Indian J. Pure Appl. Math., **51**(3): 1083-1096, September 2020

DOI: 10.1007/s13226-020-0451-3

RAMANUJAN–MORDELL TYPE FORMULAS ASSOCIATED TO CERTAIN QUADRATIC FORMS OF DISCRIMINANT 20^k or 32^k

Anup Kumar Singh* and Dongxi Ye**,[1]

**Indian Institute of Science Education and Research Berhampur, Odisha* 760 010, *India*

***School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai* 519082, *Guangdong, People's Republic of China*

e-mails: anupsinghmath@gmail.com; lawrencefrommath@gmail.com

(*Received* 28 *January* 2019; *after final revision* 21 *May* 2019; *accepted* 10 *June* 2019)

In this work, we establish Ramanujan–Mordell type formulas associated to certain quadratic forms of discriminants 20^k or 32^k. In the end, a remark is given to illustrate potential extensions to other related cases.

Key words : Eta functions; modular forms; quadratic forms; representations of integers; sums of squares; theta functions.

2010 Mathematics Subject Classification : 11F11, 11F20, 11F30, 11E25.

1. Introduction

In the long history of number theory, one of the classical problems is to determine exact formulas for the number of representations of a positive integer n as a sum of $2k$ squares, that is, the number of integral solutions of

$$x_1^2 + x_2^2 + \cdots + x_{2k-1}^2 + x_{2k}^2 = n,$$

denoted by $\mathcal{R}_k(n)$. Such a classical but interesting problem has been explored and studied by many mathematicians since it was introduced. As a result, formulas for $\mathcal{R}_k(n)$ for various cases have been

[1]Dongxi Ye is supported by the Natural Science Foundation of China (Grant No. 11901586), the Natural Science Foundation of Guangdong Province (Grant No. 2019A1515011323) and the Sun Yat-sen University Research Grant for Youth Scholars (Grant No. 19lgpy244).

found. For example, for $k = 1,\ 2,\ 3$ and 4, i.e., sums of $2,\ 4,\ 6$ and 8 squares, (reformulated) formulas for $\mathcal{R}_k(n)$ are originally due to Jacobi [11],

$$\mathcal{R}_1(n) = 4\sum_{d|n}\left(\frac{-4}{d}\right),$$
$$\mathcal{R}_2(n) = 8\sum_{d|n} d - 32\sum_{d|\frac{n}{4}} d,$$

$$\mathcal{R}_3(n) = -4\sum_{d|n}\left(\frac{-4}{d}\right)d^2 + 16\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2,$$
$$\mathcal{R}_4(n) = 16\sum_{d|n} d^3 - 32\sum_{d|\frac{n}{2}} d^3 + 256\sum_{d|\frac{n}{4}} d^3$$

where, here and throughout this work, $\left(\frac{D}{\cdot}\right)$ denotes the quadratic character for discriminant D. The result for $k = 5$, i.e., sum of 10 squares, was due (without proof) in part to Eisenstein [8], and fully described (without proof) by Liouville [13]. The results for $1 \le k \le 9$ were all proved by Glaisher [9]. Now if one sets

$$\theta(\tau) := \sum_{m=-\infty}^{\infty} q^{m^2}, \tag{1.1}$$

where, here and throughout the remainder of this work, τ denotes a complex number with positive imaginary part and $q = e^{2\pi i\tau}$, then one has the well known relation

$$\sum_{m=0}^{\infty} \mathcal{R}_k(m)q^m = \theta(\tau)^{2k},$$

where the function in τ on the right hand side is known [22, Chapter 1] to be a modular form of weight k with Nebentypus $\left(\frac{-4}{\cdot}\right)^k$ for $\Gamma_0(4)$. By such a nice relation and thanks to the theory of classical modular forms, we know that the modular form $\theta(\tau)^{2k}$ has the following interesting decomposition

$$\theta(\tau)^{2k} = E_k^*(\tau) + C_k(\tau), \tag{1.2}$$

where $E_k^*(\tau)$ is an Eisenstein series whose n-th Fourier coefficient is some divisor function in n, and $C_k(\tau)$ is a cusp form whose n-th Fourier coefficient is of order substantially lower than that of $E_k^*(\tau)$. Then as a consequence, formulas for $\mathcal{R}_k(n)$ follows from equating the Fourier coefficients on both sides of (1.2). Such a modular form theoretic intrinsic of $\theta(\tau)^{2k}$ indirectly allows Ramanujan [19], [20, Eqs. (145)-(147)] to "completely" solved (without proof) the problem in around 1916. To state Ramanujan's fascinating results, we need the Dedekind eta function, which is defined by

$$\eta(\tau) := q^{1/24}\prod_{j=1}^{\infty}(1-q^n).$$

Here and throughout the remainder of this work, we write η_m for $\eta(m\tau)$ for any positive integer m.

***Theorem* 1.1** — *(Ramanujam). Suppose k is a positive integer. Let $\theta(\tau)$ be defined by (1.1). Then*

$$\theta(\tau)^{2k} = F_k(\tau) + \theta(\tau)^{2k} \sum_{1 \le j \le \frac{(k-1)}{4}} c_{k,j} x^j \tag{1.3}$$

where $c_{k,j}$ are numerical rational constants that depend on j and k,

$$x = x(\tau) := \frac{\eta_1^{24}\eta_4^{24}}{\eta_2^{48}},$$

and $F_k(\tau)$ is an Eisenstein series defined by

$$F_1(\tau) := 1 + 4\sum_{j=1}^{\infty} \frac{q^j}{1+q^{2j}},$$

and for $k \ge 1$,

$$F_{2k}(\tau) := 1 - \frac{4k(-1)^k}{(2^{2k}-1)\mathcal{B}_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1}q^j}{1-(-1)^{k+j}q^j},$$

and

$$F_{2k+1}(\tau) := 1 + \frac{4(-1)^k}{\mathcal{E}_{2k}} \sum_{j=1}^{\infty} \left(\frac{(2j)^{2k}q^j}{1+q^{2j}} - \frac{(-1)^{k+j}(2j-1)^{2k}q^{2j-1}}{1-q^{2j-1}} \right).$$

Here $\mathcal{B}_k$ and $\mathcal{E}_k$ are the Bernoulli numbers and Euler numbers, respectively, defined by

$$\frac{u}{e^u - 1} = \sum_{k=0}^{\infty} \frac{\mathcal{B}_k}{k!} u^k \quad \text{and} \quad \frac{1}{\cosh u} = \sum_{k=0}^{\infty} \frac{\mathcal{E}_k}{k!} u^k.$$

Theorem 1.1 (formula 1.3) was proved first by Mordell [16] utilizing the theory of modular forms, and thus giving credits to Mordell as well, it is now called the Ramanujan–Mordell Theorem (resp. the Ramanujan–Mordell formula). An elementary proof was given by Cooper in [5] by making skillful use of Ramanujan's ${}_1\psi_1$ formula. Inspired by the beauty of the Ramanujan–Mordell formula, it will be very interesting if one can extend it to other $2k$-ary quadratic forms. Such extensions and developments have recently been initiated by Cooper, Kane and the second author of the present work in [6], in which they extended the Ramanujan–Mordell formula to the modular forms (theta functions) $(\theta(\tau)\theta(m\tau))^k$ associated to the $2k$-ary quadratic forms

$$x_1^2 + \cdots + x_k^2 + m\left(x_{k+1}^2 + \cdots + x_{2k}^2\right)$$

for $m \in \{3,7,11,23\}$, and analogously obtained a beautiful unified *polynomial representation* (or called Ramanujan–Mordell type representation) for the cusp form component $C_k(\tau)$. Subsequently, the second author of the present work further extends the Ramanujan–Mordell formula to the cases $m \in \{2,4\}$ in [23], and in the end, he indicates the essence of the existence of these beautiful Ramanujan–Mordell type formulas for $(\theta(\tau)\theta(m\tau))^k$ and how one can give an attempt to other cases. Based on the comments given in [23, Section 4], in this work we further explore such an interesting topic. We aim to treat the uncharted cases, $m = 5$ and $m = 8$, whose associated quadratic forms are of discriminants 20^k and 32^k, respectively, and obtain their corresponding Ramanujan–Mordell type formulas. We conclude this introduction section by stating the main results of the present work and presenting several explicit examples. To this end, we first define the (quasi) Eisenstein series of integral weight k with Nebentypus χ and ψ. Suppose that χ and ψ are primitive Dirichlet characters with conductors M and N, respectively. For $k = 1$ with χ trivial or $k \geq 2$ such that $\chi(-1)\psi(-1) = (-1)^k$, let the (quasi) Eisenstein series $E_{k,\chi,\psi}(\tau)$ be defined by

$$E_{k,\chi,\psi}(\tau) := \delta_{M,1} - \frac{2k}{B_{k,\psi}} \sum_{n\geq 1} \sigma_{k-1;\chi,\psi}(n) q^n, \tag{1.4}$$

where $\delta_{a,b}$ denotes the Kronecker delta, $B_{k,\psi}$ denotes the generalized Bernoulli number with respect to the character ψ, i.e.,

$$\sum_{a=1}^{N} \frac{\psi(a) x e^{ax}}{e^{Nx} - 1} = \sum_{k=0}^{\infty} \frac{B_{k,\psi}}{k!} x^k,$$

and

$$\sigma_{k-1;\chi,\psi}(n) := \sum_{d|n} \psi(d)\chi(n/d) d^{k-1}.$$

We write $\mathbf{1}$ for the trivial character, χ_D for the quadratic character $\left(\frac{D}{\cdot}\right)$ for discriminant D and $E_k(\tau)$ for the (quasi) Eisenstein series $E_{k,\mathbf{1},\mathbf{1}}(\tau)$ for $\mathrm{SL}_2(\mathbb{Z})$. It is known [21, Theorem 5.8] that $E_{k,\chi,\psi}(\tau)$ is a weight k modular form for $\Gamma_0(MN)$ with Nebentypus χ/ψ. We are ready to state the main result of this work in the following theorem.

***Theorem* 1.2** — *For $m \in \{5,8\}$, let $x_m = x_m(\tau)$ be defined by*

$$x_m = \begin{cases} \dfrac{\eta_1^3\eta_4\eta_5\eta_{20}^3}{\eta_2^4\eta_{10}^4} & \text{if } m = 5, \\[2ex] \dfrac{\eta_1^2\eta_4\eta_8\eta_{32}^2}{\eta_2^3\eta_{16}^3} & \text{if } m = 8. \end{cases} \tag{1.5}$$

Let $F_{k,m}(\tau)$ be defined by

$$F_{k,m}(\tau) = \begin{cases} \frac{(-1)^\ell E_{2\ell}(\tau)-(-1)^\ell 2E_{2\ell}(2\tau)+(-4)^\ell E_{2\ell}(4\tau)+5^\ell E_{2\ell}(5\tau)-5^\ell 2E_{2\ell}(10\tau)+20^\ell E_{2\ell}(20\tau)}{-(-1)^\ell+(-4)^\ell-5^\ell+20^\ell} & \text{if } k=2\ell \text{ and } m=5, \\ E_{2\ell+1,\mathbf{1},\chi_{-20}}(\tau)+(-20)^\ell E_{2\ell+1,\chi_{-20},\mathbf{1}}(\tau) & \text{if } k=2\ell+1\geq 3 \text{ and } m=5, \\ E_{1,\mathbf{1},\chi_{-20}}(\tau) & \text{if } k=1 \text{ and } m=5, \\ \frac{(-1)^\ell E_{2\ell}(\tau)-(-1)^\ell E_{2\ell}(2\tau)-8^\ell E_{2\ell}(16\tau)+32^\ell E_{2\ell}(32\tau)}{-8^\ell+32^\ell} & \text{if } k=2\ell \text{ and } m=8, \\ E_{2\ell+1,\mathbf{1},\chi_{-8}}(4\tau)+(-1)^\ell 2^{\ell-1}E_{2\ell+1,\chi_{-8},\mathbf{1}}(\tau) & \text{if } k=2\ell+1\geq 3 \text{ and } m=8, \\ \frac{1}{3}E_{1,\mathbf{1},\chi_{-8}}(\tau)+\frac{2}{3}E_{1,\mathbf{1},\chi_{-8}}(4\tau) & \text{if } k=1 \text{ and } m=8. \end{cases}$$

Furthermore, let $l(k,m)$ be defined by

$$l(k,m) = \begin{cases} 3\ell-2 & \textit{if } k=2\ell \textit{ and } m=5, \\ 3\ell+1 & \textit{if } k=2\ell+1 \textit{ and } m=5, \\ 2k-1 & \textit{if } k\geq 2 \textit{ and } m=8, \\ 2 & \textit{if } k=1 \textit{ and } m=8. \end{cases}$$

Then there exist rational numbers $c_{j,k,m}$ depending on k, j and m such that

$$(\theta(\tau)\theta(m\tau))^k = F_{k,m}(\tau) + (\theta(\tau)\theta(m\tau))^k \sum_{j=1}^{l(k,m)} c_{j,k,m}x_m^j. \tag{1.6}$$

Example 1.3 : For $k=1$ or 2, the general formula (1.6) gives the following explicit identities.

$$\theta(\tau)\theta(5\tau) = 1+\sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n}+\frac{\eta_1\eta_2\eta_{10}\eta_{20}}{\eta_4\eta_5}, \tag{1.7}$$

$$\begin{aligned}(\theta(\tau)\theta(5\tau))^2 = 1+\sum_{n=1}^{\infty}\Bigg(&2\frac{nq^n}{1-q^n}-4\frac{nq^{2n}}{1-q^{2n}}+8\frac{nq^{4n}}{1-q^{4n}} \\ &-10\frac{nq^{5n}}{1-q^{5n}}+20\frac{nq^{10n}}{1-q^{10n}}-40\frac{nq^{20n}}{1-q^{20n}}\Bigg)+2\frac{\eta_2^6\eta_{10}^6}{\eta_1\eta_4^3\eta_5^3\eta_{20}},\end{aligned} \tag{1.8}$$

$$\theta(\tau)\theta(8\tau) = 1+\frac{2}{3}\sum_{n=1}^{\infty}\left(\left(\frac{-8}{n}\right)\frac{q^n}{1-q^n}+2\left(\frac{-8}{n}\right)\frac{q^{4n}}{1-q^{4n}}\right)+\frac{4}{3}\frac{\eta_2^2\eta_{16}^2}{\eta_4\eta_8}-\frac{2}{3}\frac{\eta_1^2\eta_{32}^2}{\eta_2\eta_{16}}, \tag{1.9}$$

$$\begin{aligned}(\theta(\tau)\theta(8\tau))^2 = 1+\sum_{n=1}^{\infty}&\left(\frac{nq^n}{1-q^n}-\frac{nq^{2n}}{1-q^{2n}}+8\frac{nq^{16n}}{1-q^{16n}}-32\frac{nq^{32n}}{1-q^{32n}}\right)+3\frac{\eta_2^7\eta_{16}^7}{\eta_1^2\eta_4^3\eta_8^3\eta_{32}^2} \\ &-4\frac{\eta_2^4\eta_{16}^4}{\eta_4^2\eta_8^2}+2\frac{\eta_1^2\eta_2\eta_{16}\eta_{32}^2}{\eta_4\eta_8}.\end{aligned} \tag{1.10}$$

The identity (1.7) was given first in [3] by Berkovich and Yesilyurt making use of Ramanujan's ${}_1\psi_1$ summation formula. Equivalent forms of identities (1.8) and (1.10) were given in [2] and [1], respectively. To the best knowledge of the authors, identity (1.9) has not yet appeared in any literature.

In the next section, some preliminary results for preparing for the proof of Theorem 1.2 are given. The proof of Theorem 1.2 is given in the last section with a concluding remark regarding potential extensions to other related cases.

2. Preliminary Results

This section is devoted to presenting several preliminary results that will be used for the proof of Theorem 1.2.

The following lemma gives the transformation formulas for certain concerned $E_{k,\chi,\psi}(\tau)$ under the action of some Fricke involutions.

Lemma 2.1 — Let $E_{k,\chi,\psi}(\tau)$ be defined as in (1.4). Then for $k=1$,

$$E_{k,\mathbf{1},\chi_{-20}}\left(-\frac{1}{20\tau}\right)=\frac{(20\tau)^k}{i\sqrt{20}}E_{k,\mathbf{1},\chi_{-20}}(\tau),$$
$$E_{k,\mathbf{1},\chi_{-8}}\left(-\frac{1}{8\tau}\right)=\frac{(8\tau)^k}{i\sqrt{8}}E_{k,\mathbf{1},\chi_{-8}}(\tau),$$

and for $k\geq 3$,

$$E_{k,\mathbf{1},\chi_{-20}}\left(-\frac{1}{20\tau}\right)=\frac{(20\tau)^k}{i\sqrt{20}}E_{k,\chi_{-20},\mathbf{1}}(\tau),$$
$$E_{k,\chi_{-20},\mathbf{1}}\left(-\frac{1}{20\tau}\right)=\frac{\sqrt{20}\tau^k}{i}E_{k,\mathbf{1},\chi_{-20}}\left(\tau\right),$$
$$E_{k,\mathbf{1},\chi_{-8}}\left(-\frac{1}{8\tau}\right)=\frac{(8\tau)^k}{i\sqrt{8}}E_{k,\chi_{-8},\mathbf{1}}(\tau),$$
$$E_{k,\chi_{-8},\mathbf{1}}\left(-\frac{1}{8\tau}\right)=\frac{\sqrt{8}\tau^k}{i}E_{k,\mathbf{1},\chi_{-8}}\left(\tau\right).$$

Proof : These are well-known, see, e.g., [15, Section 7]. □

Lemma 2.2 — Under the transformation $\tau\to\tau+\frac{1}{2}$, the following identities hold.

When k is even,

$$E_k\left(\tau+\frac{1}{2}\right)=-E_k(\tau)+(2^k+2)E_k(2\tau)-2^kE_k(4\tau);$$

when k is odd,

$$E_{k,\mathbf{1},\chi_{-20}}\left(\tau+\frac{1}{2}\right) = -E_{k,\mathbf{1},\chi_{-20}}(\tau) + 2E_{k,\mathbf{1},\chi_{-20}}(2\tau),$$
$$E_{k,\chi_{-20},\mathbf{1}}\left(\tau+\frac{1}{2}\right) = -E_{k,\chi_{-20},\mathbf{1}}(\tau) + 2^k E_{k,\chi_{-20},\mathbf{1}}(2\tau),$$
$$E_{k,\mathbf{1},\chi_{-8}}\left(\tau+\frac{1}{2}\right) = -E_{k,\mathbf{1},\chi_{-8}}(\tau) + 2E_{k,\mathbf{1},\chi_{-8}}(2\tau),$$
$$E_{k,\chi_{-8},\mathbf{1}}\left(\tau+\frac{1}{2}\right) = -E_{k,\chi_{-8},\mathbf{1}}(\tau) + 2^k E_{k,\chi_{-8},\mathbf{1}}(2\tau).$$

PROOF : The proofs are similar to that of [6, Lemma 3.2], so we omit the details.

Here and throughout the remainder of this work, write $\mathrm{ord}_s(f)$ for the order of vanishing of a modular form f at a cusp s, which is defined as follows. For a cusp s, let $M_s \in \mathrm{SL}_2(\mathbb{Z})$ such that $M_s(i\infty) = s$. Suppose that f is of level Γ and $g := f|_k M_s = q_N^h + o(q_N^h)$, where $q_N = e^{2\pi i\tau/N}$ and N is the smallest positive integer such that T^N or $-T^N \in M_s^{-1}\Gamma M_s$ with $T = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Then $\mathrm{ord}_s(f) := h$. The preceding two lemmas are now used to compute the order of vanishing of $F_{k,m}(\tau)$ at the cusp $\frac{1}{2}$ and yield the following results.

Lemma 2.3 — Let $F_{k,m} = F_{k,m}(\tau)$ be defined as in Theorem 1.2. Then

$$\mathrm{ord}_{1/2}(F_{k,m}) = \begin{cases} 2 & \text{if } k \text{ even and } m = 5,\\ \frac{1}{2} & \text{if } k \text{ odd and } m = 5,\\ 1 & \text{if } k \geq 2 \text{ and } m = 8,\\ 0 & \text{if } k = 1 \text{ and } m = 8.\end{cases}$$

PROOF : One can first check that $F_{k,m}(\tau)$ is a modular form of weight k with Nebentypus $\left(\frac{-4m}{\cdot}\right)^k$ for $\Gamma_0(4m)$. Also, one can note that $\pi : X(\Gamma_0(4m)) \twoheadrightarrow X(\Gamma_0(4m) + 4m)$ is a degree 2 projective morphism, where $\Gamma_0(4m) + 4m$ denotes the discrete group generated by $\Gamma_0(4m)$ and its Fricke involution $\begin{pmatrix}0 & -1\\ 4m & 0\end{pmatrix}$, and that the cusp $\frac{1}{2}$ of the latter modular curve ramifies as two cusps in the former modular curve, each of ramification index 1. Thus, the local coordinate of the cusp $\frac{1}{2}$ remains the same, which is of width m, and one can proceed with the standard way to compute the order of vanishing of $F_{k,m}$ as a modular form for $\Gamma_0(4m)$. We give the proof to the case $k \geq 3$ odd and

$m=5$, and leave the details of the other cases to the reader. One has for $k=2\ell+1\geq 3$,

$$\begin{aligned}
&(2\tau+1)^{-k}F_k\left(\frac{\tau}{2\tau+1}\right)\\
&=(2\tau+1)^{-k}\left(E_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{\tau}{2\tau+1}\right)+(-20)^{\ell}E_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{\tau}{2\tau+1}\right)\right)\\
&=(2\tau+1)^{-k}\left(E_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{-1}{4\tau+2}+\frac{1}{2}\right)+(-20)^{\ell}E_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{-1}{4\tau+2}+\frac{1}{2}\right)\right)\\
&=(2\tau+1)^{-k}\Bigg(-E_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{-1}{4\tau+2}\right)+2E_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{-1}{2\tau+1}\right)
\end{aligned}$$

$$\begin{aligned}
&-(-20)^{\ell}E_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{-1}{4\tau+2}\right)+2^{2\ell+1}(-20)^{\ell}E_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{-1}{2\tau+1}\right)\Bigg)\text{by Lemma 2.3}\\
&=(2\tau+1)^{-k}\Bigg(-\frac{20^{2\ell+1}}{i\sqrt{20}}\left(\frac{2\tau+1}{10}\right)^{2\ell+1}E_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{2\tau+1}{10}\right)\\
&\qquad+2\frac{20^{2\ell+1}}{i\sqrt{20}}\left(\frac{2\tau+1}{20}\right)^{2\ell+1}E_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{2\tau+1}{20}\right)\\
&\qquad-(-20)^{\ell}\frac{\sqrt{20}}{i}(4\tau+2)^{2\ell+1}E_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{4\tau+2}{20}\right)\\
&\qquad+2^{2\ell+1}(-20)^{\ell}\frac{\sqrt{20}}{i}(2\tau+1)^{2\ell+1}E_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{2\tau+1}{20}\right)\Bigg)\quad\text{by Lemma 2.3}\\
&=Cq^{\frac{1}{10}}+O(q^{\frac{1}{5}})
\end{aligned}$$

for some nonzero constant C. Hence, the order of vanishing of $F_{k,m}$ for $k\geq 3$ odd and $m=5$ at the cusp $\frac{1}{2}$ is $\frac{1}{2}$. □

The next two lemmas are useful for computing the order of vanishing of $\theta(\tau)\theta(m\tau)$ and $x_m(\tau)$ at cusps.

Lemma 2.4 — If $f(\tau)=\prod_{d|N}\eta_d^{r_d}$ for some positive integer N with $k=\frac{1}{2}\sum_{d|N}r_d\in\mathbb{Z}$, with the additional properties that

$$\sum_{d|N}dr_d\equiv 0\pmod{24}$$

and

$$\sum_{d|N}\frac{N}{d}r_d\equiv 0\pmod{24},$$

then $f(\tau)$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(d)(c\tau+d)^kf(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here the character χ is defined by Jacobi symbol $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$ where $s = \prod_{d|N} d^{r_d}$.

PROOF : See Gordon and Hughes [10], or Newman [17, 18]. □

Lemma 2.5 — Let a, c and N be positive integers with $c|N$ and $\gcd(a,c) = 1$. If $f(\tau) = \prod_{d|N} \eta_d^{r_d}$ satisfies the conditions of Lemma 2.4 for N, then the order of vanishing $\mathrm{ord_{a/c}(f)}$ of $f(\tau)$ at the cusp a/c is

$$\frac{N}{24} \sum_{d|N} \frac{\gcd(c,d)^2 r_d}{\gcd(c, N/c) cd}.$$

PROOF : See Biagioli [4], Ligozat [12] or Martin [14]. □

By the eta-quotient representation [11] of $\theta(\tau)$, i.e.,

$$\theta(\tau) = \frac{\eta_2^5}{\eta_1^2 \eta_4^2},$$

and the definitions of $x_m(\tau)$, using Lemmas 2.4 and 2.5, direct computations give the order of vanishing of $\theta(\tau)\theta(m\tau)$ and x_m at the cusps 0, $\frac{1}{2}$ and $\frac{1}{4}$, which are the cusps of $X(\Gamma_0(4m) + 4m)$ for $m \in \{5, 8\}$, as follows.

Lemma 2.6 — Let $\theta(\tau)$ and $x_m = x_m(\tau)$ be defined as in (1.1) and (1.5), respectively. Then

$$\mathrm{ord}_s(\theta(\tau)\theta(m\tau)) = \begin{cases} 0 & \text{if } s = 0 \text{ or } \frac{1}{4}, \text{ and } m = 5 \text{ or } 8, \\ \frac{3}{2} & \text{if } s = \frac{1}{2} \text{ and } m = 5, \\ 2 & \text{if } s = \frac{1}{2} \text{ and } m = 8, \end{cases}$$

and for $m \in \{5, 8\}$,

$$\mathrm{ord}_s(x_m) = \begin{cases} 1 & \text{if } s = 0, \\ -1 & \text{if } s = \frac{1}{2}, \\ 0 & \text{if } s = \frac{1}{4}. \end{cases}$$

3. PROOF OF THEOREM 1.2 AND A REMARK

In this section, we give the proof to Theorem 1.2, and conclude this work with a remark regarding potential extensions to other cases.

PROOF OF THEOREM 1.2 : Let $F_{k,m}(\tau)$ and $l(k,m)$ be defined as in Theorem 1.2. Then according to the definition of $F_k(\tau)$, one can easily check that $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k}$ is a modular function for

$\Gamma_0(4m)+4m$ with no poles on the open modular curve $Y(\Gamma_0(4m)+4m)$. By Lemmas 2.3 and 2.6, one can check that $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k}$ has only poles at the cusp $\frac{1}{2}$ of order $l(k,m)$. Also, by Lemma 2.6, the modular function $x_m(\tau)$ has a simple pole at the cusp $\frac{1}{2}$ and a simple zero at the cusp 0, which is equivalent to $i\infty$ under $\Gamma_0(4m)+4m$, and thus one can tell that $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k x_m(\tau)^{l(k,m)}}$ is a modular function for $\Gamma_0(4m)+4m$ with only poles at the cusp 0 of order $l(k,m)$ and with no poles or zeros at the cusp $\frac{1}{2}$. Then inductively, there are rational numbers $b_{j,k,m}$ such that

$$\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k x_m(\tau)^{l(k,m)}} - \sum_{j=1}^{l(k,m)} b_{j,k,m} x_m^{-j}$$

is holomorphic at the cusp 0, and thus holomorphic everywhere on the compact modular curve $X(\Gamma_0(4m)+4m)$, which must be a constant C. Since x_m^{-1} has a zero at the cusp $\frac{1}{2}$, while $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k x_m(\tau)^{l(k,m)}}$ is holomorphic and nonvanishing at the cusp $\frac{1}{2}$, then the constant C is nonzero. Rearranging both sides by multiplying both sides by $x_m^{l(k,m)}$, one has

$$b_{k,l(k,m),m}\left(\theta(\tau)\theta(m\tau)\right)^k = F_{k,m}(\tau) + \left(\theta(\tau)\theta(m\tau)\right)^k \sum_{j=1}^{l(k,m)} c_{j,k,m} x_m^j$$

by setting $c_{j,k,m} = b_{k,l(k,m)-j,m}$ and $c_{k,l(k,m),m} = C$. Taking $\tau \to i\infty$, one has $b_{k,l(k,m),m} = 1$ according to the definition of F_k. This completes the proof.

Remark 3.1 : As indicated in [23, Section 4], to obtain a Ramanujan–Mordell type formula associated to the $2k$-ary quadratic form,

$$x_1^2 + \cdots + x_k^2 + m\left(x_{k+1}^2 + \cdots + x_{2k}^2\right),$$

one way is to consider a genus zero Fuchsian subgroup Γ containing $\Gamma_0(4m)$ that is commensurable with $\mathrm{SL}_2(\mathbb{Z})$ such that it has a uniformizer $\pi(\tau)$ with locations of poles the same as the locations of zeros of $\theta(\tau)\theta(m\tau)$ on $X_0(4m)$, and $\theta(\tau)\theta(m\tau)|_1\,\gamma = c(\gamma)\theta(\tau)\theta(m\tau)$ for some constant c depending on γ. The first condition is for the purpose of obtaining a unified polynomial representation for the cusp forms part, and the second condition guarantees that there is no other inequivalent class of the quadratic form in consideration involved. As we know, one of the most famous families of genus zero Fuchsian subgroups that is commensurable with $\mathrm{SL}_2(\mathbb{Z})$ is the so called moonshine discrete groups [7], which are obtained by adjoining certain Atkin–Lehner involutions to the Hecke groups $\Gamma_0(N)$ and whose uniformizers can be easily constructed by using Dedekind eta function and making use of Lemmas 2.4 and 2.5. More or less directly, we first note that $\theta(\tau)\theta(m\tau)|_1\gamma = \chi_{-4m}(\gamma)\theta(\tau)\theta(m\tau)$ for $\gamma \in \Gamma_0(4m)$ and $\theta(\tau)\theta(m\tau)|_1 \begin{pmatrix} 0 & -1 \\ 4m & 0 \end{pmatrix} = -i\theta(\tau)\theta(m\tau)$. These observations motivate us to first consider the moonshine discrete group $\Gamma_0(4m)+4m$ obtained by adjoining the Fricke involution $\begin{pmatrix} 0 & -1 \\ 4m & 0 \end{pmatrix}$ to $\Gamma_0(4m)$. There are only eight such groups [7], namely, $\Gamma_0(4m)+4m$ for $m=1,2,3,4,5,6,8$ or 9. The case for $m=1$ is the fascinating Ramanujan–Mordell formula, the cases for $m=2,3$ or 4 have been studied in [6, 23], and the results for $m=5$ or 8 have been treated in this work. Now a question will be certainly raised to ask how about the cases for $m=6$ or 9. To answer this question, we first note that by the proof of Theorem 1.2, the beautifully unified polynomial representation for the cusp form part follows from the fact that $\theta(\tau)\theta(m\tau)$ and $x_m(\tau)$ for $m=5$ or 8 do satisfy the first condition mentioned above, while one can check using Lemmas 2.4 and 2.5, $\theta(\tau)\theta(m\tau)$ for $m=6$ or 9 has zeros at two $\Gamma_0(4m)+4m$ inequivalent cusps (thus, two $\Gamma_0(4m)$ inequivalent cusps), which can never be canceled out by just one uniformizer for $\Gamma_0(4m)+4m$. So if one keeps considering $\Gamma_0(4m)+4m$ as the larger group containing $\Gamma_0(4m)$ to work on, then one needs another uniformizer to cancel out the other zeros. Such a uniformizer can be easily obtained by shifting the location of poles of the pre-chosen one through some fractional linear transformation. Following these discussions, we may obtain Ramanujan–Mordell type formulas for $m=6$ or 9 as follows. Let $x_m=x_m(\tau)$ and $y_m=y_m(\tau)$ be defined by

$$x_m=\begin{cases} \dfrac{\eta_1^3\eta_4^3\eta_6^3\eta_{24}^3}{\eta_2^5\eta_3\eta_8\eta_{12}^5} & \text{if } m=6, \\[2ex] \dfrac{\eta_1^2\eta_4\eta_6^2\eta_9\eta_{36}^2}{\eta_2^3\eta_3\eta_{12}\eta_{18}^3} & \text{if } m=9, \end{cases} \quad \text{and} \quad y_m=\begin{cases} \dfrac{\eta_1\eta_2^2\eta_3\eta_8\eta_{12}^2\eta_{24}}{\eta_4^4\eta_6^4} & \text{if } m=6, \\[2ex] \dfrac{\eta_2^2\eta_3^3\eta_{12}^3\eta_{18}^2}{\eta_4\eta_6^8\eta_9} & \text{if } m=9. \end{cases}$$

Let $F_{k,m}(\tau)$ be defined by

$$F_{k,m}(\tau)$$
$$=\begin{cases} \left(\left(-\frac{2}{3}\right)^{\ell}+1\right)\left(\left(-\frac{8}{3}\right)^{\ell}E_{2\ell}(8\tau)+E_{2\ell}(3\tau)\right) \\ \quad -\left(\left(-\frac{8}{3}\right)^{\ell}+1\right)\left(\left(-\frac{2}{3}\right)^{\ell}E_{2\ell}(4\tau)+E_{2\ell}(6\tau)\right) & \text{if } k=2\ell \text{ and } m=6, \\[2ex] E_{2\ell+1,\mathbf{1},\chi_{-24}}(\tau)+(-24)^{\ell}E_{2\ell+1,\chi_{-24},\mathbf{1}}(\tau) & \text{if } k=2\ell+1\geq 3 \text{ and } m=6, \\[2ex] 2^kE_k(4\tau)-(2^k+3^k)E_k(6\tau)+3^kE_k(9\tau) & \text{if } k=4\ell \text{ and } m=9, \\[2ex] \left(2^k+(3i)^k\right)\left(E_k(3\tau)(2i)^kE_k(12\tau)\right) \\ \quad -(1+(2i)^k)\left(2^kE_k(4\tau)+(3i)^kE_k(9\tau)\right) & \text{if } k=4\ell+2 \text{ and } m=9, \\[2ex] 2^kE_{k,\mathbf{1},\chi_{-4}}(\tau)+(-2^k+2(-1)^{\ell}3^k)E_{k,\mathbf{1},\chi_{-4}}(3\tau)-2(-1)^{\ell}3^kE_{k,\mathbf{1},\chi_{-4}}(9\tau) \\ \quad +2^kE_{k,\chi_{-4},\mathbf{1}}(\tau)+\frac{2^k}{2}(-1)^{\ell}(-2^k+2(-1)^{\ell}3^k)E_{k,\chi_{-4},\mathbf{1}}(3\tau)+\frac{12^k}{2}(-1)^{\ell}E_{k,\chi_{-4},\mathbf{1}}(9\tau) & \text{if } k=2\ell+1\geq 3 \text{ and } m=9. \end{cases}$$

Moreover, let $l_x(k,m)$, $l_y(k,m)$ and $l(k,m)$ be defined by

$$l_x(k,m) = \begin{cases} 3\ell & \text{if } k = 2\ell \text{ and } m = 6, \\ 3\ell + 1 & \text{if } k = 2\ell + 1 \text{ and } m = 6, \\ 10\ell & \text{if } k = 4\ell \text{ and } m = 9, \\ 10\ell & \text{if } k = 4\ell + 2 \text{ and } m = 9, \\ 5\ell + 2 & \text{if } k = 2\ell + 1 \text{ and } m = 9, \end{cases} \qquad l_y(k,m) = \begin{cases} \ell & \text{if } k = 2\ell \text{ and } m = 6, \\ \ell & \text{if } k = 2\ell + 1 \text{ and } m = 6, \\ 2\ell & \text{if } k = 4\ell \text{ and } m = 9, \\ 2\ell & \text{if } k = 4\ell + 2 \text{ and } m = 9, \\ \ell & \text{if } k = 2\ell + 1 \text{ and } m = 9, \end{cases}$$

and

$$l(k,m) = \begin{cases} 4\ell - 3 & \text{if } k = 2\ell \text{ and } m = 6, \\ 4\ell + 1 & \text{if } k = 2\ell + 1 \text{ and } m = 6, \\ 12\ell - 4 & \text{if } k = 4\ell \text{ and } m = 9, \\ 12\ell + 3 & \text{if } k = 4\ell + 2 \text{ and } m = 9, \\ 6\ell & \text{if } k = 2\ell + 1 \text{ and } m = 9. \end{cases}$$

Then there are rational numbers $c_{j,k,m}$ depending on j, k and m such that

$$F_{k,m}(\tau) = (\theta(\tau)\theta(m\tau))^k \, y_m^{l_y(k,m)} x_m^{l_x(k,m)} \sum_{j=0}^{l(k,m)} c_{j,k,m} x_m^{-j}. \tag{3.1}$$

We leave the proof of (3.1) as an interesting exercise to the reader.

Finally, one may also be curious about if the other moonshine discrete group containing $\Gamma_0(4m)$ for $m = 6$ or 9 would work or not. For example, for $m = 6$, there are two other moonshine discrete groups obtained from $\Gamma_0(24)$, namely, $\Gamma_0(24)+8$ and $\Gamma_0(24)+$, which both contain $\begin{pmatrix} 8 & -3 \\ 24 & -8 \end{pmatrix}$. However, one can check that under the action of $\begin{pmatrix} 8 & -3 \\ 24 & -8 \end{pmatrix}$, $\theta(\tau)\theta(6\tau)$ is transformed to its inequivalent class $\theta(2\tau)\theta(3\tau)$, and this would force one to get $\theta(2\tau)\theta(3\tau)$ involved when he tries to establish a "Ramanujan–Mordell type" formula for $\theta(\tau)\theta(6\tau)$ through $\Gamma_0(24) + 8$ or $\Gamma_0(24)+$.

ACKNOWLEDGEMENT

The authors thank the anonymous referee for his/her helpful comments, suggestions, and corrections.

REFERENCES

1. A. Alaca, S. Alaca, and M. F. Lemire, Jacobi's identity and representation of integers by certain quaternary quadratic forms, *Int. J. Modern Math.*, **2** (2007), 143-176.

2. A. Alaca, S. Alaca, and K. S. Williams, On the quaternary forms $x^2+y^2+z^2+5t^2, x^2+y^2+5z^2+5t^2$ and $x^2+5y^2+5z^2+5t^2$, *JP J. Algebra Number Theory Appl.*, **9** (2007), 37-53.

3. A. Berkovich and H. Yesilyurt, Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms, *Ramanujan J.*, **20** (2009), 375-408.

4. A. J. F. Biagioli, The construction of modular forms as products of transforms of the Dedekind eta function, *Acta Arith.*, **54** (1990), 272-300.

5. S. Cooper, On sums of an even number of squares, and an even number of triangular numbers: An elementary approach based on Ramanujan's ${}_1\psi_1$ summation formula, q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), 115-137, *Contemp. Math.*, **291**, *Amer. Math. Soc.*, Providence, RI, 2001.

6. S. Cooper, B. Kane, and D. Ye, Analogues of the Ramanujan–Mordell Theorem, *J. Math. Anal. Appl.*, **446** (2017), 568-579.

7. J. Conway, J. McKay, and A. Sebbar, On the discrete groups of moonshine, *Proc. Amer. Math. Soc.*, **132** (2004), 2233-2240.

8. G. F. Eisenstein, *Mathematische Werke,* 2nd ed., Chelsea, New York, 1988.

9. J. W. L. Glaisher, On the numbers of representations of a number as a sum of $2r$ squares, where $2r$ does not exceed eighteen, *Proc. London Math. Soc.*, **5** (1907), 479-490.

10. B. Gordon and K. Hughes, Multiplicative properties of $\eta-$products II, A tribute to Emil Grosswald: Number Theory and related analysis, *Cont. Math. of the Amer. Math. Soc.*, **143** (1993), 415-430.

11. C. G. J. Jacobi, *Gesammelte Werke. Bände I,* Herausgegeben auf Veranlassung der Königlich Preussischen Akademie der Wissenschaften. Zweite Ausgabe, Chelsea, New York, 1969.

12. G. Ligozat, Courbes modulaires de genre 1, *Bull. Soc. Math.*, France [Memoire 43] (1972), 1-80.

13. J. Liouville, Nombre des représentations d'un entier quelconque sous la forme d'une comme de dix carrés, *J. Math. Pures Appl.*, **11** (1966), 1-8.

14. Y. Martin, Multiplicative η-quotients, *Trans. Amer. Math. Soc.*, **348** (1996), 4825-4856.

15. T. Miyake, *Modular forms*, Springer, Heidelberg, 1989.

16. L. J. Mordell, On the representation of numbers as the sum of 2r squares, *Quart. J. Pure and Appl. Math.*, Oxford **48** (1917), 93-104.

17. M. Newman, Construction and application of a certain class of modular functions, *Proc. London Math. Soc.*, **7** (1956), 334-350.

18. M. Newman, Construction and application of a certain class of modular functions II, *Proc. London Math. Soc.*, **9** (1959), 373-387.

19. S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Philos. Soc.*, **22** (1916), 159-184.

20. S. Ramanujan, *Collected papers of Srinivasa Ramanujan*, edited by G. H. Hardy *et al.*, AMS Chelsea, Providence, RI, 2000.

21. W. Stein, Modular forms, a computational approach, Graduate studies in mathematics, *Amer. Math. Soc.*, **79** 2007.

22. D. Pei and X. Wang, *Modular forms with integral and half-integral weights*, Springer, Science Press Beijing, 2012.

23. D. Ye, Representations of integers by certain $2k$-ary quadratic forms, *J. Number Theory*, **179** (2017), 50-64.