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SUFFICIENT CONDITIONS FOR EXISTENCE OF INTEGRAL SOLUTION FOR NON-INSTANTANEOUS IMPULSIVE FRACTIONAL EVOLUTION EQUATIONS

Jayanta Borah and Swaroop Nandan Bora

Department of Mathematics, Indian Institute of Technology Guwahati,
Guwahati 781 039, *India*
e-mails: b.jayanta@iitg.ac.in; swaroop@iitg.ac.in

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In this article, we establish sufficient conditions for existence and uniqueness of integral solution for some non-densely defined non-instantaneous impulsive evolution equations on a Banach space involving Caputo fractional derivative. The results are obtained by means of characteristic functions based on probability density. Finally, the main results are illustrated through examples.

Key words : Fractional evolution equation; integral solution; non-instantaneous impulse; fixed point theorem.

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1. INTRODUCTION

The study of differential equations with abrupt and instantaneous impulse (processes which at certain instants change their state rapidly) has been a subject of great interest due to its wide applications in physics, economics, population dynamics, control theory etc. There exists extensive literature on the existence and qualitative properties of solutions for differential equations with instantaneous impulses. For developments in the study of mild solutions and its qualitative properties to instantaneous impulsive differential equations, we refer the readers to [3, 6, 21, 26, 30] and the references therein. However, the action of instantaneous impulses does not describe dynamics of some physical processes. For example, the introduction of drugs in the bloodstream and the consequent absorption for the body, which are gradual and continuous processes, is not appropriately explained by instantaneous impulses. In fact this situation should be characterized by a new idea of impulsive action, which starts at a certain point and the action continues for some finite interval. Hernández and O'Regan [13] initiated the study of this new class of abstract semilinear impulsive differential equations with non-instantaneous impulse in a PC-normed space of the following form:

$$x'(t) = Ax(t) + f(t, x(t)), t \in (s_i, t_{i+1}], i = 0, 1, ..., N, \tag{1.1}$$

$$x(t) = g_i(t, x(t)), t \in (t_i, s_i], i = 1, 2, ..., N, \tag{1.2}$$

$$x(0) = x_0, \tag{1.3}$$

where $A : D(A) \subset X \longrightarrow X$ is a densely defined closed linear operator on a Banach space X and $0 = s_0 < t_1 \leq s_1 \leq t_2 \cdots \leq t_N \leq s_N < t_{N+1} = T$ is a partition of the interval $J = [0, T]$, the functions $g_i \in C((t_i, s_i] \times X, X)$ for each $i = 1, 2, ...N$ and $f : [0, T] \times X \to X$ is a suitable function. Pierri *et al.* [23] studied the existence and uniqueness of (1.1)-(1.3) in the fractional power space using the theory of analytic semigroups. Yu and Wang [32] investigated the existence of solution with non-instantaneous impulses on Banach spaces by using the theory of semigroup and fixed point methods. Chen *et al.* [8] obtained the existence of PC-mild solution for the initial value problem to a class of semilinear evolution equations with a non-instantaneous conditions in ordered Banach spaces. On the other hand, the derivative of non-integer order, popularly known as fractional calculus, can be effectively used to describe the hereditary properties of various materials and memory processes. Being an alternative model for nonlinear differential equations, fractional differential equations have recently been recognized to be a strong tool in the mathematical modelling of many phenomena in physics, biology, mechanics, etc. For details, one can refer to [15, 19, 25] etc. In [2], Agarwal *et al.* have highlighted some basic points in introducing non-instantaneous impulses in Caputo fractional differential equations. Borah and Bora [7] recently studied the existence of mild solution of non-instantaneous impulsive fractional evolution equation involving Volterra-Fredholm type integral operators. More works on non-instantaneous impulsive differential equations in fractional setup can be found in [10, 11, 16, 17, 31].

Da Prato and Sinestrari [20] initiated the study of evolution equations with a non-densely defined linear operator. It was shown that the density condition was not necessary to deal with partial functional differential equations. The main method used in their work was based on integrated semigroup theory. Some results on existence of integral solution of non-densely defined evolution equation (1.1)-(1.3) without impulse have been proved under suitable hypotheses, for any X-valued continuous function f and any $x_0 \in \overline{D(A)}$. For more details and examples on non-densely defined operators and the concept of integrated semigroup, we refer the readers to [1, 5, 9]. Thieme [27] showed that the integral solution reduced to the mild solution when $f \in \overline{D(A)}$. Zhang and Liu [33] corrected the error in the formulation of integral solution for non-densely defined fractional differential equation with impulsive effects that was observed in some past works, e.g., [4, 22]. The solution was obtained by integrated semigroup theory and some probability densities.

Motivated by the above discussion, we consider the semilinear impulsive Cauchy problem with not instantaneous impulses in the following form:

$$^{C}D_t^{\alpha}x(t) = Ax(t) + f(t, x(t)),\ a.e.\ t \in (s_i, t_{i+1}], i = 0, 1, ..., N, \tag{1.4}$$

$$x(t) = g_i(t, x(t)), t \in (t_i, s_i], i = 1, 2, ..., N, \tag{1.5}$$

$$x(0) = x_0, \tag{1.6}$$

with $0 < \alpha < 1$; $A : D(A) \subset X \to X$ not necessarily a densely defined closed linear operator on the Banach space $(X, \|.\|)$; $f : J \times X \to X$ a given function; $g_i \in (C((t_i, s_i] \times X, \overline{D(A)}), i = 1, 2, ..., N$.

Denote by $C(J, X)$ the Banach space of continuous functions from J into X with the norm $\|x\|_{C(J,X)} = \sup_{t\in J}\|x(t)\|, x \in C(J, X)$ and $B(X)$ be the space of all bounded linear operators from X to X with norm $\|Q\|_{B(X)} = \sup\{\|Q(x)\| : \|x\| = 1\}, Q \in B(X), x \in X$.

To deal with impulsive conditions, we consider the space $PC(X)$, consisting of all functions $x : [0, T] \to X$ such that $x(.)$ is continuous at $t \neq t_i, x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exists for all $i = 1, 2, ..., N$ endowed with the uniform norm on $[0, T]$ denoted by $||x||_{PC(X)}$. For a function $x \in PC(X)$ and $i = 0, 1, \ldots, N$, we introduce the function $\tilde{x_i} \in C([t_i, t_{i+1}], X)$ given by

$$\tilde{x_i}(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}], \\ x(t_i^+), & t = t_i. \end{cases}$$

Let $B \subset PC(X)$ and we define $\tilde{B_i} = \{\tilde{x_i} : x \in B\}$.

Lemma 1.1 — A set $B \subseteq PC(X)$ is relatively compact in $PC(X)$ iff the set $\tilde{B_i}$ is relatively compact in $C([t_i, t_{i+1}], X)$.

We use contraction mapping principle and Krasnoselskii's fixed point theorem to prove the existence of the integral solution of problem (1.4)-(1.6).

The rest of this paper is organized as follows. In Section 2 we give some basic definition of fractional calculus, definitions of the integral solution and recall some known results on non-densely defined operator. In Section 3, we study the existence and the uniqueness of the integral solution for the fractional semilinear differential equation (1.4)-(1.6). An example is provided in Section 4 to illustrate our results.

2. PRELIMINARIES

In this section, we recall some basic notations, definitions and fundamental results which will be used throughout this article. First, we introduce some basic definitions and properties of fractional calculus.

Definition 2.1 — The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f \in C_\alpha$, is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{(\alpha-1)} f(s) ds, t > a,$$

provided the right side is point-wise defined on $[a, b]$.

Definition 2.2 — The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f \in C_\alpha^n, n \in \mathbb{N}$, is defined as

$${}^L D_t^\alpha f(t) = D^n D^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, t > a, n-1 < \alpha < n.$$

Definition 2.3 — The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C_\alpha^n, n \in \mathbb{N}$, is defined as

$${}^C D_t^\alpha f(t) = D^{\alpha-n} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s) ds, t > a, n-1 < \alpha < n.$$

If f is an abstract function with values in X, then the integrals which appear in the above definitions are taken in Bochner sense.

Lemma 2.4 — (Bochner Theorem). A measurable function $F : J \to X$ is Bochner's integrable if $\|F\|$ is Lebesgue integrable.

Lemma 2.5 — (Hölder's inequality). Assume that $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(J, X), g \in$ Ł$^q(J, X)$, then for $1 \leq p \leq \infty$,

$$fg \in L^1(J, X) \text{ and } \|fg\|_{L^1 J} \leq \|f\|_{L^p J} \|g\|_{L^q J}.$$

Lemma 2.6 — (Arzela-Ascoli's theorem). If a family $\Lambda = \{f(t)\}$ in $C(J, X)$ is uniformly bounded and equicontinuous on J, and for any $t^* \in J, \{f(t^*)\}$ is relatively compact, then Λ has a uniformly convergent subsequence $\{f_n(t)\}_{n=1}^\infty$.

Let $X_0 = \overline{D(A)}$ and A_0 be the part of A in $\overline{D(A)}$ defined by

$$D(A_0) = \{x_1 \in D(A) : Ax_1 \in \overline{D(A)}\},\ A_0(x_1) = A(x_1).$$

Throughout our analysis, the following hypotheses will be considered:

(H1) $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, that is, there exist two constants $\omega \in \mathbb{R}$ and $M_0 \geq 0$ such that $(\omega, \infty) \subset \rho(A)$ and

$$\|(\lambda I - A)^{-n}\|_{L(X)} \leq \frac{M_0}{(\lambda - \omega)^n}, \text{ for all } \lambda > \omega, n \geq 1.$$

(H2) The part A_0 of A generates a compact C_0-semigroup $\{(Q(t)\}_{t\geq 0}$ in X_0 which is uniformly bounded, that is, there exists $M \geq 1$ such that $\sup_{t\in[0,\infty)}\|Q(t)\| < M$.

Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. It is to be noted that $(S'(t))_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by A_0 and $\|S'(t)\| \leq Me^{\omega t}, t \geq 0, M$ and ω are the constants used in the Hille-Yosida condition.

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then for all $x_1 \in X_0, B_\lambda x_1 \to x_1$ as $\lambda \to \infty$. Also from Hille-Yosida condition, it is clear that $\lim_{\lambda\to\infty}\|B_\lambda\| \leq M_0$.

Lemma 2.8 — By the integral solution $x(t)$ of the non-homogenous fractional order evolution system (with continuous source f)

$$^C D^\alpha_{0+} x(t) = Ax(t) + f(t), t \in J = (0, a], \tag{2.1}$$

$$x(0) = x_0 \in X_0, \tag{2.2}$$

we mean a continuous function $x : J \to X$ which satisfies the following conditions:

(i) $I^\alpha_{0+}x(t) \in X_0$ for $t \in J$ and

(ii) $x(t) = x_0 + AI^\alpha_{0+}x(t) + I^\alpha_{0+}f(t), \quad t \in J.$

Lemma 2.9 — [12]. If x is an integral solution of (2.1)-(2.2), then for all $t \in J, x(t) \in X_0$. In particular, $x(0) = x_0 \in X_0$.

Definition 2.10 — [18]. The Wright function $M_\alpha(\theta)$ is defined by

$$M_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, 0 < \alpha < 1, \theta \in \mathbb{C}.$$

It has the following property:

$$\int_0^\infty \theta^\delta M_\alpha(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \text{ for } \delta \geq 0.$$

Lemma 2.11 — [12]. The integral solution $x(t) = x_0 + A_0 I^{\alpha}_{0+} x(t) + I^{\alpha}_{0+} f(t),$ for $t \in J, x_0 \in X_0$ of the auxiliary problem

$$\begin{aligned} {}^{C}D^{\alpha}_{0+} x(t) &= A_0 x(t) + f(t), t \in (0, a], \qquad (2.3) \\ x(0) &= x_0, \qquad (2.4) \end{aligned}$$

can be expressed as

$$x(t) = S_\alpha(t) x_0 + \lim_{\lambda \to \infty} \int_0^\infty K_\alpha(t-s) B_\lambda f(s) ds,$$

where,

$$S_\alpha(t) = I^{1-\alpha}_{0+} K_\alpha(t), K_\alpha(t) = t^{\alpha-1} P_\alpha(t), P_\alpha(t) = \int_0^\infty \alpha\theta M_\alpha(\theta) Q(t^\alpha \theta) d\theta.$$

Lemma 2.12 — [34]. $P_\alpha(t)$ is continuous in the uniform operator topology for $t > 0$.

Lemma 2.13 — [35]. For any fixed $t > 0$, $K_\alpha(t)$ and $S_\alpha(t)$ are linear operators, and for any $x_1 \in X_0$,

$$\|K_\alpha(t) x_1\| \leq \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \|x_1\| \text{ and } \|S_\alpha(t) x_1\| \leq M \|x_1\|.$$

Lemma 2.14 — [35]. $\{K_\alpha(t)\}_{t>0}$ and $\{S_\alpha(t)\}_{t>0}$ are strongly continuous.

***Theorem* 2.15** — [12]. *$x(t)$ is an integral solution of (2.1)-(2.2) if and only if*

$$x(t) = S_\alpha(t) x_0 + \lim_{\lambda \to \infty} \int_0^\infty K_\alpha(t-s) B_\lambda f(s) ds, \ for\ t \in J\ and\ x_0 \in X_0.$$

Definition 2.16 — [12]. The operator defined by

$$\phi_\alpha(t) x_1 = \lim_{\lambda \to \infty} \int_0^\infty K_\alpha(t-s) B_\lambda x_1 ds = \lim_{\lambda \to \infty} \int_0^\infty K_\alpha(s) B_\lambda x_1 ds \text{ for } x_1 \in X \text{ and } t \geq 0$$

exist as a bounded linear operator for $x \in X$ and $t \geq 0$.

Remark : [12]. We say that A generates the operator $\{\phi_\alpha(t)\}_{t \geq 0}$. When $\alpha = 1$, $\{\phi_\alpha(t)\}_{t \geq 0}$ degenerates into $\{S(t)\}_{t \geq 0}$, which is the integrated semigroup generated by A in [14].

Lemma 2.17 — (Krasnoselskii's fixed point theorem). Let B be a closed, convex and nonempty subset of a Banach space X . Let Q_1 and Q_2 be two operators such that

- $Q_1 x_1 + Q_2 x_2 \in B$ whenever $x_1, x_2 \in B$;
- Q_1 is a contraction mapping;
- Q_2 is compact and continuous.

Then there exists $z \in B$ such that $z = Q_1 z + Q_2 z$.

3. Integral Solution to a Nonlinear Cauchy Problem

Here we take $a = \frac{\alpha-1}{1-\alpha_1} \in (-1, 0)$.

Motivated by the theorem 3.13 of [12], we adopt the following concept of integral solution of our problem:

Definition 3.1 — A function $x \in PC(X)$ is said to be an integral solution of the Cauchy problem (1.4)-(1.6) if it satisfies $x(0) = x_0 \in X_0, \quad x(t) = g_i(t, x(t))$ for all $t \in (t_i, s_i],\ i = 1, 2, \ldots, N$:

$$x(t) = S_\alpha(t)x_0 + \lim_{\lambda\to\infty} \int_0^t K_\alpha(t-s)B_\lambda f(s, x(s))ds, \quad t \in [0, t_1] \text{ and}$$

$$x(t) = S_\alpha(t-s_i)g_i(s_i, x(s_i)) + \lim_{\lambda\to\infty} \int_{s_i}^t K_\alpha(t-s)B_\lambda f(s, x(s))ds, \quad t \in [s_i, t_{i+1}],\ i = 1, 2, \ldots, N.$$

To study the existence and uniqueness of the integral solution of impulsive factional evolution equation, we need the following assumptions:

(H3) $f : J \times X \to X$ is continuous and there exists a constant $\alpha_1 \in (0, \alpha)$ and a function $\mu \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, x_1) - f(t, x_2)\| \le \mu(t)\|x_1 - x_2\| \text{ for all } x_1, x_2 \in X \text{ and almost all } t \in J.$$

(H4) The functions $g_i \in C((t_i, s_i] \times X, X_0)$ and there are positive constants L_{g_i} such that $\|g_i(t, x_1) - g_i(t, x_2)\| \le L_{g_i}\|x_1 - x_2\|$ for all $x_1, x_2 \in X$, $t \in (t_i, s_i],\ i = 1, 2, ..., N$.

***Theorem* 3.2** — *Assume the hypotheses (H1)-(H4) to hold and*

$$k = \max\left\{ \max_{1\le i\le N}\left\{ ML_{g_i} + \frac{MM_0}{\Gamma(\alpha)} \frac{(t_{i+1}-s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|\mu\|_{L^{\alpha_1}([s_i,t_{i+1}])} \right\}, \frac{MM_0}{\Gamma(\alpha)} \frac{t_1^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|\mu\|_{L^{\alpha_1}([0,t_1])} \right\} < 1.$$

Then there exists a unique integral solution in $PC(X)$ of the problem (1.4)-(1.6) provided $x_0 \in X_0$.

Proof : Define the operator $F : PC(X) \to PC(X)$ by $Fx(0) = x_0, Fx(t) = g_i(t, x(t))$ for $t \in (t_i, s_i]$ and

$$Fx(t) = S_\alpha(t-s_i)g_i(s_i, x(s_i)) + \lim_{\lambda\to\infty} \int_{s_i}^t K_\alpha(t-s)B_\lambda f(s, x(s))ds, t \in [s_i, t_{i+1}],\ i \ge 0.$$

By the hypothesis $(H3)$ and Hölder's inequality, the operator F is well-defined.

Let $x, y \in PC(X)$. For $t \in [s_i, t_{i+1}], i = 1, 2, ..., N$, we get

$$\begin{aligned}
\|Fx(t) - Fy(t)\| &\leq \|S_\alpha(t - s_i)g_i(s_i, x(s_i)) - S_\alpha(t - s_i)g_i(s_i, y(s_i))\| \\
&+ \lim_{\lambda\to\infty} \int_{s_i}^{t} \|K_\alpha(t - s)B_\lambda f(s, x(s)) - K_\alpha(t - s)B_\lambda f(s, y(s))\| ds \\
&\leq ML_{g_i}\|x - y\|_{PC} + \frac{MM_0}{\Gamma(\alpha)} \int_{s_i}^{t} (t - s)^{\alpha-1}\mu(s)ds\|x - y\|_{PC} \\
&\leq ML_{g_i}\|x - y\|_{PC} + \frac{MM_0}{\Gamma(\alpha)} \left[\int_{s_i}^{t} ((t - s)^{\alpha-1})^{\frac{1}{1-\alpha_1}} ds\right]^{1-\alpha_1} \\
&\quad \|\mu\|_{L^{\alpha_1}([s_i,t_{i+1}])}\|x - y\|_{PC} \\
&\leq \left[ML_{g_i} + \frac{MM_0}{\Gamma(\alpha)} \frac{(t_{i+1} - s_i)^{(1+a)(1-\alpha_1)}}{(1 + a)^{1-\alpha_1}} \|\mu\|_{L^{\alpha_1}([s_i,t_{i+1}])}\right] \|x - y\|_{PC}.
\end{aligned}$$

For $t \in [0, t_1]$,

$$\|Fx(t) - Fy(t)\| \leq \left[\frac{MM_0}{\Gamma(\alpha)} \frac{{t_1}^{(1+a)(1-\alpha_1)}}{(1 + a)^{1-\alpha_1}} \|\mu\|_{L^{\alpha_1}([0,t_1],\mathbb{R}^+)}\right] \|x - y\|_{PC}.$$

For $t \in (t_i, s_i], i = 1, 2, ..., N$, we have

$$\|Fx(t) - Fy(t)\| \leq L_{g_i}\|x - y\|_{PC} \leq ML_{g_i}\|x - y\|_{PC}.$$

From above, we observe that

$$\|F(x) - F(y)\|_{PC} \leq k\|x - y\|_{PC},$$

which implies that $F(.)$ is a contraction and there exists a unique integral solution of (1.4)-(1.6).

To prove the next theorem , we add the following assumptions:

For $r > 0$, let B_r be a closed ball in $PC(X)$ with radius r and center at 0, that is,

$$B_r = \{x \in PC(J, X_0) : \|x\|_{PC} \leq r\}.$$

Then B_r is a closed, convex and bounded set in $PC(X)$.

(H5) For each $t \in J$, the function $f(t, .) : X \to X$ is continuous and for each $x_1 \in X$, the function $f(., x_1) : J \to X$ is strongly measurable.

(H6) There exists a constant $\alpha_1 \in (0, \alpha)$ and a function $m \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t,x)\| \leq m(t) \text{ for all } x \in B_r \text{ and almost all } t \in J.\square$$

***Theorem* 3.3** — *Assume that the conditions (H1)-(H6) are satisfied with the exception of condition* $(H3)$*, the functions* $g_i(.,0)$ *are bounded and*

$$\kappa = (M+1)L_{g_i} < 1, \quad \forall\, i = 1, 2, \ldots, N.$$

Then there exists at least one integral solution in B_r *of the problem (1.4)-(1.6).*

PROOF : Let $r > 1$ and $0 < \eta < 1$ be such that

$$M\|x_0\| + (1+M) \max_{i=1,2,\ldots,N} \|g_i(.,0)\|_{C((t_i,s_i],X)} < (1-\eta)r,$$
$$\max_{i=1,2,\ldots,N} \left\{ (1+M)L_{g_i} + \frac{1}{\nu} \frac{MM_0(t_{i+1}-s_i)^{(1+a)(1-\alpha_1)}}{\Gamma(\alpha)(1+a)^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}([s_i,t_{i+1}])} \right\} \leq \eta, \quad \nu \geq r,$$
$$\frac{1}{\nu} \frac{MM_0 t_1^{(1+a)(1-\alpha_1)}}{\Gamma(\alpha)(1+a)^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}([0,t_1])} < \eta, \quad \nu \geq r.$$

For any $x \in B_r$, we define the operator F as follows

$$F = \sum_{i=0}^{m} F_i^1 + \sum_{i=0}^{m} F_i^2 = F^1 + F^2,$$

where

$$(F_i^1 x)(t) = \begin{cases} g_i(t, x(t)), & t \in (t_i, s_i],\ i \geq 1, \\ S_\alpha(t-s_i) g_i(s_i, x(s_i)), & t \in (s_i, t_{i+1}],\ i \geq 1, \\ 0, & t \notin (t_i, t_{i+1}],\ i \geq 0, \\ S_\alpha(t)x_0, & t \in [0, t_1], \end{cases}$$
$$(F_i^2 x)(t) = \begin{cases} \lim_{\lambda\to\infty} \int_{s_i}^t K_\alpha(t-s)B_\lambda f(s, x(s))ds, & t \in (s_i, t_{i+1}],\ i \geq 0, \\ 0, \quad \text{otherwise}. \end{cases}$$

We claim that the map F is a χ-contraction map from B_r into B_r. This consists of the following steps.

Step I : To show that $F(B_r) \subset B_r$.

Let $x \in B_r$. For $i \geq 1$, and $t \in (t_i, t_{i+1}]$,

$$\begin{aligned}
\|(Fx)(t)\| &\leq L_{g_i}\|x(t)\| + \|g_i(t,0)\| + M(L_{g_i}\|x(t)\| + \|g_i(t,0)\|) + \| \\
&\quad \lim_{\lambda\to\infty}\int_{s_i}^t K_\alpha(t-s)B_\lambda f(s,x(s))ds\| \\
&\leq (M+1)L_{g_i}r + (1+M)\|g_i(.,0)\|_{C((t_i.s_i],X)} + \frac{MM_0}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}m(s)ds \\
&\leq (M+1)L_{g_i}r + (1-\eta)r + \frac{MM_0}{\Gamma(\alpha)}\left[\int_{s_i}^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right]^{1-\alpha_1}\|m\|_{L^{\frac{1}{\alpha_1}}([s_i,t_{i+1}])} \\
&\leq (M+1)L_{g_i}r + (1-\eta)r + \frac{MM_0(t_{i+1}-s_i)^{(1+a)(1-\alpha_1)}}{\Gamma(\alpha)(1+a)^{1-\alpha_1}}\|m\|_{L^{\frac{1}{\alpha_1}}([s_i,t_{i+1}]} \\
&\leq (1-\eta)r + \eta r \\
&= r.
\end{aligned}$$

Therefore,

$$\|Fx\|_{C((t_i,t_{i+1}],X)} \leq r \quad \text{for all } i \geq 1.$$

For $t \in [0, t_1]$,

$$\begin{aligned}
\|(Fx)(t)\| &\leq \|S_\alpha(t)x_0\| + \|\lim_{\lambda\to\infty}\int_0^t K_\alpha(t-s)B_\lambda f(s,x(s))ds\| \\
&\leq M\|x_0\| + \frac{MM_0t_1^{(1+a)(1-\alpha_1)}}{\Gamma(\alpha)(1+a)^{1-\alpha_1}}\|m\|_{L^{\frac{1}{\alpha_1}}([0,t_1])} \\
&\leq r.
\end{aligned}$$

Therefore,

$$\|Fx\|_{C([0,t_1],X)} \leq r.$$

Thus it follows that $\|Fx\|_{PC} \leq r$ and F maps B_r into B_r.

By the same line of argument, one can show that for any $x, y \in B_r$, $F^1x + F^2y \in B_r$.

Step II : To show that the map $F^1 = \sum_{i=0}^N F_i^1$ is a contraction on B_r.

Let $x, y \in B_r$, $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, N$. Then

$$\|(F_i^1x)(t) - (F_i^1y)(t)\| \leq (M+1)L_{g_i}\|x-y\|_{C((t_i,t_{i+1}],X)}.$$

Therefore,

$$\|\sum_{i=0}^{N} F_i^1 x - \sum_{i=0}^{N} F_i^1 y\|_{PC(X)} \le \kappa \|x-y\|_{PC(X)}$$

and F^1 is a contraction on B_r.

Step III : To show that for $i = 0, 1, \ldots, N$ and for any $s_i < s < t < t_{i+1}$, the set $V(u) = \cup_{u\in[s,t]}\{(F_i^2 x)(u) : x \in B_r\}$ is relatively compact in X. For $\forall\epsilon \in (s_i, s)$ and $\delta > 0$, we define an operator $(F_i^2)_{\epsilon,\delta}$ on B_r by the formula

$$\begin{aligned}
((F_i^2)_{\epsilon,\delta}x)(u) &= \lim_{\lambda\to\infty} \alpha \int_{s_i}^{u-\epsilon}\int_{\delta}^{\infty} \theta(u-s)^{\alpha-1} M_\alpha(\theta) Q((u-s)^\alpha\theta) B_\lambda f(s,x(s))dsd\theta \\
&= \alpha\epsilon^\alpha\delta \lim_{\lambda\to\infty} \int_{s_i}^{u-\epsilon}\int_{\delta}^{\infty} \theta(u-s)^{\alpha-1} M_\alpha(\theta) Q((u-s)^\alpha\theta - \epsilon^\alpha\delta) B_\lambda f(s,x(s))dsd\theta.
\end{aligned}$$

From the compactness of $Q(\epsilon^\alpha\delta)$, ($\epsilon^\alpha\delta > 0$), we obtain that the set $V_{\epsilon,\delta}(u) = \{\big((F_i^2)_{\epsilon,\delta}x\big)(u) : x \in B_r\}$ is relatively compact in X for $\forall\epsilon \in (s_i, s)$ and $\delta > 0$. Moreover, for any $x \in B_r$, we have

$$\begin{aligned}
\|(F_i^2 x)(u) - ((F_i^2)_{\epsilon,\delta}x)(u)\| &\le \|\alpha \lim_{\lambda\to\infty} \int_{s_i}^{u}\int_{0}^{\delta} \theta(u-s)^{\alpha-1} M_\alpha(\theta) Q((u-s)^\alpha\theta) B_\lambda f(s,x(s))dsd\theta\| \\
&\quad + \alpha \lim_{\lambda\to\infty} \| \int_{u-\epsilon}^{u}\int_{\delta}^{\infty} \theta(u-s)^{\alpha-1} M_\alpha(\theta) Q((u-s)^\alpha\theta) B_\lambda f(s,x(s))dsd\theta\| \\
&\le \alpha M M_0 \int_{s_i}^{u} (u-s)^{\alpha-1} m(s)ds \int_0^\delta \theta M_\alpha(\theta)d\theta \\
&\quad + \alpha M M_0 \int_{u-\epsilon}^{u} (u-s)^{\alpha-1} m(s)ds \int_0^\infty \theta M_\alpha(\theta)d\theta
\end{aligned}$$

$$\begin{aligned}
&\le \alpha M M_0 \frac{(u-s_i)^{(a+1)(1-\alpha_1)}}{(a+1)^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}[s_i,t_{i+1}]} \int_0^\delta \theta M_\alpha(\theta)d\theta \\
&+ \frac{\alpha M M_0}{\Gamma(1+\alpha)} \frac{\epsilon^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}[s_i,t_{i+1}]} \\
&\longrightarrow 0 \quad \text{as } \epsilon, \delta \to 0.
\end{aligned}$$

Hence the set $V(u)$ is arbitrarily close to the relatively compact set $V_{\epsilon,\delta}(u)$ and subsequently $V(u)$ is relatively compact in X.

Step IV : We prove that the set of functions $[\widetilde{F_i^2 B_r}]_i$, $i = 0, \ldots, N$, is an equicontinuous subset of $C([t_i, t_{i+1}], X)$.

Let, $l_1, l_2 \in [s_i, t_{i+1}]$, $s_i < l_1 < l_2$ and $x \in B_r$.

We have

$$\begin{aligned}\|(F_i^2)x(l_2)-(F_i^2)x(l_1)\| &= \|\lim_{\lambda\to\infty}\int_{s_i}^{l_2}(l_2-s)^{\alpha-1}P_\alpha(l_2-s)B_\lambda f(s,x(s))ds\\ &- \lim_{\lambda\to\infty}\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}P_\alpha(l_1-s)B_\lambda f(s,x(s))ds\|\\ &= \|\lim_{\lambda\to\infty}\int_{l_1}^{l_2}(l_2-s)^{\alpha-1}P_\alpha(l_2-s)B_\lambda f(s,x(s))ds\\ &+ \|\alpha\lim_{\lambda\to\infty}\int_{s_i}^{l_1}(l_2-s)^{\alpha-1}P_\alpha(l_2-s)B_\lambda f(s,x(s))ds\\ &- \lim_{\lambda\to\infty}\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}P_\alpha(l_2-s)B_\lambda f(s,x(s))ds\|\\ &+ \|\alpha\lim_{\lambda\to\infty}\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}P_\alpha(l_2-s)B_\lambda f(s,x(s))ds\\ &- \alpha\lim_{\lambda\to\infty}\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}P_\alpha((l_1-s)B_\lambda f(s,x(s))ds\|\\ &= I_1+I_2+I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \frac{MM_0}{\Gamma(\alpha)}\left|\int_{l_1}^{l_2}(l_2-s)^{\alpha-1}m(s)ds\right|,\\ I_2 &= \frac{MM_0}{\Gamma(\alpha)}\int_{s_i}^{a}[(l_1-s)^{\alpha-1}-(l_2-s)^{\alpha-1}]m(s)ds,\\ I_3 &= M_0\int_{s_i}^{l_1-\epsilon}(l_1-s)^{\alpha-1}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|m(s)ds.\end{aligned}$$

Now,

$$\begin{aligned}I_1 &\le \frac{MM_0}{\Gamma(\alpha)}\left[\int_{l_1}^{l_2}(l_2-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right]^{1-\alpha_1}\Big(\int_{l_1}^{l_2}|m(s)|^{\frac{1}{\alpha_1}}ds\Big)^{\alpha_1}\\ &\le \frac{MM_0}{\Gamma(\alpha)}\frac{(l_2-l_1)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}\|m\|_{L^{\alpha_1}([l_1,l_2],\mathbb{R}^+)}\\ &\to 0 \quad \text{as} \quad l_2\to l_1.\end{aligned}$$

Also,

$$
\begin{aligned}
I_2 &\le \frac{MM_0}{\Gamma(\alpha)}\Big(\int_{s_i}^{l_1}[(l_1-s)^{\alpha-1}-(l_2-s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}}ds\Big)^{1-\alpha_1}\\
&\le \frac{MM_0}{\Gamma(\alpha)}\Big(\int_{s_i}^{l_1}[(l_1-s)^{a}-(l_2-s)^{a}]ds\Big)^{1-\alpha_1}\|m\|_{L^{\alpha_1}([s_i,l_1])}\\
&\le \frac{MM_0}{\Gamma(\alpha)}\big((l_2-l_1)^{a+1}-\big((l_2-s_i)^{a+1}-(l_1-s_i)]^{a+1}\big)\big)^{1-\alpha_1}\|m\|_{L^{\alpha_1}([s_i,l_1],\mathbb{R}^+)}\\
&\le \frac{MM_0}{\Gamma(\alpha)}((l_2-l_1)^{a+1}\|m\|_{L^{\alpha_1}([s_i,l_1],\mathbb{R}^+)}\\
&\to 0 \quad \text{as} \quad l_2\to l_1.
\end{aligned}
$$

For $\epsilon > 0$ small enough,

$$
\begin{aligned}
I_3 &= M_0\int_{s_i}^{l_1-\epsilon}(l_1-s)^{\alpha-1}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|m(s)ds\\
&+ M_0\int_{l_1-\epsilon}^{l_1}(l_1-s)^{\alpha-1}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|m(s)ds\\
&\le \int_{s_i}^{l_1-\epsilon}(l_1-s)^{\alpha-1}m(s)ds\sup_{s\in[s_i,l_1-\epsilon]}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|\\
&+ \int_{s_i}^{l_1}(l_1-s)^{\alpha-1}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|m(s)ds\\
&- \int_{s_i}^{l_1-\epsilon}(l_1-s)^{\alpha-1}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|m(s)ds \qquad (3.1)\\
&\le M_0\int_{s_i}^{l_1-\epsilon}(l_1-s)^{\alpha-1}m(s)ds\sup_{s\in[s_i,l_1-\epsilon]}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|\\
&+ \frac{2M_0\alpha}{\Gamma(\alpha+1)}[\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}m(s)ds-\int_{s_i}^{l_1-\epsilon}(l_1-s)^{\alpha-1}m(s)ds]\\
&\le \int_{s_i}^{l_1-\epsilon}(l_1-s)^{\alpha-1}m(s)ds\sup_{s\in[s_i,l_1-\epsilon]}\|P_\alpha(l_2-s)-P_\alpha(l_1-s)\|\\
&+ \frac{2M_0M_1}{\Gamma(\alpha)}[\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}m(s)ds-\int_{s_i}^{l_1-\epsilon}(l_1-\epsilon-s)^{\alpha-1}m(s)ds]\\
&+ \frac{2M_0M_1}{\Gamma(\alpha)}\int_{s_i}^{l_1-\epsilon}[(l_1-\epsilon-s)^{\alpha-1}-(l_1-s)^{\alpha-1}]m(s)ds\\
&= I_{31}+I_{32}+I_{33}.
\end{aligned}
$$

Since $P_\alpha(t), t > 0$ is continuous in the uniform operator topology, so $I_{31} \to 0$ as $l_2 \to l_1$.

As in the proof of I_2 and I_3, $I_{32} \to 0$, $I_{33} \to 0$ as $\epsilon \to 0$, and therefore $||(T_i^2 x)(l_2) - (F_i^2 x)(l_1)|| \to 0$ independent of $x \in B_\eta(0, PC(X))$ as $l_2 \to l_1$.
Similarly the case $l_1 = s_i$ can also be verified.
Thus, $[F_i^{\tilde{2}} B_r]_i$ is an equicontinuous subset of $C([t_i, t_{i+1}], X)$.

Step V : To establish that F^2 is continuous in B_r.

Let $\{x_n\}$ be a sequence of functions in B_r such that $x_n \to x \in B_r$. By $(H3)$, we have

$$f(t, x_n(t)) \to f(t, x(t)) \text{ as } n \to \infty.$$

For each $t \in J$, we obtain

$$(t-s)^{\alpha-1}|f(s, x_n(s)) - f(s, x(s))| \leq 2(t-s)^{\alpha-1} m(s) \text{ a.e. } \in [s_i, t).$$

By Hölder's inequality, the RHS of the above inequality is integrable, for $s \in [s_i, t), , s_i \leq t \leq t_{i+1}$, and hence by Lebesegue dominated convergence theorem, we obtain

$$\int_{s_i}^t (t-s)^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| ds \to 0 \text{ as } n \to \infty.$$

Thus, for $t_i < t \leq t_{i+1}$, we obtain

$$\begin{aligned} \|(F^2 x_n)(t) - (F^2 x)(t)\| &\leq \|\lim_{\lambda \to \infty} \int_{s_i}^t K_\alpha(t-s) B_\lambda f(s, x_n(s)) - f(s, x(s)) ds\| \\ &\leq \frac{MM_0}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $F^2 x_n \to F^2 x$ is pointwise on $(t_i, t_{i+1}]$ as $n \to \infty$. Hence it follows from step IV that $F x_n \to F x$ uniformly on $(t_i, t_{i+1}]$ as $n \to \infty$. and so F is continuous.

From the above discussion, F^2 is a completely continuous operator. Then Krasnoselskii's fixed point theorem ensures that T has a fixed point which gives rise to a mild solution.

4. APPLICATIONS

As an application of our results, we consider the following fractional time partial differential equation:

$${}^C D_t^\alpha x(t, z) = \frac{\partial^2}{\partial z^2} x(t, z) + \mathcal{F}(t, x(t, z)), \ a.e. \ (t, z) \in \bigcup_{i=1}^N [s_i, t_{i+1}] \times [0, \pi], 0 < \alpha < 1, \tag{4.1}$$

$$x(t, z) = \mathcal{G}_i(t, x(t, z)), z \in [0, \pi], t \in (t_i, s_i], i = 1, 2, ..., N, \tag{4.2}$$

$$x(t, 0) = x(t, \pi) = 0, t \in [0, T] \tag{4.3}$$

$$x(0, z) = x_0(z), z \in [0, \pi], \tag{4.4}$$

where $0 = t_0 = s_0 < t_1 \leq s_1 < ... < t_N \leq s_N < t_{N+1} = T$ are fixed real numbers, $\mathcal{F} \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and $\mathcal{G}_i \in C((t_i, s_i] \times \mathbb{R}, \mathbb{R})$ for all $i = 1, ..., N$.

Let

$$\begin{aligned} x(t)z &= x(t,z), t \in [0,T], z \in [0,\pi], \\ f(t,x)(z) &= \mathcal{F}(t, x(t,z)), t \in [0,T], z \in [0,\pi], \\ g_i(t,x)(z) &= \mathcal{G}_i(t, x(t,z)), z \in [0,\pi], t \in (t_i, s_i],\ i = 1, 2, ..., N. \end{aligned}$$

We choose $X = C([0,\pi], \mathbb{R})$ endowed with the uniform topology and consider the operator $A : D(A) \subset x \to X$ defined by

$$D(A) = \{x \in C^2([0,\pi], \mathbb{R}) : x(0) = x(\pi) = 0\}, Ax = x''.$$

This shows that the problem (1.4)-(1.6) is an abstract formulation of the problem (4.1)-(4.4).

From [20], $\rho(A) = (0, \infty)$ and for $\lambda > 0, \|R(\lambda, A)\| \leq \frac{1}{\lambda}$ and

$$\overline{D(A)} = \{x \in X : x(0) = x(\pi) = 0\} \neq X.$$

This implies that A satisfies $(H1)$ with $M_0 = 1$. Since it is well known that A generates a compact C_0-semigroup $\{(Q(t))\}_{t>0}$ on X_0 such that $\|Q(t)\| \leq 1$, hence $(H2)$ is satisfied with $M = 1$.

For the validation of Theorem 3.2, let us take,

$$\begin{aligned} f(t,x)(z) &= \frac{e^{-t}}{e^t + e^{-t}} \left(\frac{|x(t,z)|}{1 + |x(t,z)|} \right) + e^{-t}, \\ g_i(t,x) &= \frac{\cos t |x(t,z)|}{5(1 + |x(t,z)|)}, t \in (t_i, s_i], i = 1, ..., N, x \in X, z \in [0,\pi]. \end{aligned}$$

Then clearly $f : [0,T] \times X \to X$ is a continuous function and

$$\|f(t,x) - f(t,y)\| \leq \frac{e^{-t}}{e^t + e^{-t}} \|x - y\|, \text{for all } x, y \in X.$$

And if we let $\mu(t) = \frac{e^{-t}}{e^t + e^{-t}}$, it follows that $\mu \in L^{\frac{1}{\alpha_1}}([0,T], \mathbb{R}^+)$. Also $g_i : (t_i, s_i] \times X \to X$ are continuous functions such that

$$\|g(t,x) - g_i(t,y\|) \leq L_{g_i} \|x - y\|,$$

with $L_{g_i} = \frac{1}{5}$. Thus the functions f and g_i satisfied the hypotheses $(H3)$ and $(H4)$ respectively. We deduce that that the system (4.1)-(4.4) has unique integral solution.

On the other hand, for the validation of Theorem 3.3, let us take, $\alpha = \frac{1}{3}$, and

$$\begin{aligned} f(t, x(t)) &= t^{-\frac{1}{4}} \sin x(t), \\ g_i(t, x) &= \frac{\cos t |x(t, z)|}{5(1 + |x(t, z)|)}, t \in (t_i, s_i], i = 1, ..., N, x \in X, z \in [0, \pi]. \end{aligned}$$

Choose $m(t) = t^{-\frac{1}{4}}$. Then the function g_i satisfies $(H4)$ and f satisfies the assumptions $(H5)$ and $(H6)$. Thus the problem (4.1)-(4.4) has a solution.

5. Conclusion

The main purpose of this paper is to extend the existence results of the non-instantaneous impulsive evolution equations to the case when the operator A is not dense and satisfies a Hille-Yosida condition. Under a set of sufficient conditions, the existence of integral solutions are obtained. The result is supported by a suitable example.

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