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THE SPHERE METHOD FOR THE INVERSE PROBLEMS OF THE RADON TRANSFORMS

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Using a different method - the sphere method, which is based on the technique of changing the integral on a plane into the integral on a hemisphere, we give some concise inverse formulas of the Radon transforms of functions with support in a cone with vertex at the origin and flare angle less than $\pi/2$, or with compact support. These formulas are easy for the computer to operate and thus can be applied in the imaging techniques of computerized tomography.

Key words : Radon transform; inverse problems; the sphere method.

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1. Introduction

The hyperplane in $\mathbb{R}^n$, with the normal vector $\theta \in \mathbb{S}^{n-1}$ and the distance $|t|(t \in \mathbb{R})$ from the origin, is define by

$$H(\theta, t) = \{x \in \mathbb{R}^n : x \cdot \theta = t\}, \tag{1}$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^n$. The Radon transform [1, 3-5] of a function f on $\mathbb{R}^n$ is defined by

$$Rf(\theta, t) = \int_{H(\theta,t)} f(y) dy_H, \tag{2}$$

where $(\theta, t) \in \mathbb{S}^{n-1} \times \mathbb{R}$ and dy_H is the Lebesgue measure on $H(\theta, t)$.

The Radon transform has many applications, covering from mathematical theories to practical areas, for example, the integral geometry [2, 3] and the remarkable computerized tomography [1, 2, 5, 6]. In computerized tomography, a variety of inverse formulas of the Radon transforms are applied for the image reconstructing. There have been several classical methods for the inverse problems of the Radon transforms, for example, the method of mean value operators [10, 12], the method of Riesz potentials [6, 10], the convolution-backprojection [1, 2, 5, 6, 9, 11] and continuous ridgelet transforms [11]. Next, we give an overview of the three kinds of methods.

For the method of mean value operators [10], as the preliminaries, the spherical mean of function f on $\mathbb{R}^n$ is defined by

$$\left(\mathcal{M}_t f\right)(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} f(x+t\theta)d\theta \; for \; x \in \mathbb{R}^n, \; t>0,$$

where σ_{n-1} is the Legesgue measure on $\mathbb{S}^{n-1}$. Then by the following operator [10]

$$\breve{\varphi}_r(x) = \int_{SO(n)} \varphi\left(\gamma\mathbb{R}^k + x + r\gamma e_n\right) d\gamma = \int_{SO(n)} \varphi\left(\gamma\tau_r + x\right) d\gamma$$

for $x \in \mathbb{R}^n$ and $r \geq 0$, the k-plane transform $(1 \leq k < n)$ [10] of f, denoted by $\hat{f}$, can be represented by the spherical mean $\mathcal{M}_t f$ of function f. If $k = n-1$, then the k-plane transform becomes the Radon transform. Finally, function f can be reconstructed from $\mathcal{M}_t f$ by the properties of the spherical mean of functions, where we recommend the readers refer to Corollary 5.3 and Theorem 5.4 in [10], and Theorem 3.4 and 3.5 in [12] for details.

The main idea of the method of Riesz potentials [6, 10] is as follows. First by the dual Radon transform R^* [5, 6, 10], defined by

$$R^*\varphi(x) = \int_{\mathbb{S}^{n-1}} \varphi(\theta, x\cdot\theta)d\theta,$$

the Radon transform Rf of function f can be transformed to its Riesz potential $I^\alpha f$ [13]. Then by the inversion methods of Riesz potentials [6, 10, 13], one can obtain f from $I^\alpha f$, that is, in form

$$I^{-\alpha} I^\alpha f = f.$$

The convolution-backprojection method [5] is the most important and popular one for the inversion of the Radon transforms. In this method, first it is given a bounded and Lebesgue measurable function ω on $\mathbb{S}^{n-1} \times \mathbb{R}$. Then by the dual Radon transform R^*, a relationship is derived, that is

$$R^*\left(\omega * Rf\right) = K * f,$$

where $K = R^*\omega$. Thus, the Radon transform Rf can be converted to a convolution $K * f$. Finally, by the identity approximation theorems of the convolution type operators, f can be obtained from $K * f$.

The method of continuous ridgelet transforms [11] is a generalization of the method of the convolution-backprojection. And its main idea is as follows. First, by two sufficiently good and even given functions u and v on $\mathbb{R}$, two convolution-dilation types of operators U_a and V_a^* [11] are defined by

$$(U_a f)(\theta, t) = \frac{1}{a}\int_{\mathbb{R}^n} f(x) u\left(\frac{t - x\cdot\theta}{a}\right) dx$$
$$(V_a^* \varphi)(x) = \frac{1}{a}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}} \varphi(\theta, t) v\left(\frac{t - x\cdot\theta}{a}\right) dt d\theta,$$

where $x \in \mathbb{R}^n$, $(\theta, t) \in \mathbb{S}^{n-1}\times\mathbb{R}$. Then by U_a and V_a^*, Rf can be expressed by a convolution $K_a * f$, where

$$K = u * v, \text{ and } K_a = \frac{1}{a^n} k\left(\frac{\cdot}{a}\right).$$

Finally, function f can be obtained from $K_a * f$.

In this article, we apply a different method, we call it sphere method, for the inverse problems of the Radon transforms. We find that this method is suitable for the inverse problems of the Radon transforms for functions with special support. The main idea of this method depends on the fact that the integral on a plane, not passing through the origin, can be transformed to the integral on a hemisphere [8] [6, VII.2]. By this reason, we can rewrite the Radon transform as the integral transform on a hemisphere. In cooperation with the the special properties of some functions, for example, the continuity and boundedness, we obtain some inverse formulas of the Radon transform of functions with special supports. Our inverse formulas are concise and easy to operate for the computers. Thus they may be applied in the imaging techniques of computerized tomography [1, 2, 5, 6]. And actually, in practical applications, the functions waiting to be reconstructed are always compactly supported.

This article mainly includes three parts. In Section 2, we introduce the inverse formulas of the Radon transforms of functions with support in a cone with vertex at the origin and flare angle less than $\pi/2$. Our results are in the point-wise and L^p-norm senses. In Section 3, we offer the inverse formulas of the Radon transforms of functions with general compact support. In the last Section 4, we show that our method is not suitable for deriving the inverse formulas of the Radon transforms for general functions without support contained in any cone, or without any compact support, and a counterexample is given in this section.

2. THE INVERSE PROBLEMS OF THE RADON TRANSFORMS OF FUNCTIONS SUPPORTED IN A CONE WITH VERTEX AT THE ORIGIN

In this section, using a different method, which is named sphere method by us, we derive the inverse formulas of the Radon transforms of functions supported in a cone with vertex at the origin and flare angle less than π, in the point-wise and L^p-norm senses. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$. For $\theta \in \mathbb{S}^{n-1}$, denote

$$\mathbb{S}_\theta^+ = \{\xi \in \mathbb{S}^{n-1} : \xi \cdot \theta > 0\}. \tag{3}$$

Then by [8][6, VII.2], the Radon transform of function f on $\mathbb{R}^n$ can be written as

$$Rf(\theta, t) = \int_{\mathbb{S}_\theta^+} f\left(\xi \frac{t}{\xi \cdot \theta}\right) \frac{|t|^{n-1}}{|\xi \cdot \theta|^n} d\xi \tag{4}$$

for $(\theta, t) \in \mathbb{S}^{n-1} \times \mathbb{R}^1$, which is the core of the sphere method, and also is the origin of the name "sphere method" since it changes the integral on plane into the integral on a hemisphere by (4). Let $\Gamma_{o,\omega,\alpha}$ the cone in $\mathbb{R}^n$, with vertex at the origin and flare angle 2α $(0 < 2\alpha < \pi/2)$, that is (see Fig. 1)

$$\Gamma_{o,\omega,\alpha} = \{y \in \mathbb{R}^n : y \cdot \omega > |y| \cos \alpha\}, \tag{5}$$

where the vector $\omega \in \mathbb{S}^{n-1}$ is in the axis of $\Gamma_{o,\omega,\alpha}$.

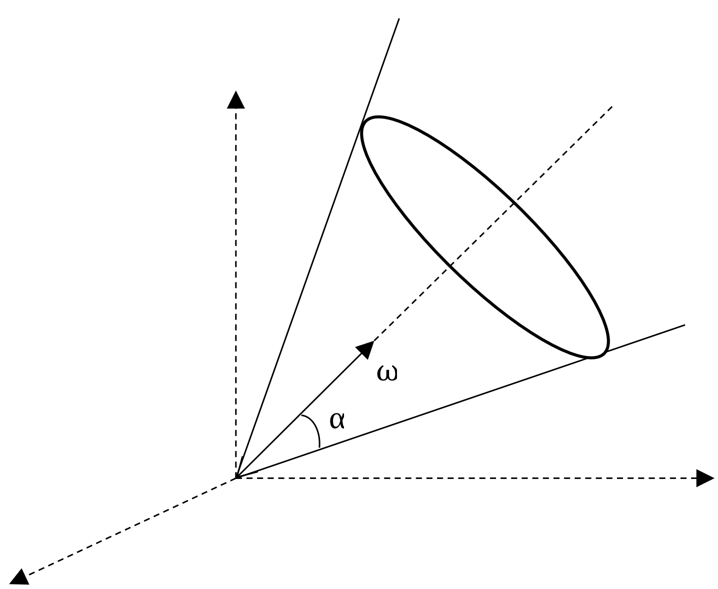

Figure 1 : A cone with vertex at the origin, axis direction ω and and flare angle 2α $(0 < 2\alpha < \pi)$.

Note that the cone $\Gamma_{o,\omega,\alpha}$ is an open set in $\mathbb{R}^n$ since it does not contain the points in its surface. Our results are stated as follows.

***Theorem* 2.1** — *Suppose that* $\text{supp}f \subset \Gamma_{o,\omega,\alpha}(0 < \alpha < \pi/4)$, $D = \mathbb{S}^{n-1} \cap \Gamma_{o,\omega,\alpha}$ *and* $\varphi = Rf$. *For* $x \in \Gamma_{o,\omega,\alpha}$, *let*

$$M_t^* \varphi(x) = t^{1-n} \int_D \varphi(\theta, t + x \cdot \theta) d\theta \ for\ t > 0. \tag{6}$$

Then

(1) If f is bounded on $\Gamma_{o,\omega,\alpha}$ and continuous at x, then

$$M_t^* \varphi(x) \to Cf(x)\ as\ t \to 0^+; \tag{7}$$

(2) If $f \in L^p(\mathbb{R}^n)(1 \leq p < \infty)$ then in the $L^p(\mathbb{R}^n)$-norm sense,

$$M_t^* \varphi(x) \to Cf(x)\ as\ t \to 0^+. \tag{8}$$

Above, the constant

$$C = \int_D \int_D \frac{1}{|y' \cdot \theta|^n} dy' d\theta \tag{9}$$

depends only on the flare angle 2α of $\Gamma_{o,\omega,\alpha}$, where $D = \mathbb{S}^{n-1} \cap \Gamma_{o,\omega,\alpha}$.

Next, we use the sphere method to prove this theorem.

PROOF (1). By the translation-shift theorem of the Radon transform [5, Sec. 2.3], we know that

$$Rf(\theta, t + x \cdot \theta) = Rf_x(\theta, t), \tag{10}$$

where $f_x(y) = f(x + y)$. Hence,

$$\varphi(\theta, t + x \cdot \theta) = \int_{H(\theta,t)} f(x + y) dy_H. \tag{11}$$

Substituting (11) into (6) we get

$$M_t^* \varphi(x) = t^{1-n} \int_D \int_{H(\theta,t)} f(x + y) dy_H d\theta, \tag{12}$$

where $D = \mathbb{S}^{n-1} \cap \Gamma_{o,\omega,\alpha}$.

Because supp$f \subset \Gamma_{o,\omega,\alpha}$, supp$f_x \subset \Gamma_{-x,\omega,\alpha}$. By (4) we have for $\theta \in D$,

$$\int_{H(\theta,t)} f(x+y)dy_H = \int_{\mathbb{S}_\theta^+} f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{t^{n-1}}{|y'\cdot\theta|^n}dy'$$
$$= \int_D f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{t^{n-1}}{|y'\cdot\theta|^n}dy'. \quad (13)$$

Therefore, (12) and (13) gives

$$M_t^*\varphi(x) = t^{1-n}\int_D\int_D f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{t^{n-1}}{|y'\cdot\theta|^n}dy'd\theta$$
$$= \int_D\int_D f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta. \quad (14)$$

It is easily seen that under the assumptions of (i)

$$\int_D\int_D f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta \to Cf(x) \text{ as } t\to 0^+, \quad (15)$$

where C is defined by (9), and is independent of the direction ω of axis of the cone $\Gamma_{o,\omega,\alpha}$. Combination of (14) and (15) gives (7).

(ii) If $1 \le p < \infty$, then $Rf(\theta,t)$ exists and is finite for almost all $t \in (0,\infty)$, for $\theta \in D$. Under the assumptions of (ii), by (14) and the generalized Minkowski inequality, we obtain

$$\left(\int_{\Gamma_{0,\omega,\alpha}} |M_t^*\varphi(x) - Cf(x)|^p dx\right)^{1/p}$$
$$= \left(\int_{\Gamma_{0,\omega,\alpha}}\left|\int_D\int_D f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta - \int_D\int_D f(x)\frac{1}{|y'\cdot\theta|^n}dy'd\theta\right|^p dx\right)^{1/p}$$
$$\le \int_D\int_D\left(\int_{\Gamma_{0,\omega,\alpha}}\left|f\left(x + y'\frac{t}{y'\cdot\theta}\right) - f(x)\right|^p dx\right)^{1/p}\frac{1}{|y'\cdot\theta|^n}dy'd\theta. \quad (16)$$

By the Lebesgue dominated convergence theorem and the integral continuity of L^p functions, it follows that

$$\int_D\int_D\left(\int_{\Gamma_{0,\omega,\alpha}}\left|f\left(x + y'\frac{t}{y'\cdot\theta}\right) - f(x)\right|^p dx\right)^{1/p}\frac{1}{|y'\cdot\theta|^n}dy'd\theta \to 0 \text{ as } t\to 0^+. \quad (17)$$

Finally, (8) can be obtained from (16) and (17). □

Remark 1 : The constant

$$C = \int_D\int_D \frac{1}{|y'\cdot\theta|^n}dy'd\theta \quad (18)$$

exists and makes sense, because for $y' \in D$ and $\theta \in D$, $|y'\cdot\theta|$ belongs to $[\cos 2\alpha, 1](0 < \cos 2\alpha < 1)$. And moreover $C \le |D|^2/(\cos 2\alpha)^n$, where $|D|$ denotes the Lebesgue measure of D in the $n-1$ dimensional Euclidean space.

3. THE INVERSE PROBLEMS OF THE RADON TRANSFORMS OF FUNCTIONS WITH COMPACT SUPPORT

In this section, we using the sphere method to deal with the inverse problems of the Radon transforms of functions with compact support. First, by translation we rewrite the functions with compact support as the functions supported in a cone with vertex at the origin and flare angle less than π. Then by the similar procedures of the proof of Theorem 2.1, we obtain the inverse formulas of the Radon transforms of functions with compact support. Denote by $B(0, r)$ the open ball centering at the origin with radius $r > 0$ in $\mathbb{R}^n$. Our results are as follows, which can be regarded as the corollary of Theorem 2.1.

Corollary 3.1 — Suppose that supp$f \subset B(0, M)(M > 0)$, function f is bounded on $\mathbb{R}^n$ and continuous at x in $B(0, M)$; and that the cone $\Gamma_{o,\omega,\alpha}$ satisfies $B(ra, M) \subset \Gamma_{o,\omega,\alpha}(M > 0)$ with $a = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1) \in \mathbb{S}^n$ for some $r > 0$. Then

$$M_t^* \varphi(x) \to Cf(x) \text{ as } t \to 0^+, \tag{19}$$

where M_t^* is defined by (1), $\varphi = Rf$, $D = \mathbb{S}^{n-1} \cap \Gamma_{o,\omega,\alpha}(0 < \alpha \leq \pi/4)$, and

$$C = \int_D \int_D \frac{1}{|y' \cdot \theta|^n} dy' d\theta. \tag{20}$$

PROOF : Let

$$F(y) = f_{-ra}(y) = f(y - ra) \text{ for } y \in \mathbb{R}^n \text{ and } r > 0. \tag{21}$$

Due to that supp$f \subset B(0, M)$, there exist $r > 0$ and $0 < \alpha < \pi/2$ such that

$$\text{supp}F \subset B(ra, M) \subset \Gamma_{o,\omega,\alpha}$$

for $a = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1) \in \mathbb{S}^n$.

Let

$$\varphi_1(\theta, t) = RF(\theta, t). \tag{22}$$

Then by the translation-shift theorem of the Radon transform [5, Sec. 2.3],

$$\varphi_1(\theta, t) = \varphi(\theta, t - ra \cdot \theta). \tag{23}$$

By (6) and (23), for $y \in \Gamma_{o,\omega,\alpha}$

$$M_t^*\varphi_1(y) = t^{1-n}\int_D \varphi_1(\theta, t + y\cdot\theta)d\theta = t^{1-n}\int_D \varphi(\theta, t + (y - ra)\cdot\theta)d\theta. \tag{24}$$

By (i) of Theorem 2.1,

$$M_t^*\varphi_1(y) \to CF(y) \text{ as } t \to 0^+, \tag{25}$$

where

$$C = \int_D\int_D \frac{1}{|y'\cdot\theta|^n}dy'd\theta. \tag{26}$$

Hence,

$$M_t^*\varphi_1(x + ra) \to CF(x + ra) \text{ as } t \to 0^+ \text{ for } x \in B(0, M). \tag{27}$$

It follows from (21) that

$$F(x + ra) = f(x) \text{ for } x \in B(0, M). \tag{28}$$

Combination of (27) and (28) gives

$$M_t^*\varphi_1(x + ra) \to Cf(x) \text{ as } t \to 0^+ \text{ for } x \in B(0, M). \tag{29}$$

Letting $y = x + ra$ for $x \in B(0, M)$ in (24), leads to

$$M_t^*\varphi_1(y) = M_t^*\varphi(x). \tag{30}$$

Finally, combination of (29) and (30) gives

$$M_t^*\varphi(x) \to Cf(x) \text{ as } t \to 0^+ \text{ for } x \in B(0, M), \tag{31}$$

which completes the proof.

Corollary 3.2 — Suppose that supp$f \subset B(0, M)(M > 0)$, $f \in L^p(\mathbb{R}^n)(1 \le p < n/(n-1))$ and M_t^* is defined by (6). Then in the $L^p(\mathbb{R}^n)$-norm sense,

$$M_t^*\varphi(x) \to Cf(x) \text{ as } t \to 0^+, \tag{32}$$

where C is defined by (20).

PROOF : The main idea of the proof follows from that of Theorem 2.1 (ii) and Corollary 3.1. □

4. The Sphere Method is Invalid for The Inverse Problems of The Radon Transforms of General Functions

In this section, we give a counterexample to illustrate that the sphere method is not suitable for finding the inverse formulas of the Radon transforms of general functions without any support contained in a cone, or without any compact support.

Let

$$f(x) = \exp(-|x|) \text{ for } x \in \mathbb{R}^n. \tag{33}$$

Next let us try to use the sphere method of section 2 to reconstruct f from its Radon transforms φ.

Imitating the process of Section 2, put x in a cone $\Gamma_{0,\alpha}(0 < \alpha < \pi/4)$, and let $D = \mathbb{S}^{n-1} \cap \Gamma_{0,\alpha}$. By (4) we have

$$\int_{H(\theta,t)} f(x+y)dy_H = \int_{\mathbb{S}_\theta^+} f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{t^{n-1}}{|y'\cdot\theta|^n}dy'. \tag{34}$$

By (10), (12) and (34), we have

$$\begin{aligned} M_t^*\varphi(x) &= t^{1-n}\int_D\int_{\mathbb{S}_\theta^+} f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{t^{n-1}}{|y'\cdot\theta|^n}dy'd\theta \\ &= \int_D\int_{\mathbb{S}_\theta^+} f\left(x + y'\frac{t}{y'\cdot\theta}\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta \\ &= \int_D\int_{\mathbb{S}_\theta^+} \exp\left(-\left|x + y'\frac{t}{y'\cdot\theta}\right|\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta. \end{aligned} \tag{35}$$

Next we estimate $M_t^*\varphi(x)$.

First, from (35) it follows that

$$\frac{M_t^*\varphi(x)}{f(x)} = \int_D\int_{\mathbb{S}_\theta^+} \exp\left(|x| - \left|x + y'\frac{t}{y'\cdot\theta}\right|\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta. \tag{36}$$

The right-hand side of (36) is no less than

$$\begin{aligned} &\int_D\int_{\mathbb{S}_\theta^+} \exp\left(|x| - \left(|x| + \frac{t}{y'\cdot\theta}\right)\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta \\ &= \int_D\int_{\mathbb{S}_\theta^+} \exp\left(-\frac{t}{y'\cdot\theta}\right)\frac{1}{|y'\cdot\theta|^n}dy'd\theta. \end{aligned} \tag{37}$$

Now we need using a formula in [6, VII.2], stated as follows. Suppose that f is a function on $[-1,1]$. Then

$$\int_{\mathbb{S}^{n-1}} f(\theta\cdot\xi)\,d\xi = \left|\mathbb{S}^{n-2}\right| \int_{-1}^{1} f(t)\left(1-t^2\right)^{(n-3)/2} dt \tag{38}$$

for all $\theta \in \mathbb{S}^{n-1}$, provided that at least one of the integrals exists. According to (38), we obtain

$$\int_{\mathbb{S}_\theta^+} \exp\left(-\frac{t}{y'\cdot\theta}\right) \frac{1}{|y'\cdot\theta|^n} dy' = |\mathbb{S}^{n-2}| \int_0^1 \exp\left(-\frac{t}{s}\right) \frac{(1-s^2)^{(n-3)/2}}{s^n} ds. \tag{39}$$

Thus by (37) and (39), we have the right-hand side of (36) is no less than

$$|\mathbb{S}^{n-2}||D| \int_0^1 \exp\left(-\frac{t}{s}\right) \frac{(1-s^2)^{(n-3)/2}}{s^n} ds \tag{40}$$

For $t>0$,

$$\lim_{s\to 0^+} \exp\left(-\frac{t}{s}\right)\frac{1}{s^n} = \lim_{s\to+\infty} \frac{s^n}{\exp(ts)} = \lim_{s\to+\infty} \frac{n!}{t^n \exp(ts)} = 0. \tag{41}$$

It can be seen from (41) that the integral of (40) is finite for $t \in (0,1)$, but is not bounded on $(0,1)$ with respect to t, and it tends to positive infinite as t tends to 0. Hence, in cooperation with (36), (4) and (41), we have

$$\frac{M_t^*\varphi(x)}{f(x)} \to +\infty \text{ as } t \to 0^+. \tag{42}$$

Therefore, there does not exist any constant C such that

$$M_t^*\varphi(x) \to Cf(x) \text{ as } t \to 0^+. \tag{43}$$

5. Conclusion

In Section 2 and 3, using a different method, called sphere method by us, we obtained the inverse formulas of the Radon transforms of functions with support in a cone with vertex at the origin and flare angle less than $\pi/2$, or with compact support. In the last Section 4, an example illustrates that the sphere method is invalid for the inverse problems of the Radon transforms of general functions, such as ones without any support contained in a cone or without any compact support.

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