappa d_0=0\}, & e^T\bar{x}=\kappa\bar{t}.\end{cases}$$

PROOF : We can obtain straightforwardly the proof using (2.7) and Theorem 4.1. □

5. Critical Cone and Its Affine Hull

Let $\bar{\mathcal{C}}$ stand for the critical cone of Problem (1.2). Using (2.4), (2.11), and Theorem 4.1, we can obtain the results concerning the critical cone $\bar{\mathcal{C}}$. Then, we can find its affine hull $aff(\bar{\mathcal{C}})$.

5.1 *The results about critical cone and its affine hull*

Theorem 5.1 — *Let $(x,t)\in\mathcal{R}^n\times\mathcal{R}$ be a given point and $\Pi_{\mathcal{C}}(x,t)=(\bar{x},\bar{t})$. α,β,γ are defined in (2.5) and $\alpha^=,\alpha^{\neq},\gamma^=,\gamma^{\neq}$ are defined in (2.6). The critical cone $\bar{\mathcal{C}}\subseteq\mathcal{R}^n\times\mathcal{R}$ of $\mathcal{C}$ at (x,t) associated with Problem (1.2) can be described by the following rule:*

(a) *if $(x,t)\in\mathcal{C}$, then $\bar{\mathcal{C}}=T_{\mathcal{C}}(\bar{x},\bar{t})$, specifically,*

(i) *when $(x,t)=(0,0)$, $\bar{\mathcal{C}}=\mathcal{C}$;*

(ii) *when* $(x,t)\in int(\mathcal{C})$, $\bar{\mathcal{C}}=\mathcal{R}^n\times\mathcal{R}$;

(iii) *when* $(x,t)\in bd(\mathcal{C})\setminus\{(0,0)\}$,

$$\bar{\mathcal{C}}=\begin{cases}\{(d,d_0): d_\alpha\geq 0, d_\gamma\leq d_0e\}, e^Tx\neq\kappa t.\\ \{(d,d_0): -d_\alpha\leq 0, d_\gamma\leq d_0e, e^Td-\kappa d_0\leq 0\}, e^Tx=\kappa t;\end{cases}$$

(b) *if* $(x,t)\in int(\mathcal{C}^o)$, *then* $\bar{\mathcal{C}}=\{(0,0)\}$;

(c) *if* $(x,t)\in bd(\mathcal{C}^o)\backslash\{(0,0)\}$, *then*

$$(x,t)\in bd(\mathcal{C}^o)\setminus\{(0,0)\}\Leftrightarrow\max\{x^Tz: z\in\Omega\}+t=0 \text{ and } t\leq 0, \tag{19}$$

and $\bar{\mathcal{C}}$ *is given by*

$$\begin{aligned}&\{(d,d_0): d_0\geq 0, \frac{d}{d_0}\in argmax_{z\in\Omega}x^Tz \text{ when } d_0>0\}\\ =&\begin{cases}\{(d,d_0): d_{\alpha^{\neq}}=0, d_{\gamma^{\neq}}=d_0e, 0\leq d_{\alpha^=}\leq d_0e, e^Td_{\alpha^=}\leq(\kappa-|\gamma^{\neq}|)d_0\}, & x^{\downarrow}_{\lfloor\kappa\rfloor+1}\leq 0\\ \{(d,d_0): d_{\alpha^{\neq}}=0, d_{\gamma^{\neq}}=d_0e, 0\leq d_{\alpha^=}\leq d_0e, e^Td_{\alpha^=}=(\kappa-|\gamma^{\neq}|)d_0\}, & x^{\downarrow}_{\lfloor\kappa\rfloor+1}>0.\end{cases}\end{aligned} \tag{20}$$

(d) *otherwise,* $\bar{\mathcal{C}}$ *is can be computed by*

$$\begin{cases}\{(d,d_0): \kappa d_0-e^Td=0, d_{\alpha^{\neq}}=0, d_{\gamma^{\neq}}=d_0e, d_{\alpha^=}\geq 0, d_{\gamma^=}\leq d_0e\}, \sigma^*\neq 0;\\ \{(d,d_0): d_{\alpha^{\neq}}=0, d_{\gamma^{\neq}}=d_0e, d_{\alpha^=}\geq 0, d_{\gamma^=}\leq d_0e\}, \sigma^*=0, e^T\bar{x}-\kappa\bar{t}<0;\\ \{(d,d_0): \kappa d_0-e^Td\geq 0, d_{\alpha^{\neq}}=0, d_{\gamma^{\neq}}=d_0e, d_{\alpha^=}\geq 0, d_{\gamma^=}\leq d_0e\}, \sigma^*=0, e^T\bar{x}-\kappa\bar{t}=0.\end{cases}$$

***Theorem* 5.2** — *Let* $(x,t)\in\mathcal{R}^n\times\mathcal{R}$ *be a given point and* $\Pi_{\mathcal{C}}(x,t)=(\bar{x},\bar{t})$. *The affine hull of* $\bar{\mathcal{C}}$, *i.e.,* $aff(\bar{\mathcal{C}})\subseteq\mathcal{R}^n\times\mathcal{R}$, *can be computed as follows:*

(a) *if* $(x,t)\in\mathcal{C}$, *then* $aff(\bar{\mathcal{C}})=\mathcal{R}^n\times\mathcal{R}$;

(b) *if* $(x,t)\in int(\mathcal{C}^o)$, *then* $aff(\bar{\mathcal{C}})=\{(0,0)\}$;

(c) *if* $(x,t)\in bd(\mathcal{C}^o)\backslash\{(0,0)\}$, *then* $aff(\bar{\mathcal{C}})$ *is given by*

$$aff(\bar{\mathcal{C}})=\begin{cases}\{(d,d_0)\in\mathcal{R}^n\times\mathcal{R}: d_{\alpha^{\neq}}=0; d_{\gamma^{\neq}}=d_0e\}, & x^{\downarrow}_{\lfloor\kappa\rfloor+1}\leq 0\\ \{(d,d_0)\in\mathcal{R}^n\times\mathcal{R}: d_{\alpha^{\neq}}=0; d_{\gamma^{\neq}}=d_0e; e^Td_{\alpha^=}=(\kappa-|\gamma^{\neq}|)d_0\}, & x^{\downarrow}_{\lfloor\kappa\rfloor+1}>0.\end{cases}$$

(d) *otherwise,*

$$aff(\bar{\mathcal{C}})=\begin{cases}\{(d,d_0): d_{\alpha^{\neq}}=0, d_{\gamma^{\neq}}=d_0e, \kappa d_0-e^Td=0\}, \sigma^*\neq 0;\\ \{(d,d_0): d_{\alpha^{\neq}}=0, d_{\gamma^{\neq}}=d_0e\}, \sigma^*=0.\end{cases}$$

5.2 *Proofs of Theorems 5.1 and 5.2*

Theorem 5.2 can be obtained from Theorem 5.1 and the definition of the affine hull (2.7). Now we prove Theorem 5.1.

PROOF : Considering (2.11) and the position of (x,t), we take account of the critical cone $\overline{\mathcal{C}}$ by considering the following four cases.

Case 1 : $(x,t) \in \mathcal{C}$. In this case, it is obvious that $(\overline{x},\overline{t}) = (x,t)$, and then $\overline{\mathcal{C}} = T_{\mathcal{C}}(x,t)$.

Case 2 : $(x,t) \in int(\mathcal{C}^o)$. In this case, $(\overline{x},\overline{t}) = (0,0)$, the tangent cone $T_{\mathcal{C}}(\overline{x},\overline{t}) = \mathcal{C}$, and $\overline{\mathcal{C}} = \mathcal{C} \cap (x,t)^{\perp}$. Now, we will show that $\mathcal{C} \cap (x,t)^{\perp} = \{(0,0)\}$. For any $(d,d_0) \in \mathcal{C} \cap (x,t)^{\perp}$, it holds that $d_0 \geq 0$. We will verity that $d_0 = 0$. In fact, if $d_0 > 0$, noticing that $(x,t) \in int(\mathcal{C}^o)$, we have

$$0 = x^T d + t d_0 = d_0 (x^T (\frac{d}{d_0}) + t) \leq d_0 (\max\{x^T z : z \in \Omega\} + t) < 0,$$

which is a contradiction. That is, d_0 must be equal to 0. Noticing that $(d,d_0) \in \mathcal{C}$, we then have $d = d_0 = 0$, which yields $\overline{\mathcal{C}} = \{(0,0)\}$.

Case 3 : $(x,t) \in bd(\mathcal{C}^o) \setminus \{(0,0)\}$. There are two possibilities:

(i) $t \leq 0, max\{x^T z : z \in \Omega\} + t = 0$.

(ii) $t = 0, max\{x^T z : z \in \Omega\} + t \leq 0$. It is clear that $max\{x^T z : z \in \Omega\} \leq 0$. Considering description (3.17) of $argmax_{z\in\Omega} x^T z$, we have $max\{x^T z : z \in \Omega\} \geq x^T 0 = 0$. Then it holds that $0 = max\{x^T z : z \in \Omega\} = max\{x^T z : z \in \Omega\} + t$.

Thus, by the above analysis of (i) and (ii), we find that

$$(x,t) \in bd(\mathcal{C}^o) \setminus \{(0,0)\} \Leftrightarrow t \leq 0 \text{ and } max\{x^T z : z \in \Omega\} + t = 0.$$

That is, (5.2) holds.

Now, we compute $\bar{\mathcal{C}}$. Since $(x,t) \in bd(\mathcal{C}^o) \setminus \{(0,0)\}$, it holds that $(\overline{x},\overline{t}) = (0,0)$ and $\bar{\mathcal{C}} = \mathcal{C} \cap (x,t)^{\perp}$. For any $(d,d_0) \in \mathcal{C} \cap (x,t)^{\perp}$, we have $d_0 \geq 0$. If $d_0 = 0$, then $d = 0$. If $d_0 > 0$, noticing that $(x,t) \in bd(\mathcal{C}^o) \setminus \{(0,0)\}$ and (5.21), we have

$$0 = x^T d + t d_0 = d_0 (x^T (\frac{d}{d_0}) + t) \leq d_0 (\max\{x^T z : z \in \Omega\} + t) = 0,$$

which implies $\dfrac{d}{d_0} \in argmax_{z\in\Omega} x^T z$. Then, by (3.17),we have

$$\bar{\mathcal{C}} = \begin{cases} \{(d,d_0) : d_{J_<} = 0, d_{J_>} = d_0 e, 0 \leq d_{J_=} \leq d_0 e, e^T d_{J_=} \leq (\kappa - |J_>|) d_0\}, & x^{\downarrow}_{\lfloor\kappa\rfloor+1} \leq 0 \\ \{(d,d_0) : d_{J_<} = 0, d_{J_>} = d_0 e, 0 \leq d_{J_=} \leq d_0 e, e^T d_{J_=} = (\kappa - |J_>|) d_0\}, & x^{\downarrow}_{\lfloor\kappa\rfloor+1} > 0. \end{cases}$$

Simultaneously, considering $(x,t) \in bd(\mathcal{C}^o) \setminus \{(0,0)\}$, we have

$$\beta = \emptyset, \alpha_{\neq} = J_<, \gamma_{\neq} = J_>, \alpha_= = \gamma_= = J_=.$$

Thus, it can be concluded that (5.22) holds.

Case 4 : Otherwise, we have $\bar{t} > 0$. By the KKT conditions of (1.2), we have

$$\begin{cases} 0 \;=\; \overline{x} - x + ue + v - w; \quad 0 = \bar{t} - t - \kappa u - e^T v; \\ 0 \;\leq\; u \perp (\kappa\bar{t} - e^T\overline{x}) \geq 0; \quad 0 \leq v \perp (\bar{t}e - \overline{x}) \geq 0; \quad 0 \leq w \perp \overline{x} \geq 0. \end{cases} \tag{21}$$

Combining (5.24) with Lemma 5.1, we find that

$$u = \sigma^*, \quad v_\alpha = v_\beta = w_\beta = w_\gamma = 0, \tag{22}$$

and

$$w_\alpha = ue - x_\alpha = \sigma^* e - x_\alpha, v_\gamma = x_\gamma - \bar{t}e - ue = x_\gamma - \bar{t}e - \sigma^* e, t - \bar{t} = -\kappa u - e^T v. \tag{23}$$

We know that

$$w_\alpha \geq 0, v_\gamma \geq 0 \tag{24}$$

For any $(d, d_0) \in T_{\mathcal{C}}(\overline{x}, \bar{t}) \cap (x - \overline{x}, t - \bar{t})^\perp$, considering (5.24), (5.26) and (2.4), we have

$$\begin{aligned} 0 &= \langle (d, d_0), (x - \overline{x}, t - \bar{t}) \rangle \\ &= \langle d_\alpha, x_\alpha - \overline{x}_\alpha \rangle + \langle d_\beta, x_\beta - \overline{x}_\beta \rangle + \langle d_\gamma, x_\gamma - \overline{x}_\gamma \rangle + d_0(t - \bar{t}) \\ &= \langle d_\alpha, \sigma^* e - w_\alpha \rangle + \langle d_\beta, \sigma^* e \rangle + \langle d_\gamma, \sigma^* e + v_\gamma \rangle + d_0(-\kappa\sigma^* - e^T v) \\ &= \sigma^* e^T d - \kappa\sigma^* d_0 - \langle d_\alpha, w_\alpha \rangle + \langle d_\gamma, v_\gamma \rangle - \langle d_0 e, v_\gamma \rangle \\ &= -\sigma^*(\kappa d_0 - e^T d) - \langle w_\alpha, d_\alpha \rangle - \langle v_\gamma, -d_\gamma + d_0 e \rangle. \end{aligned} \tag{25}$$

Subcase 4.1 : $\sigma^* > 0$. Then, we have $\kappa\bar{t} - e^T\overline{x} = 0$ by (5.24). In this case, by Theorem 4.1, we have $T_{\mathcal{C}}(\overline{x}, \bar{t}) = \{(d, d_0) : d_\alpha \geq 0, d_0 e - d_\gamma \geq 0, , \kappa d_0 - e^T d \geq 0\}$. Considering $(d, d_0) \in T_{\mathcal{C}}(\overline{x}, \bar{t}) \cap (x - \overline{x}, t - \bar{t})^\perp$, by (5.27) and (5.28), we find that $\kappa d_0 - e^T d = 0$, $\langle w_\alpha, d_\alpha \rangle = 0$ and $\langle v_\gamma, -d_\gamma + d_0 e \rangle = 0$. Together with (2.6) and (5.26), we have $w_{\alpha^=} = v_{\gamma^=} = 0$, $w_{\alpha^{\neq}} > 0$ and $v_{\gamma^{\neq}} > 0$. Hence, we obtain

$$\overline{\mathcal{C}} = \{(d, d_0) \in \mathcal{R}^n \times \mathcal{R} : \kappa d_0 - e^T d = 0, d_{\alpha^{\neq}} = 0, d_{\gamma^{\neq}} = d_0 e, d_{\alpha^=} \geq 0, d_{\gamma^=} \leq d_0 e\}$$

Subcase 4.2 : $\sigma^* = 0$ and $\kappa\bar{t} - e^T\overline{x} > 0$. In this case, by Theorem 4.1, we have $T_{\mathcal{C}}(\overline{x}, \overline{t}) = \{(d, d_0) : 0 \leq d_\alpha, d_\gamma \leq d_0 e\}$. By a similar process of deduction to that used in Subcase 4.1, we can degenerate (5.28) into $\langle w_\alpha, d_\alpha\rangle + \langle v_\gamma, -d_\gamma + d_0 e\rangle = 0$. Then

$$\overline{\mathcal{C}} = \{(d, d_0) \in \mathcal{R}^n \times \mathcal{R} : d_{\alpha^{\neq}} = 0, d_{\gamma^{\neq}} = d_0 e, d_{\alpha^{=}} \geq 0, d_{\gamma^{=}} \leq d_0 e\}.$$

Subcase 4.3 : $\sigma^* = 0$ and $\kappa\bar{t} - e^T\overline{x} = 0$. In this case, by Theorem 4.1, we have $T_{\mathcal{C}}(\overline{x}, \overline{t}) = \{(d, d_0) : 0 \leq d_\alpha, d_\gamma \leq d_0 e, e^T d - \kappa d_0 \leq 0\}$. By a process of deduction similar to that used in Subcase 4.1, we can obtain

$$\overline{\mathcal{C}} = \{(d, d_0) \in \mathcal{R}^n \times \mathcal{R} : d_{\alpha^{\neq}} = 0, d_{\gamma^{\neq}} = d_0 e, d_{\alpha^{=}} \geq 0, d_{\gamma^{=}} \leq d_0 e, e^T d \leq \kappa d_0\}.$$

We have now obtained expressions for $\bar{\mathcal{C}}$ under all possible positions of (x, t). This completes the proof. □

4. Conclusions

This paper studied the geometric properties of a class of nonsymmetric cones $\mathcal{C}$, including its dual, polar, tangent cone, and critical cone. These results are necessary for computing the directional derivative and B-subdifferential of the metric projector over cone $\mathcal{C}$. Moreover, these geometric properties are helpful for investigating the strong second-order sufficient condition and constraint nondegeneracy of the related optimization problems. On the basis of the results obtained in this paper, we will devote study to the differential properties of the metric projector over $\mathcal{C}$ and perform sensitivity analysis of Problem (1) in our next research work.

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References

1. S. Boyd, P. Diaconis, and L. Xiao, Fasted mixing Markov chain on a graph, *SIAM Review*, **46**(4) (2004), 667-689.
2. S. Boyd, P. Diaconis, P. A. Parrilo *et al.*, Fasted mixing Markov chain on graphs with symmetries, *SIAM J. Optimiz.*, **20**(2) (2009), 792-891.
3. D. F. Sun, The strong second order sufficient condition and constraint nondegeneracy in nonlinear semidefinite programming and their implications, *Math Oper Res.*, **31**(4) (2006), 761-776.

4. M. T. Chu, R. E. Funderlic, and R. J. Plemmons, Structured low rank approximation, *Linear Algebar Appl*., **336** (2003), 157-172.

5. B. Wu, C. Ding, and D. F. Sun *et al*., On the Moreau-Yoshida regularization of the vector k-norm related functions, *SIAM J. Optimiz*., **24**(2) (2014), 766-794.

6. R. Helgason, J. Kennington, and H. Lall, A polynomially bounded algorithm for a singly constrained quadratic program, *Math Program*., **18**(1) (1980), 338-343.

7. P. H. Calamai and J. J. More, Quasi-Newton updates with bounds, *SIAM J. Numer Anal*., **24**(6) (1987), 1434-1441.

8. N. Maculan, C. P. Santiago, and Macambira *et al*., An $O(n)$ algorithm for projecting a vector on the intersection of a hyperplane and a box, *J Optimiz Theorey App*., **117**(3) (2003), 553-574.

9. Y. J. Liu, S. Y. Wang, and J. H. Sun, Finding the projection onto the intersection of a closed half-space and a variable box, *Oper. Res. Lett*., **41**(3) (2013), 259-264.

10. F. H. Clarke, *Optimization and Nonsmooth Analysis*, New York: John Wiley and Sons; 1990. Chapter 2, Generalized Gradients, p.50-62.