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HALANAY INEQUALITY ON TIME SCALES WITH UNBOUNDED COEFFICIENTS AND ITS APPLICATIONS

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In the present paper, we obtain a Halanay inequality on time scales with unbounded coefficient for a dynamic problem, which extends a result of Wen *et al.* (*J. Math. Anal. Appl.*, **347** (2008), 169-178.) to the inequality of integral type on time scales. Moreover, we list two dynamic problems to which the Halanay inequality obtained above can be applied and prove the zero solution of two delay dynamic problems are asymptotically stable. Moreover, it is worth mentioning that the Halanay inequality obtained in the present paper is more precise than the results in [3, 14, 17].

Key words : Halanay inequality of integral type; time scales; unbounded coefficients; asymptotically stable.

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1. Introduction

In 1966, Halanay [7] proved that:

Lemma 1.1 — [7]. Let $x(t)$ be any non-negative solutions of

$$x'(t) \leq -\alpha x(t) + \beta \sup_{s\in[t-\tau,t]} x(x),\ t \geq t_0$$

and $\alpha > \beta > 0$, then there exist two positive constants $\gamma > 0$ and $K > 0$ such that

$$x(t) \leq Ke^{-\gamma(t-t_0)} \text{ for } t \geq t_0.$$

In 1996, Baker and Tang [3] obtained that:

Lemma 1.2 — [3]. Let $x(t)$ be any non-negative solution of

$$\begin{cases} x'(t) \le -a(t)x(t) + b(t) \sup\limits_{s\in[q(t),t]} x(s),\ t > t_0, \\ x(t) = |\phi(t)| \text{ for } t \le t_0, \end{cases}$$

where $\phi(t) : (-\infty, t_0] \to \mathbb{R}$ is a bounded and continuous function, $a(t) \ge 0$, $b(t) \ge 0$ for any $t \ge t_0$, $q(t) \le t$, and $q(t) \to \infty$ as $t \to \infty$.

If there exists $\sigma > 0$ such that

$$-a(t) + b(t) \le -\sigma < 0 \text{ for } t \le t_0,$$

then

$$\begin{cases} x(t) \le \|\phi\|^{(-\infty,t_0]}, t \ge t_0, \\ x(t) \to 0,\ \text{as } t \to \infty, \end{cases}$$

where $\|\phi\|^{(-\infty,t_0]} = \sup\limits_{t\le t_0} |\phi(t)|$.

Wen *et al*. [17] proved the following Halanay inequality with unbounded coefficients.

Lemma 1.3 — [17]. Let $x(t)$ be any non-negative solutions of

$$\begin{cases} x'(t) \le \gamma(t) + \alpha(t)x(t) + \beta(t) \sup\limits_{\xi\in[t-\tau(t),t]} x(\xi),\ t \ge t_0, \\ x(t) = |\psi(t)| \text{ for } t \le t_0, \end{cases}$$

where $\psi(t) : (-\infty, t_0] \to \mathbb{R}$ is a bounded and continuous function, continuous functions $\gamma(t) \ge 0$, $\alpha(t) \le 0$, $\beta(t) \ge 0$ for $t \in [t_0, \infty)$, $\tau(t) \ge 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$, and if there exists $\sigma > 0$ such that

$$\alpha(t) + \beta(t) \le -\sigma < 0 \text{ for } t \le t_0,$$

then, we have

(i)

$$u(t) \le \frac{\gamma^*}{\sigma} + G,\ t \ge t_0.$$

If we assume further that there exists $0 < \delta < 1$ such that

$$\delta\alpha(t) + \beta(t) < 0 \text{ for } t \ge t_0,$$

then, we have

(ii) for any given $\epsilon > 0$, there exists $\hat{t} = G > t_0$ such that

$$u(t) \leq \frac{\gamma^*}{\sigma} + \epsilon,\ t \geq \hat{t},$$

where

$$G = \sup_{t \in (-\infty, t_0]} |\psi(t)| \text{ and } \gamma^* = \sup_{t \in [t_0, +\infty)} \gamma(t).$$

Recently, Jia *et al*. [14] proved the following Halanay inequality of integral type on times scales.

Lemma 1.4 — [14]. Let $x(t)$ be any non-negative solution of

$$\begin{cases} x^{\Delta}(t) \leq -a(t)x(t) + b(t) \sup\limits_{s \in [t-\tau(t), t]} x(s) + c(t) + d(t) \int_0^{\infty} K(t,s)x(t-s)\Delta s,\ t \geq t_0, \\ x(t) = |\psi(t)| \text{ for } t \leq t_0, \end{cases}$$

where $\psi(t) : (-\infty, t_0]_{\mathbb{T}} \to \mathbb{R}$ is a bounded and rd-continuous function with $M = \sup\limits_{t \leq t_0} |\phi(t)|$, rd-continuous functions $a(t) \leq 0$, $b(t) \geq 0$, $c(t) \leq 0$, $d(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$, $c(t)$ is bounded and

$$\sup_{t \leq t_0} c(t) = \bar{c}, \lim_{t \to \infty} (t - \tau(t)) = +\infty$$

If there exists $\sigma > 0$ such that

$$a(t) - b(t) - d(t) \int_0^{\infty} K(t,s)x(t-s)\Delta s > \delta > 0$$

for $t \in [t_0, -\infty)_{\mathbb{T}}$ and with non-negative, rd-continuous function $K(t,s)$ satisfying

$$\int_0^{\infty} K(t,s)x(t-s)\Delta s < \infty \text{ for } \forall t \in \mathbb{T},$$

then, we have

(i)

$$x(t) \leq \frac{\bar{c}}{\delta} + M,\ t \in [t_0, \infty)_{\mathbb{T}}.$$

If we assume further that $d(t) = 0$ and there exists $0 < \kappa < 1$ such that

$$\kappa a(t) - b(t) > 0 \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

then, we have

(ii) for any given $\epsilon > 0$, there exists $\hat{t} = \hat{M}, \epsilon > t_0$ such that

$$x(t) \leq \frac{\bar{c}}{\delta} + \epsilon,\ t \geq \hat{t}.$$

More results about the Halanay inequality can be referred in [12, 13]. One can analyze the stability of the solutions of the differential and difference equations via the Halanay inequalities see [2, 9, 11, 16]. More comments on the Halanay inequalities can be found in [3, 4].

To the best of the author's knowledge, there is few papers considering Halanay inequalities of integral type on times scales with unbounded coefficients except [14]. Motivated by the work of [14, 17], in the present paper, we discuss the Halanay inequalities of integral type on times scales with unbounded coefficients and the stability of two dynamical equations.

In order to state the main result of the present paper, we introduce the following concepts related to the notions of time scales. We refer to [1] for more details on time scales. In the present paper, we say $t \leq (\geq) t_0$ in the sense that $t \in (-\infty, t_0]_{\mathbb{T}}([t_0, +\infty)_{\mathbb{T}})$.

Definition 1.6 — [1]. Let $\mu(t) = \sigma(t) - t$, A function $h : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if for any $t \in \mathbb{T}^k$, we have $1 + \mu(t)h(t) \neq 0$. Let $\mathcal{S}$ be the set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ and

$$\mathcal{S}_1 = \{h \in \mathcal{S} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

For any $\varphi \in \mathcal{S}$, the *exponential function* $e_\varphi(t,s) : \mathbb{T} \mapsto \mathbb{R}$ is defined by

$$e_\varphi(t,s) = \exp\Big(\int_s^t \xi_{\mu(r)}(\varphi(r))\triangle r\Big)$$

where $\xi_{\mu(s)}$ is a cylinder transformation given by

$$\xi_{\mu(s)} = \begin{cases} \frac{1}{\mu(r)} \log(1 + \mu(r)\varphi(r)), & \mu(r) > 0, \\ \varphi(r), & \mu(r) = 0. \end{cases}$$

We have the following two basic properties of the regressive and exponential function,

Lemma 1.7 — [1]. For any $\varphi \in \mathcal{S}$, the following statements hold,

(i) $e_0(s,t) \equiv 1,\ e_\varphi(t,t) \equiv 1$ and $e_\varphi(\sigma(t), s) = (1 + \mu(t)\varphi(t))e_\varphi(t,s)$;

(ii) $e_\varphi(t,s)e_{\ominus\varphi}(t,s) \equiv 1$ where $\ominus\varphi(t) = -\frac{\varphi(t)}{1+\mu(t)\varphi(t)}$;

(iii) $(\frac{1}{e_\varphi(t,s)})^\triangle = -\frac{\varphi(t)}{e_\varphi(\sigma(t),s)}$.

Lemma 1.8 — [6]. Let φ be a non-negative function satisfying $-\varphi \in \mathcal{S}_1$, then

$$1 - \int_s^t \varphi(u)du \leq e_{-\varphi}(t,s) \leq \exp\Big\{ - \int_s^t \varphi(u)du\Big\} \text{ for all } t \geq s.$$

If φ is a rd-continuous and non-negative function, then

$$1+\int_s^t \varphi(u)du \le e_\varphi(t,s) \le \exp\Big\{\int_s^t \varphi(u)du\Big\} \text{ for all } t \ge s.$$

Remark 1.9 : If $\varphi \in \mathcal{S}_1$ and $\varphi(r) > 0$ for all $s \le r \le t$, then

$$e_\varphi(t,r) \le e_\varphi(t,s),\ e_\varphi(a,b) < 1 \text{ and } e_{-\varphi}(b,a) < 1, \tag{1.1}$$

for any $s \le a < b \le t$.

In the present paper, we assume that the following conditions hold:

(H_1) Let $x(t)$ be a non-negative function satisfying

$$\begin{cases} x^{\triangle}(t) & \le -a(t)x(t) + b(t) \sup\limits_{t-\tau(t)\le s\le t} x(s) + c(t) \\ & \quad + d(t)\int_0^\infty K(t,s)x(t-\tau(s))\triangle s, \quad t \ge t_0; \\ x(s) & = |\varphi(s)|,\ s \le t_0. \end{cases}$$

where $\varphi(s)$ is a bounded continuous function in $(\infty, t_0]_{\mathbb{T}}$ and let

$$\sup_{s\le t_0} |\phi(t)| = M. \tag{1.2}$$

(H_2) $a(t)$, $b(t)$, $c(t)$, $d(t)$, $\tau(t)$ are non-negative, continuous functions in $[t_0,\infty)_{\mathbb{T}}$ and $c(t)$ is bounded and let

$$\sup_{t\ge t_0} c(t) = \bar{c}, \tag{1.3}$$

and

$$\lim_{t\to\infty} t-\tau(t) = +\infty.$$

(H_3) There exists a positive constant δ such that for any $t \in [t_0,\infty)_{\mathbb{T}}$,

$$a(t) - b(t) - d(t)\int_0^\infty K(t,s)\Delta s \ge \delta > 0.$$

(H_4) The delay kernel $K(t,s)$ is a non-negative and continuous function in $[t_0,\infty)_{\mathbb{T}} \times [0,\infty)$ satisfying

$$\int_0^\infty K(t,s)\Delta s < \infty, \text{ for any } t \ge t_0.$$

(H_5) Let $a(t)$ be a function satisfying $-a(t) \in \mathcal{S}_1$. For any fixed $t \ge t_0$, if

$$\int_0^\infty K(t,s)e_p(t, t-\tau(s))\Delta s < \infty,$$

then for any $t \in [t_0, \infty)_{\mathbb{T}}$, there exists a continuous function $p = p(t)$ such that

$$p(t) - a(t) + b(t)e_p(t, t - \tau(s)) + d(t) \int_0^{\infty} K(t, s)e_p(t, t - \tau(s))\Delta s = 0.$$

The main result of the present paper can be stated as follow:

***Theorem* 1.10** — *Assume that $(H_1) \sim (H_5)$ hold. Then we have*

$$x(t) \leq \frac{\bar{c}}{\delta} + Me_{\ominus \underline{p}}(t, t_0), \tag{1.4}$$

as $t \geq t_0$, where $\bar{c}$ and M is defined in (1.3) and (1.2) respectively, and $\underline{p}$ is defined as follow

$$\underline{p} = \inf_{t \geq t_0} \{p(t) : p(t) - a(t) + b(t)e_p(t, t - \tau(t)) + d(t) \int_0^{\infty} K(t, s)e_p(t, t - \tau(s))\triangle s = 0\}.$$

Remark 1.11 : (i) When $c(t) = d(t) = 0$, the Halanay inequality (1.4) is the main result of [3], Theorem 2.3 and Theorem 2.4 of [17], which shows that Theorem 1.10 in the present paper can be seemed as a generalization of the result in [3, 17].

(ii) The Halanay inequality (1.4) in the present paper is more precise than the results in [3, 14, 17].

The paper is organized as follow. In Section 2, we give the proof of the Halanay inequality. Two dynamic problems are listed to which the Halanay inequality obtained in Theorem 1.10 can be applied in Section 3. As a corollary, we see that the zero solutions of two delay dynamic problems that we list in Section 3 are asymptotically stable.

2. The Proof of Theorem 1.10

In this section, we prove the Theorem 1.10. Firstly, we need the following lemma.

Lemma 2.1 — [14]. Assume that $(H_1) \sim (H_4)$ hold and $-a(t) \in \mathcal{S}_1$, then

(i)

$$x(t) \leq \frac{\bar{c}}{\delta} + M, \text{ for } t \geq t_0.$$

If we assume further that $d(t) = 0$ in (H_1) and (H_3), and there exists $0 < \kappa < 1$ such that

$$\kappa a(t) - b(t) > 0,$$

Then, we have

(ii) for any given $\epsilon > 0$, there exists $\widetilde{t} = \widetilde{t}(M, \epsilon) > t_0$, such that

$$x(t) \le \frac{\overline{c}}{\delta} + \epsilon, \text{ for } t \ge \widetilde{t}.$$

THE PROOF OF THEOREM 1.10 : It is easy to infer from Lemma 2.1 that the inequality (1.4) holds when $M = 0$. As a result, it suffices to consider the case when $M \ne 0$. It follows from the definition and the basic property of the exponential function that for any $(p, t) \in [\underline{p}, \infty) \times [t_0, \infty)_{\mathbb{T}}$,

$$0 < K(t, s) \le K(t, s)e_p(t, t - \tau(s)).$$

Therefore, for any fixed $t \ge t_0$, if

$$\int_0^\infty K(t, s)e_p(t, t - \tau(s))\triangle s < \infty,$$

then,

$$\int_0^\infty K(t, s)\triangle s < \infty.$$

We may assume that for any fixed $t \ge t_0$,

$$\int_0^\infty K(t, s)e_p(t, t - \tau(s))\triangle s < \infty.$$

For any $p \ge 0$, we define $F(t, p) : [t_0, \infty)_{\mathbb{T}} \times [0, \infty) \mapsto \mathbb{R}$ as follow:

$$F(t, p) = p - a(t) + b(t)e_p(t, t - \tau(t)) + d(t)\int_0^\infty K(t, s)e_p(t, t - \tau(s))\triangle s.$$

Let

$$F(t, 0) = -a(t) + b(t) + d(t)\int_0^\infty K(t, s)\triangle s.$$

It follows from the condition (H_3) that

$$F(t, 0) \le -\delta < 0.$$

Then,

$$\lim_{p\to\infty} F(t, p) = +\infty.$$

For any fixed $t \ge t_0$, it follows from the definition of the exponential function $e_p(t, s)$ that for any $p \ge 0$,

$$e_p(t, t - \tau(s)) \text{ and } \int_0^\infty K(t, s)e_p(t, t - \tau(s))\Delta s$$

are monotonically increasing functions. Thus,$F(\cdot,p)$ is a strictly increasing function in $[t_0,\infty)_{\mathbb{T}}$. It follows from the Implicit Function Theorem that for any fixed $t \geq t_0$, there exists a unique positive function $p = p(t)$, such that

$$F(t, p(t)) = 0.$$

Let

$$\underline{p} = \inf_{t \geq t_0} p(t),$$

we claim that

$$\underline{p} \geq 0. \tag{2.1}$$

Indeed, suppose on the contrary that $\underline{p} < 0$. From the definition of infimum, there exists a positive constant $t' \geq t_0$, such that

$$p(t') < 0. \tag{2.2}$$

From the definition of $\underline{p}$ and (1.1), we have

$$\begin{aligned} p(t') &= a(t') - b(t')e_p(t', t' - \tau(t')) - d(t') \int_0^\infty K(t', s) e_p(t', t' - \tau(s)) \triangle s \\ &\geq a(t') - b(t') - d(t') \int_0^\infty K(t', s) \triangle s \geq \delta > 0, \end{aligned}$$

which reduces a contradiction to (2.2). Hence, we get the desired inequality (2.1).

Now we divide the proof into the following two steps.

Step 1 : $\underline{p} = 0$. From Lemma 2.1, we have

$$x(t) \leq \frac{\bar{c}}{\delta} + M = \frac{\bar{c}}{\delta} + Me_{\ominus \underline{p}}(t, t_0), \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

then, we obtain the desired estimate (1.4).

Step 2 : $\underline{p} > 0$. For any $\epsilon > 0$, we claim that

$$x(t) < \frac{\bar{c} + \varepsilon}{\delta} + Me_{\ominus \underline{p}}(t, t_0), \text{ for any } t \geq t_0. \tag{2.3}$$

Indeed, let

$$y(t) = \begin{cases} \frac{\bar{c}+\varepsilon}{\delta} + Me_{\ominus \underline{p}}(t, t_0), & \text{for } t \geq t_0, \\ |\varphi(t)|, & \text{for } t \leq t_0, \end{cases}$$

for any $t \geq t_0$,

$$y^{\Delta}(t) = -\frac{M\underline{p}}{e_{\underline{p}}(\sigma(t), t_0)} \leq 0.$$

This implies that $y(t)$ is a decreasing function in $[t_0,\infty)_{\mathbb{T}}$. Therefore, if $\bar{t}-\tau(\bar{t})\geq t_0$, we have

$$\sup_{\bar{t}-\tau(\bar{t})\leq s\leq \bar{t}} x(s)\leq \sup_{\bar{t}-\tau(\bar{t})\leq s\leq \bar{t}} y(s)=y(\bar{t}-\tau(\bar{t}))=\frac{\overline{c}+\varepsilon}{\delta}+Me_{\ominus \underline{p}}(\bar{t}-\tau(\bar{t}),t_0), \tag{2.4}$$

and

$$x(\bar{t}-\tau(s))\leq y(\bar{t}-\tau(s))=\frac{\overline{c}+\varepsilon}{\delta}+Me_{\ominus \underline{p}}(\bar{t}-\tau(s),t_0). \tag{2.5}$$

If $\bar{t}-\tau(\bar{t})<t_0$, we see that

$$\sup_{\bar{t}-\tau(\bar{t})\leq s\leq \bar{t}} x(s)\leq \max\left\{\sup_{s\leq t_0} x(s);\ \sup_{t_0\leq s\leq \bar{t}} x(s)\right\}\leq \frac{\overline{c}+\varepsilon}{\delta}+M; \tag{2.6}$$

and

$$x(\bar{t}-\tau(s))\leq y(\bar{t}-\tau(s))=\frac{\overline{c}+\varepsilon}{\delta}+Me_{\ominus \underline{p}}(\bar{t}-\tau(s),t_0). \tag{2.7}$$

For any $t\geq t_0$, let

$$z(t)=x(t)-y(t).$$

Since

$$y^{\Delta}(t)=-\frac{M\underline{p}e_{\ominus \underline{p}}(t,t_0)}{1+\underline{p}\mu(t)}\geq -M\underline{p}e_{\ominus \underline{p}}(t,t_0),$$

it follows from the assumption (H_1) that

$$z^{\Delta}(t)\leq M\underline{p}e_{\ominus \underline{p}}(t,t_0)-a(\bar{t})x(\bar{t}) \qquad +b(\bar{t})\sup_{\bar{t}-\tau(\bar{t})\leq s\leq \bar{t}} x(s)+c(\bar{t}) \tag{2.8}$$

$$+d(\bar{t})\int_0^{\infty}K(\bar{t},s)x(\bar{t}-\tau(s))\triangle s. \tag{2.9}$$

Let

$$\bar{t}=\sup\{t: z(s)=x(s)-y(s)<0, s\in[t_0,t]_{\mathbb{T}}\}.$$

We firstly show that

$$\bar{t}=\infty.$$

Suppose on the contrary that $\bar{t}<\infty$. Then, we claim that

$$z(\bar{t})=x(\bar{t})-y(\bar{t})\leq 0.$$

Indeed, on the contrary, we may assume that

$$z(\bar{t})=x(\bar{t})-y(\bar{t})>0. \tag{2.10}$$

If $\bar{t}$ is left-dense, there exists a time sequences $\{t_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ such that

$$t_n < \bar{t},\ t_n \to \bar{t},\ \text{as } n \to \infty, \tag{2.11}$$

and

$$z(t_n) = x(t_n) - y(t_n) \leq 0. \tag{2.12}$$

Let $n \to \infty$ in (2.11), it follows from the continuity of x and y that

$$z(\bar{t}) = \lim_{n\to\infty} z(t_n) = \lim_{n\to\infty} x(t_n) - \lim_{n\to\infty} y(t_n) = x(\bar{t}) - y(\bar{t}) \leq 0, \tag{2.13}$$

which reaches a contridiction to (2.9).

If $\bar{t}$ is left-scattered, there exists $\rho(\bar{t})$ such that

$$\rho(\bar{t}) < \bar{t} \text{ and } z(\rho(\bar{t})) = x(\rho(\bar{t})) - y(\rho(\bar{t})) \leq 0, \tag{2.14}$$

and

$$z(\bar{t}) = x(\bar{t}) - y(\bar{t}) > 0, \tag{2.15}$$

then, we have

$$\sup\{t : z(s) = x(s) - y(s) < 0, s \in [t_0, t]_{\mathbb{T}}\} = \rho_1(\bar{t}) < \bar{t}. \tag{2.16}$$

This also yields a contradiction to the definition of $\bar{t}$.

Therefore, we continue the proof under the assumption that

$$\bar{t} < \infty \text{ and } z(\bar{t}) = x(\bar{t}) - y(\bar{t}) \leq 0.$$

Next, we also will discuss the following two cases.

Case 2.1 : We may assume that $\bar{t} > t_0$ and

$$z(\bar{t}) = x(\bar{t}) - y(\bar{t}) = 0. \tag{2.17}$$

Then, for any $t_0 \leq t \leq \bar{t}$, we see that

$$z(t) = x(t) - y(t) \leq 0, \tag{2.18}$$

and

$$z^{\triangle}(\bar{t}) \geq 0. \tag{2.19}$$

If

$$\bar{t} - \tau(\bar{t}) \geq t_0,$$

putting (2.4), (2.5) into (2.8), we see that

$$\begin{aligned}
z^{\triangle}(\bar{t}) &\leq -a(\bar{t})\Big(\frac{\bar{c}+\varepsilon}{\delta} + Me_{\ominus\underline{p}}(\bar{t}, t_0)\Big) + b(\bar{t})\Big(\frac{\bar{c}+\varepsilon}{\delta} + Me_{\ominus\underline{p}}(\bar{t} - \tau(\bar{t}), t_0)\Big) \\
&\quad + \bar{c} + d(\bar{t}) \int_0^{\infty} K(\bar{t}, s)\Big(\frac{\bar{c}+\varepsilon}{\delta} + Me_{\ominus\underline{p}}(\bar{t} - \tau(s), t_0)\Big)\triangle s + M\underline{p}e_{\ominus\underline{p}}(\bar{t}, t_0) \\
&= \frac{\bar{c}+\varepsilon}{\delta}\Big(- a(\bar{t}) + b(\bar{t}) + d(\bar{t}) \int_0^{\infty} K(\bar{t}, s)\triangle s\Big) + \bar{c} \\
&\quad + Me_{\ominus\underline{p}}(\bar{t}, t_0)\Big(\underline{p} - a(\bar{t}) + b(\bar{t})e_{\underline{p}}(\bar{t}, \bar{t} - \tau(t)) \\
&\quad + d(\bar{t}) \int_0^{\infty} K(\bar{t}, s)e_{\underline{p}}(\bar{t}, \bar{t} - \tau(s))\triangle s\Big).
\end{aligned}$$

Recalling that the assumption (H_3), we have

$$\begin{aligned}
z^{\triangle}(\bar{t}) \leq Me_{\ominus\underline{p}}(\bar{t}, t_0)\Big(&\underline{p} - a(\bar{t}) + b(\bar{t})e_{\underline{p}}(\bar{t}, \bar{t} - \tau(t)) \\
&+ d(\bar{t}) \int_0^{\infty} K(\bar{t}, s)e_{\underline{p}}(\bar{t}, \bar{t} - \tau(s))\triangle s\Big) - \varepsilon.
\end{aligned} \tag{2.20}$$

From the definition of $\underline{p}$, we see that

$$\begin{aligned}
&\underline{p} - a(\bar{t}) + b(\bar{t})e_{\underline{p}}(\bar{t}, \bar{t} - \tau(\bar{t})) + d(\bar{t}) \int_0^{\infty} K(\bar{t}, s)e_{\underline{p}}(\bar{t}, \bar{t} - \tau(s))\triangle s \\
&= \underline{p} - a(\underline{t}) + b(\underline{t})e_{\underline{p}}(\underline{t}, \underline{t} - \tau(\underline{t})) + d(\underline{t}) \int_0^{\infty} K(\underline{t}, s)e_{\underline{p}}(\underline{t}, \underline{t} - \tau(s))\Delta s \\
&- \Big(p(\underline{t}) - a(\underline{t}) + b(\underline{t})e_{p}(\underline{t}, \underline{t} - \tau(\underline{t})) + d(\underline{t}) \int_0^{\infty} K(\underline{t}, s)e_{p}(\underline{t}, \underline{t} - \tau(s))\Delta s\Big) \\
&= (\underline{p} - p(\bar{t})) + b(\bar{t})\Big[e_{\underline{p}}(\bar{t}, \bar{t} - \tau(\bar{t})) - e_{p}(\bar{t}, \bar{t} - \tau(\bar{t}))\Big] \\
&+ d(\bar{t}) \int_0^{\infty} K(\bar{t}, s)\Big[e_{\underline{p}}(\bar{t}, \bar{t} - \tau(s)) - e_{p}(\bar{t}, \bar{t} - \tau(s))\Big]\triangle s \leq 0.
\end{aligned} \tag{2.21}$$

Putting (2.20) into (2.19), we see that

$$z^{\triangle}(\bar{t}) = x^{\triangle}(\bar{t}) - y^{\triangle}(\bar{t}) < 0, \tag{2.22}$$

which yields a contradiction to (2.18).

If $\bar{t} - \tau(\bar{t}) < t_0$, adopting a similar argument, we also see that

$$z^{\triangle}(\bar{t}) < 0, \tag{2.23}$$

which also reduces a contradiction to (2.18).

Step 2.2 : We may assume that $\bar{t} > t_0$ and

$$z(\bar{t}) = x(\bar{t}) - y(\bar{t}) < 0.$$

We claim that $\bar{t}$ must be right-scattered. Otherwise, if $\bar{t}$ is right-dense, for any $t_0 \le t \le \bar{t}$, we have

$$z(t) = x(t) - y(t) < 0.$$

Therefore, there exists some ϵ sufficiently small, for any $t_0 \le t \le \bar{t} + \epsilon$, we have

$$z(t) = x(t) - y(t) \le 0.$$

However, this reduces a contradiction to the definition of $\bar{t}$.

Hence, $\bar{t}$ is right-scattered and there exists $\sigma(\bar{t})$ such that

$$z(\sigma(\bar{t})) = x(\sigma(\bar{t})) - y(\sigma(\bar{t})) > 0 \tag{2.24}$$

and

$$z(t) = x(t) - y(t) < 0, \text{ for } \bar{t} \in [t, \sigma(\bar{t}))_{\mathbb{T}}. \tag{2.25}$$

We also have

$$\begin{aligned} z^{\triangle}(\bar{t}) \le{}& M\underline{p}e_{\ominus \underline{p}}(t, t_0) - a(\bar{t})x(\bar{t}) + b(\bar{t}) \sup_{\bar{t}-\tau(\bar{t}) \le s \le \bar{t}} x(s) \\ &+ c(\bar{t}) + d(\bar{t}) \int_0^{\infty} K(\bar{t}, s) x(\bar{t} - \tau(s)) \triangle s. \end{aligned} \tag{2.26}$$

It follows from (2.23), (2.24) and (2.25) that

$$\begin{aligned} z(\sigma(\bar{t})) \le{}& \Big(1 - \mu(\bar{t})a(\bar{t})\Big)x(\bar{t}) - y(\bar{t}) + \mu(\bar{t})M\underline{p}e_{\ominus \underline{p}}(t, t_0) \\ &+ \mu(\bar{t})\Big(b(\bar{t}) \sup_{\bar{t}-\tau(\bar{t}) \le s \le \bar{t}} x(s) + c(\bar{t}) + d(\bar{t}) \int_0^{\infty} K(\bar{t}, s) x(\bar{t} - \tau(s)) \triangle s\Big). \end{aligned} \tag{2.27}$$

If $\bar{t} - \tau(\bar{t}) \ge t_0$, recalling that $-a(t) \in \mathcal{S}_1$ and putting (2.4), (2.5), (2.20), (2.23) and (2.24) into

(2.26), we see that

$$\begin{aligned} z(\sigma(\bar{t})) &< \Big(1-\mu(\bar{t})a(\bar{t})\Big)\Big(\frac{\bar{c}+\varepsilon}{\delta}+Me_{\ominus\underline{p}}(\bar{t},t_0)\Big)-\Big(\frac{\bar{c}+\varepsilon}{\delta}+Me_{\ominus\underline{p}}(\bar{t},t_0)\Big) \\ &\quad+\mu(\bar{t})M\underline{p}e_{\ominus\underline{p}}(t,t_0)+\mu(\bar{t})\Big(b(\bar{t})[\frac{\bar{c}+\varepsilon}{\delta}+Me_{\ominus\underline{p}}(\bar{t}-\tau(\bar{t}),t_0)] \\ &\quad+d(\bar{t})\int_0^\infty K(\bar{t},s)[\frac{\bar{c}+\varepsilon}{\delta}+Me_{\ominus\underline{p}}(\bar{t}-\tau(s),t_0)]\triangle s\Big)+\bar{c}\mu(\bar{t}) \\ &= \frac{\bar{c}+\varepsilon}{\delta}\mu(\bar{t})\Big(-a(\bar{t})+b(\bar{t})+d(\bar{t})\int_0^\infty K(\bar{t},s)\triangle s\Big)+\bar{c}\mu(\bar{t}) \\ &\quad+M\mu(\bar{t})e_{\ominus\underline{p}}(\bar{t},t_0)\Big(\underline{p}-a(\bar{t})+b(\bar{t})e_{\underline{p}}(\bar{t},\bar{t}-\tau(t)) \\ &\quad+d(\bar{t})\int_0^\infty K(\bar{t},s)e_{\underline{p}}(\bar{t},\bar{t}-\tau(s))\triangle s\Big) \\ &\le M\mu(\bar{t})e_{\ominus\underline{p}}(\bar{t},t_0)\Big(\underline{p}-a(\bar{t})+b(\bar{t})e_{\underline{p}}(\bar{t},\bar{t}-\tau(t)) \\ &\quad+M\mu(\bar{t})e_{\ominus\underline{p}}d(\bar{t})\int_0^\infty K(\bar{t},s)e_{\underline{p}}(\bar{t},\bar{t}-\tau(s))\triangle s\Big)-\varepsilon\mu(\bar{t})<0, \end{aligned}$$

which reduces a contradiction to (2.22).

If $\bar{t}-\tau(\bar{t})<t_0$, adopting a similar argument, we also see that

$$z(\sigma(\bar{t}))<0.$$

which contradicts to (2.22).

Hence, we see that $\bar{t}=\infty$. Let $\varepsilon\to 0$ in (2.3), we obtain the desired estimate (1.4). This completes the proof of Theorem 1.10.

3. Applications to Two Cases

In this section, we list two examples to which the Halanay inequality we get in Theorem 1.10 can be applied. As a consequence, we obtain that the zero solutions of two delay dynamic problems are asymptotically stable.

Firstly, we consider a difference inequality on time scales with unbounded coefficients,

$$\begin{cases} x^{\triangle}(t)\le -a(t)x(t)+b(t)\sup\limits_{t-1\le s\le t}x(s)+c(t) \\ \qquad +d(t)\displaystyle\int_0^\infty K(t,s)x(t-\tau(s))\triangle s, t\ge 0; \\ x(t)=|\varphi(t)|, t\le 0, \end{cases} \tag{3.1}$$

where $\varphi(s)$ is a bounded continuous function for $t\le t_0$,

$$\sup_{s\le t_0}|\varphi(s)|=M,\quad \sup_{t\ge t_0}c(t)=\bar{c},\quad \lim_{t\to\infty}(t-\tau(t))=+\infty. \tag{3.2}$$

and

$$a(t) = t + \frac{1}{t+2}, b(t) = \frac{t^2+t}{e(t+2)}, c(t) = \Big(\frac{t+2}{t+1}\Big)^t, d(t) = \frac{2t}{\sqrt{\pi}(2-e^{-t^2})};$$
$$K(t,s) = (2-\cos 2ts)e^{-s^2-1}; \ \ (t,s) \in [0,\infty) \times [0,\infty).$$

It is easy to show that the conditions $(H_1) \sim (H_5)$ holds where $\overline{c} = e$, $\delta = \frac{1}{2}$, $\underline{p} = \inf_{t\geq 0} p(t)$. It follows from Theorem 1.10 that for any $t \geq t_0$,

$$x(t) \leq \frac{\overline{c}}{\delta} + Me_{\ominus\underline{p}}(t,0) = 2e + Me^{-\underline{p}t}. \tag{3.3}$$

Taking $c(t) \equiv 0$, then $\bar{c} = 0$, this implies that the zero solution of the problem (3.1) is asymptotically stable.

Next, we consider the following difference equation,

$$\triangle x(n) = -a(n)x(n+1) + b(n)x(n-\tau(n)) + c(n), \qquad n, \tau(n) \geq 0, \tag{3.4}$$

where

$$a(n) = 2(n+1),\ b(n) = \frac{n^2}{2n+1}, c(n) = \frac{5n}{\sqrt[n]{n!}}, p = \frac{n+2}{n+1}, \tau(n) = 2.$$

It is easy to show that the conditions $(H_1) \sim (H_5)$ holds where $\bar{c} = 5e$, $\delta = 5$, $\underline{p} = 1$. It follows from Theorem 1.10 that for $n \geq 2$,

$$x(n) \leq \frac{\overline{c}}{\delta} + Me_{\ominus\underline{p}}(n,2) = e + Me_{\ominus\underline{p}}(n,2). \tag{3.5}$$

Taking $c(n) \equiv 0$, then $\bar{c} = 0$, this implies that the zero solution of the problem (3.4) is asymptotically stable.

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