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HARDY-TYPE OPERATORS IN LORENTZ-TYPE SPACES DEFINED ON MEASURE SPACES[1]

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Weight criteria for the boundedness and compactness of generalized Hardy-type operators

$$Tf(x) = u_1(x) \int_{\{\phi(y)\le\psi(x)\}} f(y)u_2(y)v_0(y)\, d\mu(y), \quad x \in X, \tag{0.1}$$

in Orlicz-Lorentz spaces defined on measure spaces is investigated where the functions $\phi,\ \psi,\ u_1,\ u_2,\ v_0$ are positive measurable functions. Some sufficient conditions of boundedness of $T:\ \Lambda_{v_0}^{G_0}(w_0) \to \Lambda_{v_1}^{G_1}(w_1)$ and $T:\ \Lambda_{v_0}^{G_0}(w_0) \to \Lambda_{v_1}^{G_1,\infty}(w_1)$ are obtained on Orlicz-Lorentz spaces. Furthermore, we achieve sufficient and necessary conditions for T to be bounded and compact from a weighted Lorentz space $\Lambda_{v_0}^{p_0}(w_0)$ to another $\Lambda_{v_1}^{p_1,q_1}(w_1)$. It is notable that the function spaces concerned here are quasi-Banach spaces instead of Banach spaces.

Key words : Hardy operator; Orlicz-Lorentz spaces; weighted Lorentz spaces; boundedness; compactness.

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1. INTRODUCTION

For the Hardy operator S defined by $Sf(x) = \int_0^x f(t)dt$, the weighted Lebesgue-norm inequalities have been characterized by many authors (e.g. [3, 12, 30, 32]). Sawyer [36] characterized the weights

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u, v such that $S: L^{p,q}(u) \to L^{r,s}(v)$ is bounded, under certain restriction on exponents $p,\ q,\ r$ and s. Later on Carro and Soria [5] described the exponents $p_0,\ p_1$, the weights $u_0,\ u_1,\ w_0,\ w_1$ such that $S: \Lambda^{p_0}_{v_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$ or $S: \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1}_{u_1}(w_1)$ is bounded.

For the Hardy operator A by $Af(x) = \frac{1}{x}\int_0^x f(t)dt$, Sawyer [36] analyzed the weights v and w such that $A: L^r(v) \to L^{p,q}(w)$ is bounded under some assumptions on exponents $p,\ q,\ r$. Given non-negative measurable functions ψ and ϕ on $\mathbb{R}_+$ define the operator

$$H_1 f(x) = \psi(x)\int_0^x \phi(t)f(t)dt,\ x > 0.$$

Ferreyra [10] gave a characterization of boundedness of $H_1: L^{r_1}(u_1) \to L^{p_1,q_1}(w_1)$ under the assumptions $1 \le r_1 \le \min(p_1, q_1)$ and normability of $L^{p_1,q_1}(w_1)$. Edmund, Gurka and Pick in [7, Theorems 3-4] obtained characterization of boundedness and compactness of

$$H_1: L^{r_0,s_0}(v_0) \to L^{p_0,q_0}(w_0),$$

when $\max(r_0, s_0) \le \min(p_0, q_0)$ and the Lorentz spaces $L^{r_0,s_0}(v_0)$ and $L^{p_0,q_0}(w_0)$ are normable. The result in [7] can also be used to the description of boundedness and compactness of the high dimensional Hardy operator

$$Hf(x) = \psi(x)\int_{B(0,|x|)} \phi(y)f(y)dy,\ x \in \mathbb{R}^n, \tag{1.1}$$

from $L^{r,s}(u)$ to $L^{p,q}(w)$ where $\psi,\ \phi$ are non-negative measurable functions on $\mathbb{R}^n$, $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$, $B(0,t)$ is the ball of radius t in $\mathbb{R}^n$ centered at 0 when $\max(r,s) \le \min(p,q)$ and the Lorentz spaces $L^{r,s}(u)$ and $L^{p,q}(w)$ are normable. Martín-Reyes, Ortega Salvador and Sarrión Gavilán [28] discovered the conditions for boundedness of

$$H: \Lambda^{p_0}_{v_0}(w_0) \to \Lambda^{p_1,q_1}_{u_1}(w_1) \tag{1.2}$$

when $0 < p_0 \le p_1 \le q_1 \le \infty$, w_1 is a non-increasing. Li and Kaminska [24] study boundedness and compactness of

$$H: \Lambda^{G_0}_{v_0}(w_0) \to \Lambda^{G_1}_{u_1}(w_1) \text{ and } H: \Lambda^{G_0}_{v_0}(w_0) \to \Lambda^{G_1,\infty}_{u_1}(w_1)$$

on the Orlicz-Lorentz spaces and (1.2) on the weighted Lorentz spaces, improving the results in [28] through enlarging the range of weights and indices.

Edmunds, Kokilashvili, and Meskhi [8] (see also [9]) considered the Hardy-type operator T on a $\sigma-$finite measure space (X,μ) defined as

$$Tf(x) = u_1(x)\int_{\{\phi(y)\le\psi(x)\}} f(y)u_2(y)v_0(y)\,d\mu(y), \quad x \in X. \tag{1.3}$$

where the functions $\phi,\ \psi,\ u_1,\ u_2,,\ v_0$ are positive measurable functions on $(X,\ \mu)$ and for every $t_1,\ t_2$ with $0 < t_1 < t_2 < \infty$ the conditions:

$$0 < \mu\{y \in X : t_1 < \phi(y) < t_2\} < \infty,$$

$$0 < \mu\{x \in X : t_1 < \psi(x) < t_2\} < \infty,$$

are fulfilled. Obviously, the operator T is a generalization of the operators H_1 and H. The authors found a characterization of boundedness and compactness of T from a Banach function space (X_1, μ, v_0) to another (X_2, μ, v_1) on which some special conditions are required.

The first main result in the present paper is Proposition 3.1 which gives necessary and sufficient conditions of the modular inequalities,

$$\|G(Tf)\|_{\Lambda^{p_1}_{v_1}(w_1)} \le C\|G(f)\|_{\Lambda^{p_0}_{v_0}(w_0)} \text{ and } \|G(Tf)\|_{\Lambda^{p_1,\infty}_{v_1}(w_1)} \le C\|G(f)\|_{\Lambda^{p_0}_{v_0}(w_0)},$$

when $0 < p_0 \le p_1 < \infty$. It leads to Corollary 3.1 which states sufficient conditions of the boundedness of the operators

$$T : \Lambda^{Gp_0}_{v_0}(w_0) \to \Lambda^{Gp_1}_{v_1}(w_1) \text{ and } T : \Lambda^{Gp_0}_{v_0}(w_0) \to \Lambda^{Gp_1,\infty}_{v_1}(w_1)$$

between Orlicz-Lorentz spaces. These results generalize the result of [24] since the operator H is a special case of T and also improves [8, 9] due to no restriction of the spaces to be Banach spaces. When discussing the sufficiency of Proposition 3.1 we exploit a method quite different from [8, 9] since the spaces studied here are not Banach function spaces and thus we can not use the principle of duality. Specifically, the principle of duality says that if X and Y are Banach function spaces on a measure space (Ω, μ), then the boundedness of

$$T : X \to Y \tag{1.4}$$

is equivalent to the boundedness of $T' : Y' \to X'$ or the establishment of the inequality

$$\left|\int (Tf)g\right| \le C\|f\|_X\|g\|_{Y'}, \tag{1.5}$$

where T' is the conjugate operator of T defined by the formula $\int (Tf)g = \int fT'g$ and X' represents the Köthe dual of X,

$$X' = \left\{g : \int |fg| < \infty \text{ for all } f \in X\right\},$$

and the associate norm of g for $g \in X'$ is endowed by

$$\|g\|_{X'} = \sup_{\|f\|_X \le 1} \int |fg|.$$

[8, 9] transfered the question (1.4) to the proof of (1.5). But we indicate that in the background of quasi-Banach function spaces this method is false. For example, if $Y = L^{1,q}$, $1 < q \leq \infty$, then $Y' = \{0\}$ and thus we can not change the question (1.4) to (1.5).

Particularly, for $Y = \Lambda_{v_1}^{p_1}(w_1)$ or $\Lambda_{v_1}^{p_1,\infty}(w_1)$ under some assumptions on exponents and weights, the present Corollary 3.2 gives a sufficient and necessary condition of the boundedness of

$$T: \ \Lambda_{v_0}^{p_0}(w_0) \to Y. \tag{1.6}$$

Under the conditions of Corollary 3.2, Theorem 3.1 presents the characterization of the boundedness of (1.6) which is analogous to [9, Theorem 1.6] but due to weaker assumptions it is also an improvement. At the same time, Theorem 3.1 extends [24, Theorem 3.9] since the operator T is more general than H. Meanwhile, we get a sufficient condition of the boundedness of

$$P : \Lambda_{v_0}^{p_0}(w_0) \to Y$$

where

$$Pf(x) = u_1(x) \int_{\{a\psi(x)<\phi(y)<b\psi(x)\}} f(y)u_2(y)v_0(y)d\mu(y)$$

which extends the corresponding results in [8, 9] since the weighted Lorentz spaces are only quasi-Banach spaces. Similarly to the results on the boundedness of the operator H, a characterization of the compactness of (1.6) is given in Theorem 4.1 under quasi-Banach spaces. This result extends [7, Theorem 4] and [24, Theorem 4.1] as well. Finally we provide simpler characterizations of compactness of H in the spirit of Prokhorov's results on the Lebesgue spaces [33, Theorem 3] between two weighted Lorentz spaces with one being $L^{p,q}$ when either u_1 or u_2 and ϕ are power functions. All the results can also be applied to the case when the underlying space X is a homogeneous or nonhomogeneous space which is concisely demonstrated in the end.

2. Preliminaries

Let (X, μ) be a σ-finite measure space and $\mathcal{M}(X, \mu)$ be the space of all μ-measurable real valued functions on X. The decreasing rearrangement f_μ^* of $f \in \mathcal{M}(X, \mu)$ is defined in [2] by

$$f_\mu^*(t) = \inf\{s : \lambda_f^\mu(s) \leq t\}, \ \ t \geq 0,$$

where

$$\lambda_f^\mu(s) = \mu\{x \in X : |f(x)| > s\}, \ s \geq 0,$$

is a distribution function of f. The function $w : X \to \mathbb{R}_+$ is called a weight function, or simply a weight, whenever w is measurable, not identically equal to zero and integrable on sets of finite measure. If w is a weight on $\mathbb{R}_+$, then we denote $W(t) = \int_0^t w(s)\,ds$, and we always have that $W(t) < \infty$, $t > 0$. Letting $0 < p,\ q < \infty$, we say that $f \in \mathcal{M}(X,\mu)$ belongs to the Lorentz space $L^{p,q}(X)$ [2, 14] if $\|f\|_{L^{p,q}(X)} = \left(\int_0^\infty (t^{1/p} f_\mu^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty$, and for $0 < p \le \infty$, the space $L^{p,\infty}(X)$ is defined as a class of $\mathcal{M}(X,\mu)$ such that $\|f\|_{L^{p,\infty}(X)} = \sup_{t>0} t^{1/p} f_\mu^*(t) < \infty$. If $(X,\mu) = (\mathbb{R}_+, w dx)$, we use the notation $L^{p,q}(X) = L^{p,q}(w)$ and $\mu(E) = w(E)$ for every Lebesgue measurable subset E of $\mathbb{R}_+$.

Let w be a weight on $\mathbb{R}_+$. Define for $0 < p,\ q < \infty$ the weighted Lorentz space $\Lambda_X^{p,q}(w)$ (see [4] or [5]) as a class of $f \in \mathcal{M}(X,\mu)$ such that

$$\|f\|_{\Lambda_X^{p,q}(w)} = \|f_\mu^*\|_{L^{p,q}(w)} = p^{1/q} \left\| y \left(\int_0^{\lambda_f^\mu(y)} w(t)dt \right)^{\frac{1}{p}} \right\|_{L^q(\frac{dy}{y})} < \infty,$$

where $\|g\|_{L^q(\frac{dy}{y})} = (\int_0^\infty |g(y)|^q \frac{dy}{y})^{1/q}$, and the weighted Lorentz space $\Lambda_X^{p,\infty}(w)$ consisting of $f \in \mathcal{M}(X,\mu)$ with

$$\|f\|_{\Lambda_X^{p,\infty}(w)} = \|f_\mu^*\|_{L^{p,\infty}(w)} = \sup_{y>0} y \left(\int_0^{\lambda_f^\mu(y)} w(t)dt \right)^{\frac{1}{p}} < \infty.$$

We agree on the convention $\Lambda_X^p(w) = \Lambda_X^{p,p}(w)$ and note that for $0 < p,\ q < \infty$, $\Lambda_X^{p,q}(w) = \Lambda_X^q(\bar{w})$ where $\bar{w} = W^{\frac{q}{p}-1} w$.

Let $L_{dec}^p(w)$ be the cone of all non-increasing functions in $L^p(w)$ where w is a weight on $[0,\infty)$, $0 < p < \infty$ and the operator A is defined by $Af(t) = \frac{1}{t}\int_0^t f(s)ds$ for all nonnegative measurable functions f on $\mathbb{R}_+$. Ariño and Muckenhoupt [1] gave a characterization of the boundedness of $A :\ L_{dec}^p(w) \to L^p(w)$ in terms of the inequality on w called condition B_p, that is, w satisfies the following condition:

$$r^p \int_r^\infty \frac{w(x)}{x^p} dx \le C \int_0^r w(x)dx, \quad r > 0,$$

for some $C > 0$. Carro and Soria [6] obtained a characterization of boundedness of $A :\ L_{dec}^p(w) \to L^{p,\infty}(w)$ showing that A is bounded whenever $w \in B_{p,\infty}$ that is there exists $C > 0$ such that if $p > 1$ then

$$\left(\int_0^r \left(\frac{1}{x} \int_0^x w(t)dt \right)^{-p'} w(x)dx \right)^{1/p'} \left(\int_0^r w(x)dx \right)^{1/p} \le Cr, \quad r > 0,$$

and if $p \le 1$ then

$$\frac{1}{r^p} \int_0^r w(x)dx \le \frac{C}{s^p} \int_0^s w(x)dx, \quad 0 < s < r.$$

For other characterizations of $B_p,\ B_{p,\infty}$, we refer to [4, 18, 37, 40]. We know [4, Theorem 2.2.5] that if $p \ge 1$ and $w \in B_{p,\infty}$ then $\Lambda_X^p(w)$ is normable and if $w \in B_p$ then $\Lambda_X^{p,\infty}(w)$ is normable.

Let $G : [0,\infty) \to [0,\infty)$ be an Orlicz function [31], in symbol $G \in \mathfrak{F}$, that is G is continuous and strictly increasing on $\mathbb{R}_+$, such that $\lim_{t\to\infty} G(t) = \infty$ and $G(0) = 0$. Given $G \in \mathfrak{F}$ and a weight w on $\mathbb{R}_+$, the Orlicz-Lorentz space $\Lambda_X^G(w)$ (resp. $\Lambda_X^{G,\infty}(w)$) [15, 16, 23, 27, 29, 31] is the set of $f \in \mathcal{M}(X,\mu)$ such that for some $\lambda > 0$, we have $I_{X,w}^G(\lambda f) < \infty$ (resp. $I_{X,w}^{G,\infty}(\lambda f) < \infty$), where

$$I_{X,w}^G(f) = \int_0^\infty G(f_\mu^*(t))w(t)dt, \quad \left(\text{resp. } I_{X,w}^{G,\infty}(f) = \sup_{t>0} G(f_\mu^*(t))W(t)\right),$$

and we let

$$\|f\|_{\Lambda_X^G(w)} = \inf\left\{\epsilon > 0:\ I_{X,w}^G\left(\frac{f}{\epsilon}\right) \le 1\right\} \ \left(\text{resp. } \|f\|_{\Lambda_X^{G,\infty}(w)} = \inf\left\{\epsilon > 0:\ I_{X,w}^{G,\infty}\left(\frac{f}{\epsilon}\right) \le 1\right\}\right).$$

We will assume further, without loss of generality, that the weight w vanishes on the interval $[\mu(X),\infty)$ if $\mu(X) < \infty$.

For $G \in \mathfrak{F}$, define its lower and upper Matuszewska-Orlicz indices [27] as follows:

$$\alpha_G = \sup\{r > 0 : \sup_{0<a\le 1,\ t>0} \frac{G(at)}{G(t)a^r} < \infty\}, \ \ \beta_G = \inf\{r > 0 : \inf_{0<a\le 1,\ t>0} \frac{G(at)}{G(t)a^r} > 0\}.$$

We say that a function $G : [0,\infty) \to [0,\infty)$ satisfies condition Δ_2, in symbol $G \in \Delta_2$, whenever $\sup_{t>0} G(2t)/G(t) < \infty$. It is well known that $\beta_G < \infty$ if and only if $G \in \Delta_2$. Kamińska and Raynaud showed in [20, Proposition 4.5] that if $\alpha_G > 0$ and $W \in \Delta_2$, then $\|\cdot\|_{\Lambda_X^G(w)}$ and $\|\cdot\|_{\Lambda_X^{G,\infty}(w)}$ are quasi-norms.

If $G(t) = t^p$, $0 < p < \infty$, then $\Lambda_X^G(w) = \Lambda_X^p(w)$ and $\Lambda_X^{G,\infty}(w) = \Lambda_X^{p,\infty}(w)$ are weighted Lorentz spaces (see [4, 26]). If a measure $vd\mu(y)$ is given on X, we denote $\Lambda_X^G(w) = \Lambda_v^G(w)$, $\Lambda_X^{G,\infty}(w) = \Lambda_v^{G,\infty}(w)$, $\Lambda_X^p(w) = \Lambda_v^p(w)$, $\Lambda_X^{p,\infty}(w) = \Lambda_v^{p,\infty}(w)$ and if $v = 1$ then $L^{p,q}(X) = L^{p,q}$.

Letting $0 < p < \infty$, $f_\mu^{**}(t) = A(f_\mu^*)(t)$, $t > 0$, for $f \in \mathcal{M}(X,\mu)$, define the space [4, Section 2.2.4]

$$\Gamma_X^p(w) = \{f \in \mathcal{M}(X,\mu) : \|f\|_{\Gamma_X^p(w)} = \int_0^\infty f_\mu^{**p}(t)w(t)dt < \infty\},$$

and if Φ is a non-negative function on $\mathbb{R}_+$, define

$$\Gamma_X^{p,\infty}(d\Phi) = \{f \in \mathcal{M}(X,\mu) : \|f\|_{\Gamma_X^{p,\infty}(d\Phi)} = \sup_{t>0} f_\mu^{**}(t)\Phi^{1/p}(t) < \infty\}.$$

If a measure $vd\mu(y)$ is given on X, denote $\Gamma_X^p(w) = \Gamma_v^p(w)$ and $\Gamma_X^{p,\infty}(d\Phi) = \Gamma_v^{p,\infty}(d\Phi)$.

A function $G \in \mathfrak{F}$ is said to satisfy Δ' (resp. ∇') condition [22, 35], in symbol $G \in \Delta'$ (resp. ∇') if there exists $C > 0$ such that

$$G(xy) \le CG(x)G(y), \ \ x, y \ge 0 \text{ (resp. } G(Cxy) \ge G(x)G(y), \ \ x,\ y \ge 0).$$

We have that $G \in \nabla'$ if and only if there exists $C > 0$ such that $G(xy) \geq CG(x)G(y), x, y \geq 0$. The explanation can be found in [24]. It is also easy to see that $G \in \Delta'$ yields $G \in \Delta_2$.

A strictly increasing positive sequence $\{x_j\}_{j\in\mathbb{Z}}$ is called a covering sequence [13] if the sequence is of the form $\{x_j\}_{j=-\infty}^{j=\infty}$ or of the form $\{x_j\}_{j=N}^{j=M}$, where M and/or N is finite. In the latter case we define $x_{N-1} = 0$ and/or $x_{M+1} = \infty$.

Throughout the paper, we assume that the expressions of the form $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are equal to zero. Given $1 \leq p < \infty$ denote by p' its conjugate index that is $\frac{1}{p} + \frac{1}{p'} = 1$. The notation $f \approx g$ indicates the existence of a universal constant $C > 0$ independent of all parameters involved, so that $(1/C)f \leq g \leq Cf$. The symbol $f \downarrow$ indicates that f is a non-negative non-increasing function in $(0, b)$ for given $b > 0$ with $0 < b \leq \infty$. Note that the constant C, unless specifically stated otherwise, may differ from one occurrence to another.

3. Boundedness of Hardy Operators

Let for the rest of the paper $G \in \mathfrak{F}$, w_0, w_1 be the weights on $\mathbb{R}_+$ and v_0, v_1 the weights on (X, μ). The operator T is defined by (1.3). Let us consider first when the operator $T: \Lambda_{v_0}^{Gp_0}(w_0) \to \Lambda_{v_1}^{Gp_1}(w_1)$ (resp. $\Lambda_{v_1}^{Gp_1,\infty}(w_1)$) is bounded. We begin from looking for necessary and sufficient conditions of the modular inequality

$$\|G(Tf)\|_Y \leq C\|G(|f|)\|_{\Lambda_{v_0}^{p_0}(w_0)}, \quad f \in \Lambda_{v_0}^{Gp_0}(w_0),$$

where $Y = \Lambda_{v_1}^{p_1}(w_1)$ or $\Lambda_{v_1}^{p_1,\infty}(w_1)$. The following lemma is [24, Lemma 3.1] essentially however they have different pattern for representation and we omit the proof.

Lemma 3.1 — (i) Let $0 < p_0 \leq \sigma$. There exists $C > 0$ such that for every $\{t_k\}_k \subset \mathbb{R}_+$ with $\sum_k t_k \leq v_0(X)$,

$$\sum_k \left(\int_0^{t_k} w_0(t)dt\right)^{\frac{\sigma}{p_0}} \leq C \left(\int_0^{\sum_k t_k} w_0(t)dt\right)^{\frac{\sigma}{p_0}} \tag{3.1}$$

if and only if there exists a constant $C > 0$ such that for every collection of measurable sets E_k with $\sum_k \chi_{E_k} \leq c$ and a function $f \in \Lambda_{v_0}^{p_0}(w_0)$,

$$\left\| f\chi_{\cup_k E_k} \right\|_{\Lambda_{v_0}^{p_0}(w_0)}^{\sigma} \geq C \sum_k \|f\chi_{E_k}\|_{\Lambda_{v_0}^{p_0}(w_0)}^{\sigma}. \tag{3.2}$$

(ii) Let $p_1 \geq \sigma > 0$. There exists $C > 0$ such that for all $\{t_k\}_k \subset \mathbb{R}_+$ satisfying $\sum_k t_k < v_1(X)$,

$$\sum_k \left(\int_0^{t_k} w_1(t)dt\right)^{\frac{\sigma}{p_1}} \geq C \left(\int_0^{\sum_k t_k} w_1(t)dt\right)^{\frac{\sigma}{p_1}}, \tag{3.3}$$

if and only if there exists a constant $C > 0$ such that for every collection of measurable sets E_k with $\sum_k \chi_{E_k} \leq c$ and a function $f \in \Lambda^{p_1}_{v_1}(w_1)$,

$$\left\| f\chi_{\cup_k E_k} \right\|^{\sigma}_{\Lambda^{p_1}_{v_1}(w_1)} \leq C \sum_k \|f\chi_{E_k}\|^{\sigma}_{\Lambda^{p_1}_{v_1}(w_1)}. \tag{3.4}$$

(iii) Let $p_1,\ \sigma > 0$. Then w_1 satisfies (3.3) if and only if there exists a constant $C > 0$ such that for every collection of measurable sets E_k with $\sum_k \chi_{E_k} \leq c$ and a function $f \in \Lambda^{p_1,\infty}_{u_1}(w_1)$,

$$\left\| f\chi_{\cup_k E_k} \right\|^{\sigma}_{\Lambda^{p_1,\infty}_{u_1}(w_1)} \leq C \sum_k \|f\chi_{E_k}\|^{\sigma}_{\Lambda^{p_1,\infty}_{u_1}(w_1)}.$$

Remark 3.1 : (1) We say that $l \in B_\Psi$ [11, Definition 1.2] if $l:\ \mathbb{R}_+ \to \mathbb{R}_+$, $l(1) = 1$, $l \in C^1$ (the class of functions having continuous first derivatives), and

$$0 < \xi_l = \inf_{t>0} \frac{tl'(t)}{l(t)} \leq \sup_{t>0} \frac{tl'(t)}{l(t)} = \eta_l < 1.$$

If $l \in B_\Psi$, $1 < p_0 \leq \sigma$, $1 \leq \sigma(1-\eta_l)$, then $w_0(t) = \left(\frac{t^{1/p_0'}}{l(t)}\right)^{p_0}$ satisfies (3.1) [5, Proposition 4.3]. The classes of weights satisfying (3.3) can be found in [24, Remark 3.2].

(2) If $\|\cdot\|_{\Lambda^{p_0}_{v_0}(w_0)}$ (resp. $\|\cdot\|_{\Lambda^{p_1}_{v_1}(w_1)}$) is a quasi-norm then (3.2) (resp. (3.4)) means that the space $\Lambda^{p_0}_{v_0}(w_0)$ (resp. $\Lambda^{p_1}_{v_1}(w_1)$) satisfies the lower (resp. upper) σ-estimate for $\sigma \geq p_0$ (resp. $p_1 \geq \sigma$). There exist other characterizations of $\Lambda^{p_0}_{v_0}(w_0)$ (resp. $\Lambda^{p_1}_{v_1}(w_1)$) satisfying the lower (resp. upper) σ-estimate for $\sigma \geq p_0$ (resp. $p_1 \geq \sigma$) [18].

Proposition 3.1 — Let $0 < p_0 \leq p_1 < \infty$ and $Y = \Lambda^{p_1}_{v_1}(w_1)$ or $Y = \Lambda^{p_1,\infty}_{v_1}(w_1)$. Assuming that $G \in \nabla'$, the inequality

$$\|G(|Tf|)\|_Y \leq C\|G(|f|)\|_{\Lambda^{p_0}_{v_0}(w_0)}, \quad f \in \Lambda^{Gp_0}_{v_0}(w_0), \tag{3.5}$$

implies that there is a constant $C > 0$ such that for all $a > 0$, $f \in \Lambda^{Gp_0}_{v_0}(w_0)$,

$$\frac{G(|\int_X f(y)u_2(y)v_0(y)\chi_{\{\phi(y)\leq t\}}(y)d\mu(y)|)}{\|G(|f|)\|_{\Lambda^{p_0}_{v_0}(w_0)}} \left\| G\left(u_1\chi_{\{\psi(x)\geq t\}}\right)\right\|_Y \leq C. \tag{3.6}$$

Conversely, if $G \in \Delta'$ and there exists $p_0 \leq \sigma \leq p_1$ when $Y = \Lambda^{p_1}_{u_1}(w_1)$, and $\sigma \geq p_0$ when $Y = \Lambda^{p_1,\infty}_{u_1}(w_1)$ such that (3.1) and (3.3) hold, then (3.6) implies (3.5).

PROOF : Let $G \in \nabla'$ and (3.5) hold. Without loss of generality, let $f \geq 0$. Since for each $a > 0$ and $\psi(s) \geq a$,

$$Tf(s) \geq u_1(s) \int_{\{\phi(y)\leq a\}} u_2(y)f(y)v_0(y)d\mu(y),$$

by the modular inequality,

$$\begin{aligned} \|G(f)\|_{\Lambda^{p_0}_{v_0}(w_0)} &\geq C\|G(Tf)\|_Y \geq C\left\|G\left(u_1\chi_{\psi(s)\geq a}\int_{\{\phi(y)\leq a\}} u_2(y)f(y)v_0(y)d\mu(y)\right)\right\|_Y \\ &\geq C\left\|G\left(\int_{\{\phi(y)\leq a\}} u_2(y)f(y)d\mu(y)\right)G(u_1\chi_{\psi(s)\geq a})\right\|_Y, \text{ by } \mathrm{G}\in\nabla' \\ &= CG\left(\int_{\{\phi(y)\leq a\}} u_2(y)f(y)d\mu(y)\right)\left\|G(u_1\chi_{\psi(s)\geq a})\right\|_Y \end{aligned}$$

which is (3.6).

Let

$$I(s) = \int_{\{\phi(y)\leq s\}} f(y)u_2(y)v_0(y)d\mu(y).$$

Then $I(s)$ is right continuous. Let

$$\int_{\{\phi(y)\neq 0\}} f(y)u_2(y)v_0(y)d\mu(y) \in [2^m, 2^{m+1})$$

for $m \in Z$ and $s_j = \sup\{s : I(s) < 2^j\},\ j \leq m$.

Then

$$I(s_j) \geq 2^j,\ \ and\ \ I(s) < 2^j,\ s < s_j.$$

Let $J_m = \{j \leq m+1 : s_{j+1} > s_j\}$ and $\beta = \lim_{j\to-\infty} s_j$ and $s_{m+1} = \infty$. Then

$$(0,\infty) = (\cup_{j\in J_m}[s_j, s_{j+1})) \cup (0,\beta)$$

if J_m is finite, and

$$(0,\infty) = (\cup_{j\in J_m}[s_j, s_{j+1})) \cup (0,\beta]$$

if J_m is infinite.

Thus

$$\{x \in X : \psi(x) \neq 0\} = (\cup_{j\in J_m} E_j) \cup F,$$

where $E_j = \{s_j \leq \psi(x) < s_{j+1}\}$, $F = \{0 < \psi(x) \leq \beta\}$. Let $F_1 = \{x : \psi(x) = 0\}$. If $s \in E_j$, then $2^j \leq I(s) < 2^{j+1}$; if $x \in F$, then $I(\psi(x)) = 0$, that is, $Tf(x) = u_1(x)I(\psi(x)) = 0$. Now

$$\begin{aligned}
\|G(|Tf|)\|_Y^\sigma &= \left\|G\left(\sum_{j\in J_m}(Tf)\chi_{E_j} + (Tf)\chi_{F_1}\right)\right\|_Y^\sigma \\
&= \left\|\sum_{j\in J_m} G((Tf)\chi_{E_j}) + G((Tf)\chi_{F_1})\right\|_Y^\sigma \\
&\leq C\left[\sum_{j\in J_m}\|G((Tf)\chi_{E_j})\|_Y^\sigma + \|G((Tf)\chi_{F_1})\|_Y^\sigma\right], \text{ Lemma 3.1 } (ii) \text{ and } (iii) \\
&\leq C\left[\sum_{j\in J_m}\|G(u_1 2^{j+1})\chi_{E_j}\|_Y^\sigma + C\|G((Tf)\chi_{F_1})\|_Y^\sigma\right] \\
&\leq CI_1 + CI_2.
\end{aligned}$$

But

$$\begin{aligned}
I_1 &\leq C\sum_{j\in J_m} G^\sigma(2^{j+1})\|G(u_1\chi_{E_j})\|_Y^\sigma, \\
&\leq C\sum_{j\in J_m} G^\sigma(2^{j-1})\|G(u_1\chi_{E_j})\|_Y^\sigma, \text{ by G} \in \Delta', \\
&\leq C\sum_{j\in J_m} G^\sigma\left(\int_{\{s_{j-1}\leq\phi(y)\leq s_j\}} f(y)u_2(y)v_0(y)d\mu(y)\right)\|G(u_1)\chi_{E_j}\|_Y^\sigma \\
&\leq C\sum_{j\in J_m} G^\sigma\left(\int_{\{\phi(y)\leq s_j\}} f(y)\chi_{\{s_{j-1}\leq\phi(y)\leq s_j\}}u_2(y)v_0(y)d\mu(y)\right)\|G(u_1)\chi_{\{\psi(x)\geq s_j\}}\|_Y^\sigma \\
&\leq C\sum_{j\in J_m}\|G(f\chi_{\{s_{j-1}\leq\phi(y)\leq s_j\}})\|_{\Lambda_{v_0}^{p_0}(w_0))}^\sigma, \text{ by (3.6)} \\
&\leq C\left\|\sum_{j\in J_m} G(f\chi_{\{s_{j-1}\leq\phi(y)\leq s_j\}})\right\|_{\Lambda_{v_0}^{p_0}(w_0))}^\sigma, \text{ by Lemma 3.3 } (i) \\
&\leq C\|G(f)\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq \|G(u_1(x)\chi_{\{\psi(x)=0\}})\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma G(\int_{\{\phi(y)=0\}} f(y)u_2(y)v_0(y))^\sigma \\
&\leq \|G(u_1(x)\chi_{\{\psi(x)\geq 0\}})\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma G(\int_{\{\phi(y)\leq 0\}} f(y)u_2(y)v_0(y))^\sigma \\
&\leq C\|G(f)\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma, \text{ by (3.6).}\square
\end{aligned}$$

We may express the inequality in Proposition 3.1 in different terms by using the following lemma which can be obtained analogously to [13, Theorem 3.1 and Corollary 3.2].

Lemma 3.2 — Let $b > 0$. For a non-negative function g on $(0, b)$ and $0 < p_0 < \infty$ we have

$$\sup_{f\downarrow} \frac{G(\int_0^b f(t)g(t)dt)}{\|G(f)\|_{L^{p_0}(w_0)}} \approx A^{1/p_0},$$

where

$$A = \sup \left\{ \frac{G_0(\sum_j \epsilon_j \int_{x_j}^{x_{j+1}} g(t)dt)}{\sum_j G_0(\epsilon_j) \int_{x_j}^{x_{j+1}} w_0(t)dt} : \{x_j\} \text{ is a covering sequence, } \int_0^{x_j} w_0 = 2^k,\ k \in \mathbb{Z},\ \epsilon_j \downarrow \right\} \tag{3.7}$$

with $G_0 = G^{p_0}$.

If additionally

$$G_0\Big(\sum a_j\Big) \le C \sum G_0(a_j) \quad \text{for all non-negative sequences } \{a_j\}_{j\in\mathbb{Z}}, \tag{3.8}$$

then

$$A \approx \sup_{0<r,\epsilon<b} \frac{G_0(G_0^{-1}(\frac{\epsilon}{W_0(r)}) \int_0^r g(t)dt)}{\epsilon}. \tag{3.9}$$

Applying Lemmas 3.1 and 3.2, we get the following result.

Lemma 3.3 — Let $0 < p_0 \le p_1 < \infty$, $Y = \Lambda_{v_1}^{p_1}(w_1)$ or $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$. Let also $B(a)$ be the right hand side of (3.7) and if additionally (3.8) holds, then $B(a)$ is the right hand side of (3.9), with $g = (u_2\chi_{\{\phi(y)\le a\}})_{v_0}^*$ and $b = v_0(X)$. If $G \in \nabla'$, then (3.5) implies

$$\sup_{a>0} B(a)^{1/p_0} \left\|G\left(u_1\chi_{\psi(x)>a}\right)\right\|_Y < \infty. \tag{3.10}$$

Conversely if $G \in \Delta'$ and there exists $p_0 \le \sigma \le p_1$ when $Y = \Lambda_{v_1}^{p_1}(w_1)$, and $\sigma \ge p_0$ when $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, such that (3.1) and (3.3) hold, then (3.10) implies (3.5).

PROOF : By the property of rearrangement of function [2, Theorem 2.7], which implies

$$\sup_{f\in\Lambda_{v_0}^{p_0}(w_0)} \frac{G(|\int_X f(y)u_2(y)v_0(y)\chi_{\{\phi(y)\le a\}}(y)d\mu(y)|)}{\|G(f)\|_{\Lambda_{v_0}^{p_0}(w_0)}} = \sup_{f\in\Lambda_{v_0}^{p_0}(w_0)} \frac{G(\int_0^\infty f_{v_0}^*(t)(u_2\chi_{\phi(y)\le a})_{v_0}^*(t)dt)}{\|G(f_{v_0}^*)\|_{L^{p_0}(w_0)}}$$

$$= \sup_{f\downarrow} \frac{G(\int_0^\infty f(t)(u_2\chi_{\phi(y)\le a})_{v_0}^*(t)dt)}{\|G(f)\|_{L^{p_0}(w_0)}},$$

and Lemmas 3.1-3.2, the Lemma holds. □

In view of Lemma 3.3 and the fact that modular inequalities can deduce norm inequalities, we get the following sufficient condition of the boundedness of T between Orlicz-Lorentz spaces.

Corollary 3.1 — Let $0 < p_0 \le p_1 < \infty$, $G \in \Delta'$. Then

(i) If (3.10) holds with $Y = \Lambda_{v_1}^{p_1}(w_1)$, and there exists $p_0 \le \sigma \le p_1$ such that (3.1), (3.3) hold, then $T: \Lambda_{v_0}^{Gp_0}(w_0) \to \Lambda_{v_1}^{Gp_1}(w_1)$ is bounded.

(ii) If (3.10) holds with $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, and there exists $\sigma \ge p_0$ such that (3.1), (3.3) hold, then $T: \Lambda_{v_0}^{Gp_0}(w_0) \to \Lambda_{v_1}^{Gp_1,\infty}(w_1)$ is bounded.

Letting $G(t) = t^\alpha,\ \alpha > 0$, clearly $G \in \Delta' \cap \nabla'$. Taking without loss of generality $\alpha = 1$ we obtain the next corollary as a consequence of Lemma 3.3.

Corollary 3.2 — Let $0 < p_0 \le p_1 < \infty$, (X, μ) be a nonatomic resonance measure space, $Y = \Lambda_{v_1}^{p_1}(w_1)$ or $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, and there exists $p_0 \le \sigma \le p_1$ when $Y = \Lambda_{v_1}^{p_1}(w_1)$, $\sigma \ge p_0$ when $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, such that (3.1) and (3.3) hold. Then a necessary and sufficient condition for $T : \Lambda_{v_0}^{p_0}(w_0) \to Y$ is

$$\sup_{a>0} D(a)^{1/p_0} \left\| u_1 \chi_{\{\psi(x)>a\}} \right\|_Y < \infty,$$

where $D(a)$ is the right hand side of (3.7) and if $p_0 \le 1$, then $D(a)$ can be the right hand side of (3.9), with $g = (u_2 \chi_{\{\phi(y)\le a\}})^*_{v_0}$ and $G_0(t) = t^{p_0}$.

Recall now the formulas of the associate spaces of $\Lambda_{v_0}^{p_0}(w_0)$, for $0 < p_0 < \infty$ [4, Definition 2.4.1]:

$$\left(\Lambda_{v_0}^{p_0}(w_0)\right)' = \begin{cases} \Gamma_{v_0}^{p_0'}(\widetilde{w_0}), & \text{if } w_0 \notin L^1,\ p_0 > 1, \\ \Gamma_{v_0}^{p_0'}(\widetilde{w_0}) \cap L^1, & \text{if } w_0 \in L^1,\ p_0 > 1, \\ \Gamma_{v_0}^{1,\infty}(d\Phi), & \text{if } 0 < p_0 \le 1, \end{cases}$$

where $\widetilde{w_0}(t) = t^{p_0'} W_0^{-p_0'}(t) w_0(t)$, $\Phi(t) = t W_0^{-1/p_0}(t)$ (see [4, Theorem 2.4.7]). In the rest of this paper, the notation $\left(\Lambda_{v_0}^{p_0}(w_0)\right)'$ is meant as above.

The next theorem is another characterization of boundedness of the operator T between weighted Lorentz spaces. It has a simpler pattern than that in Corollary 3.2 and it generalizes [8, Theorem 2.3] and [9, Theorem 1.1.3] by weakenning its assumptions.

***Theorem* 3.1** — *(Charaterization of Boundedness) Let $0 < p_0 \le p_1 < \infty$, $Y = \Lambda_{v_1}^{p_1}(w_1)$ or $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, and there exists $p_0 \le \sigma \le p_1$ when $Y = \Lambda_{v_1}^{p_1}(w_1)$, $\sigma \ge p_0$ when $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, such that (3.1) and (3.3) hold. Then $T : \Lambda_{v_0}^{p_0}(w_0) \to Y$ is bounded if and only if there exists $C > 0$ such that for all $a > 0$,*

$$I(a) := \left\| u_2 \chi_{\{\phi(y)\le a\}} \right\|_{(\Lambda_{v_0}^{p_0}(w_0))'} \cdot \left\| u_1 \chi_{\{\psi(x)\ge a\}} \right\|_Y \le C. \tag{3.11}$$

PROOF : The theorem establishes by using Proposition 3.1 and the following equality

$$\sup_{f\in\Lambda^{p_0}_{v_0}(w_0)} \frac{\int_X f(y)\phi(y)v_0(y)\chi_{\{\phi(y)\le a\}}(y)d\mu(y)}{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}} = \|\phi(y)\chi_{\{\phi(y)\le a\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'}.$$

Remark 3.2 : (i) [24, Theorem 3.9] gave a characterization of the boundedness of

$$H : \Lambda^{p_0}_{v_0}(w_0) \to Y$$

when the underlying space is $X = \mathbb{R}^n$, that is,

$$I(a) := \|u_2\chi_{B(0,a)}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} \cdot \|u_1\chi_{\mathbb{R}^n\setminus B(0,a)}\|_Y \le C. \tag{3.12}$$

Clearly, it is a special case in Theorem 3.1 when $X = \mathbb{R}^n$, $\phi(y) = |y|$ and $\psi(x) = |x|$.

(ii) [8, 9] gave a characterization of the boundedness of $H : X \to Y$ when X, Y are Banach function spaces. But Since the weighted Lorentz spaces in Theorem 3.1 are not required to be Banach spaces, our results remain true for a wider class of spaces than before. Furthermore, it is obvious that the method of the proof of Theorem 3.1 is quite different from that of [8, 9].

Let us consider the dual operator of T

$$T^*f(x) = u_1(x)\int_{\{\phi(y)\ge\psi(x)\}} f(y)u_2(y)v_0(y)\,d\mu(y), \quad x \in X.$$

Similarly to Proposition 3.1 and Theorem 3.1, we get the following results.

Corollary 3.3 — Let $0 < p_0 \le p_1 < \infty$ and $Y = \Lambda^{p_1}_{v_1}(w_1)$ or $Y = \Lambda^{p_1,\infty}_{v_1}(w_1)$. Assuming that $G \in \nabla'$, the inequality

$$\|G(|T^*f|)\|_Y \le C\|G(|f|)\|_{\Lambda^{p_0}_{v_0}(w_0)}, \quad f \in \Lambda^{Gp_0}_{v_0}(w_0), \tag{3.13}$$

implies that there is a constant $C > 0$ such that for all $a > 0$, $f \in \Lambda^{Gp_0}_{v_0}(w_0)$,

$$\frac{G(|\int_X f(y)u_2(y)v_0(y)\chi_{\{\phi(y)\ge t\}}(y)d\mu(y)|)}{\|G(|f|)\|_{\Lambda^{p_0}_{v_0}(w_0)}}\left\|G\left(v_1\chi_{\{\psi(x)\le t\}}\right)\right\|_Y \le C. \tag{3.14}$$

Conversely, if $G \in \Delta'$ and there exists $p_0 \le \sigma \le p_1$ when $Y = \Lambda^{p_1}_{v_1}(w_1)$, and $\sigma \ge p_0$ when $Y = \Lambda^{p_1,\infty}_{v_1}(w_1)$ such that (3.1) and (3.3) hold, then (3.14) implies (3.13).

Corollary 3.4 — Let the assumptions be as in Theorem 3.1. Then $T^* : \Lambda^{p_0}_{v_0}(w_0) \to Y$ is bounded if and only if there exists $C > 0$ such that for all $a > 0$,

$$I(a) := \|u_2\chi_{\{\phi(y)\ge a\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} \cdot \|u_1\chi_{\{\psi(x)\le a\}}\|_Y \le C. \tag{3.15}$$

Next we investigate the boundedness of the operator P defined by

$$Pf(x) = u_1(x) \int_{\{a\psi(x)<\phi(y)<b\psi(x)\}} f(y)u_2(y)v_0(y)d\mu(y).$$

Theorem 3.2 gives a characterization of the boundedness of $P : \Lambda_{v_0}^{p_0}(w_0) \to Y$ which extends [8, Theorem 1.1.5] since the weighted Lorentz spaces considered here are only quasi-Banach spaces. First introduce two operators

$$P_1 f(x) = \chi_{\{a<\psi(x)<b\}}(x)u_1(x) \int_{\{\lambda\psi(x)<\phi(y)<\lambda b\}} f(y)u_2(y)v_0(y)d\mu(y),$$

$$P_2 f(x) = \chi_{\{a<\psi(x)<b\}}(x)u_1(x) \int_{\{\lambda a<\phi(y)<\lambda\psi(x)\}} f(y)u_2(y)v_0(y)d\mu(y).$$

Lemma 3.4 — Let $0 < p_0 \le p_1 < \infty$, $Y = \Lambda_{v_1}^{p_1}(w_1)$ or $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, and there exists $p_0 \le \sigma \le p_1$ when $Y = \Lambda_{v_1}^{p_1}(w_1)$, $\sigma \ge p_0$ when $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$, such that (3.1) and (3.3) hold.

Let

$$\mu\{\phi(y) = t\} = 0 \; for \; every \; t \in [\lambda a, \lambda b],$$

$$\mu\{\psi(x) = t\} = 0 \; for \; every \; t \in [a, b].$$

Let

$$A_{ab}^1 = \sup_{a<t<b} \|u_2\chi_{\{\lambda t<\phi(y)<\lambda b\}}\|_{\Lambda_{v_0}^{p_0}(w_0)'} \|u_1\chi_{\{a<\psi(x)<t\}}\|_Y.$$

Then there exist a constant $C > 0$ such that

$$\|P_1 f\|_Y \le C\|f\chi_{\{\lambda a<\phi(y)<\lambda b\}}\|_{\Lambda_{v_0}^{p_0}(w_0)} \tag{3.16}$$

if and only if

$$A_{ab}^1 < \infty.$$

PROOF : It is apparent that

$$\begin{aligned} P_1 f(x) &= u_1(x)\chi_{\{\lambda a<\lambda\psi(x)<\lambda b\}}(x) \int_{\{\lambda\psi(x)<\phi(y)<\lambda b\}} f(y)u_2(y)v_0(y)d\mu(y) \\ &= u_1(x)\chi_{\{\lambda a<\lambda\psi(x)<\lambda b\}}(x) \int_{\{\phi(y)>\lambda\psi(x)\}} \chi_{\{\lambda\psi(x)<\phi(y)<\lambda b\}}(x) f(y)u_2(y)v_0(y)d\mu(y) \\ &= u_1(x)\chi_{\{\lambda a<\psi_1(x)<\lambda b\}}(x) \int_{\{\phi(y)>\psi_1(x)\}} \chi_{\{\lambda a<\phi(y)<\lambda b\}}(x) f(y)u_2(y)v_0(y)d\mu(y), \end{aligned}$$

where $\psi_1(x) = \lambda\psi(x)$. Thus by Corollary 3.4 we get that (3.16) holds if and only if

$$\sup_{s\geq 0} \|u_2\chi_{\{\phi(y)>s\}}\chi_{\{\lambda a<\phi(y)<\lambda b\}}\|_{\Lambda^{p_0}_{v_0}(w_0)'}\|u_1\chi_{\{\lambda a<\psi_1(x)<\lambda b\}}\chi_{\{\psi_1(x)<s\}}\|_Y < C,$$

i.e.,

$$\sup_{a<t<b} \|u_2\chi_{\{\lambda t<\phi(y)<\lambda b\}}\|_{\Lambda^{p_0}_{v_0}(w_0)'}\|u_1\chi_{\{a<\psi(x)<t\}}\|_Y < C,$$

which complete the proof.

Similarly to Lemma 3.4, it follows that

Lemma 3.5 — Let $\phi,\ \psi$ satisfy the conditions of Lemma 3.4. Let

$$A^2_{ab} = \sup_{a<t<b} \|u_2\chi_{\{\lambda a<\phi(y)<\lambda t\}}\|_{\Lambda^{p_0}_{v_0}(w_0)'}\|u_1\chi_{\{t<\psi(x)<b\}}\|_Y.$$

Then there exists a constant $C > 0$ such that

$$\|P_2 f\|_Y \leq C\|f\chi_{\{\lambda a<\phi(y)<\lambda b\}}\|_{\Lambda^{p_0}_{v_0}(w_0)}$$

if and only if

$$A^2_{ab} < \infty.$$

Let

$$A_{1k} = \sup_{(\frac{b}{a})^k<t<(\frac{b}{a})^{k+1}} \left\|u_1\chi_{\{(\frac{b}{a})^k<\psi(x)<t\}}\right\|_Y \left\|u_2\chi_{\{at<\phi(x)<a(\frac{b}{a})^{k+1}\}}\right\|_{\Lambda^{p_0}_{v_0}(w_0)'},$$

$$A_{2k} = \sup_{(\frac{b}{a})^k<t<(\frac{b}{a})^{k+1}} \left\|u_1\chi_{\{t<\psi(x)<(\frac{b}{a})^{k+1}\}}\right\|_Y \left\|u_2\chi_{\{b(\frac{b}{a})^k<\phi(x)<bt\}}\right\|_{\Lambda^{p_0}_{v_0}(w_0)'},$$

$$A_1 = \sup_{k\in Z}\{A_{1k}\},\ \ A_2 = \sup_{k\in Z}\{A_{2k}\},\ \ A = \max\{A_1, A_2\}.$$

***Theorem* 3.2** — *Let* $0 < p_0 \leq p_1 < \infty$, $Y = \Lambda^{p_1}_{v_1}(w_1)$ *or* $Y = \Lambda^{p_1,\infty}_{v_1}(w_1)$, *and there exists* $p_0 \leq \sigma \leq p_1$ *when* $Y = \Lambda^{p_1}_{v_1}(w_1)$, $\sigma \geq p_0$ *when* $Y = \Lambda^{p_1,\infty}_{v_1}(w_1)$, *such that (3.1) and (3.3) hold. Suppose that*

$$\mu\{x : \phi(y) = t\} = \mu\{x : \psi(x) = t\} = 0$$

for any $t \in [0,\infty)$. *Then the operator* $P : \Lambda^{p_0}_{v_0}(w_0) \to Y$ *is bounded if and only if* $A < \infty$.

PROOF : Let $F_k = \{x : (\frac{b}{a})^k \le \psi(x) < (\frac{b}{a})^{k+1}\}$. Then $X = \cup_{k\in Z} F_k$. Thus

$$\begin{aligned}
&\|Pf\|_Y^\sigma = \left\|\sum_k (Pf)\chi_{F_k}\right\|_Y^\sigma \le \sum_k \|(Pf)\chi_{F_k}\|_Y^\sigma, \text{ by Lemma 3.1 } (ii) \text{ and } (iii)\\
= &\sum_k \left\|u_1(x)\left(\int_{\{a\psi(x)<\phi(y)<b\psi(x)\}} f(y)u_2(y)v_0(y)d\mu(y)\right)\chi_{\{(\frac{b}{a})^k\le\psi(x)<(\frac{b}{a})^{k+1}\}}\right\|_Y^\sigma\\
= &\sum_k \left\|u_1(x)\left[\left(\int_{\{a\psi(x)<\phi(y)<b(\frac{b}{a})^k\}} + \int_{\{a(\frac{b}{a})^{k+1}<\phi(y)<b\psi(x)\}}\right) f(y)u_2(y)v_0(y)d\mu(y)\right]\right.\\
&\left.\chi_{\{(\frac{b}{a})^k\le\psi(x)<(\frac{b}{a})^{k+1}\}}\right\|_Y^\sigma\\
\le &\sum_k \left\|u_1(x)\left[\left(\int_{\{a\psi(x)<\phi(y)<b(\frac{b}{a})^k\}} + \int_{\{a(\frac{b}{a})^{k+1}<\phi(y)<b\psi(x)\}}\right) f(y)u_2(y)v_0(y)d\mu(y)\right]\right.\\
&\left.\chi_{\{(\frac{b}{a})^k\le\psi(x)<(\frac{b}{a})^{k+1}\}}\right\|_Y^\sigma\\
\le &C\sum_k \left\|u_1(x)\left[\int_{\{a\psi(x)<\phi(y)<b(\frac{b}{a})^k\}} f(y)u_2(y)v_0(y)d\mu(y)\right]\chi_{\{(\frac{b}{a})^k\le\psi(x)<(\frac{b}{a})^{k+1}\}}\right\|_Y^\sigma\\
&+C\sum_k \left\|u_1(x)\left[\int_{\{a(\frac{b}{a})^{k+1}<\phi(y)<b\psi(x)\}} f(y)u_2(y)v_0(y)d\mu(y)\right]\chi_{\{(\frac{b}{a})^k\le\psi(x)<(\frac{b}{a})^{k+1}\}}\right\|_Y^\sigma,\\
&\text{since Y is a quasi} - \text{Banach space}\\
= &\sum_k \tau_{1k} + \sum_k \tau_{2k} = I_1 + I_2.
\end{aligned}$$

Estimate I_1 as follows:

$$\begin{aligned}
I_1 &= \sum_k \left\|u_1(x)\left[\int_{\{a\psi(x)<\phi(y)<b(\frac{b}{a})^k\}} f(y)u_2(y)v_0(y)d\mu\right]\chi_{\{(\frac{b}{a})^k\le\psi(x)<(\frac{b}{a})^{k+1}\}}\right\|_Y^\sigma\\
&\le \sum_k CA_{1k}^\sigma \|f\chi_{\{a(\frac{b}{a})^k\le\psi(x)<a(\frac{b}{a})^{k+1}\}}\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma, \text{ by Lemma 3.4}\\
&\le CA^\sigma \sum_k \|f\chi_{\{a(\frac{b}{a})^k\le\psi(x)<a(\frac{b}{a})^{k+1}\}}\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma\\
&\le CA^\sigma \|f\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma, \text{ by Lemma 3.1 } (i).
\end{aligned}$$

Similarly, by Lemma 3.5,

$$I_2 \le CA^\sigma \|f\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma.$$

Thus

$$\|Pf\|_Y^\sigma \le CA^\sigma \|f\|_{\Lambda_{v_0}^{p_0}(w_0)}^\sigma.$$

4. COMPACTNESS OF HARDY OPERATORS

We next consider the compactness of T on weighted Lorentz spaces. Edmund, Kokilashvili and Meskhi [8, 9] gave a characterization for T to be compact from $\widetilde{X}$ to $\widetilde{Y}$ where $\widetilde{X}$ and $\widetilde{Y}$ are Banach function spaces when the spaces $\widetilde{X},\ \widetilde{Y}$ satisfy certain conditions. This part studies the compactness of $T : \Lambda^{p_0}_{v_0}(w_0) \to Y$ when $\Lambda^{p_0}_{v_0}(w_0)$ is a quasi-Banach space. Thus Theorem 4.1 improves (7)-(9) since the normability of $\Lambda^{p_0}_{v_0}(w_0)$ is not compulsory. On the other hand, if $X = \mathbb{R}^n$, $\phi(y) = |y|$ and $\psi(x) = |x|$ Theorem 4.1 reduces to [24, Theorem 4.1].

In this section, we let

$$\mu\{x \in X : \phi(x) = t\} = \mu\{x \in X : \psi(x) = t\} = 0.$$

For $0 < a < b < \infty$, let

$$T_a f(x) = \chi_{\{\psi(x)<a\}}(x)u_1(x)\int_{\{\phi(y)<\psi(x)\}} \chi_{\{\phi(y)<a\}} f(y)u_2(y)v_0(y)d\mu(y),$$

$$T_b f(x) = \chi_{\{\psi(x)>b\}}(x)u_1(x)\int_{\{\phi(y)<\psi(x)\}} \chi_{\{\phi(y)>b\}} f(y)u_2(y)v_0(y)d\mu(y),$$

$$T_{ab} f(x) = \chi_{\{a<\psi(x)\le b\}}(x)u_1(x)\int_{\{\phi(y)<\psi(x)\}} \chi_{\{a<\phi(y)<b\}} f(y)u_2(y)v_0(y)d\mu(y).$$

We first give characterization of the compactness of the operator $T_{ab} : \Lambda^{p_0}_{v_0}(w_0) \to Y$ inspired by [9].

Lemma 4.1 — Let $W_0 \in \Delta_2$, $1 \le p_1 < \infty$ and $w_1 \in B_{p_1,\infty}$ if $Y = \Lambda^{p_1}_{v_1}(w_1)$; and $w_1 \in B_{p_1}$ if $Y = \Lambda^{p_1,\infty}_{v_1}(w_1)$. Let $\|u_1\chi_{\{a<\psi(x)<b\}}\|_Y = C_1 < \infty,\ \ \|u_2\chi_{\{a<\phi(y)<b\}}\|_{\Lambda^{p_0}_{v_0}(w_0)'} = C_2 < \infty$. Then the operator $T_{ab} : \Lambda^{p_0}_{v_0}(w_0) \to Y$ is compact if and only if for every $\alpha \in [a, b]$,

$$\lim_{R\to\alpha^+} \|u_1\chi_{\{\alpha<\psi(x)<R\}}\|_Y = 0 \ \text{ or } \ \lim_{R\to\alpha^+} \|u_2\chi_{\{\alpha<\phi(y)<R\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} = 0 \tag{4.1}$$

and

$$\lim_{R\to\alpha^-} \|u_1\chi_{\{R<\psi(x)<\alpha\}}\|_Y = 0 \ \text{ or } \ \lim_{R\to\alpha^-} \|u_2\chi_{\{R<\phi(y)<\alpha\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} = 0. \tag{4.2}$$

PROOF : *Sufficiency*. The operator T_{ab} is bounded from $\Lambda^{p_0}_{v_0}(w_0)$ to Y since

$$\begin{aligned}\|T_{ab}f\|_Y &\le \|u_1\chi_{\{a<\psi(x)<b\}}\|_Y \int_{\{a<\phi(y)<b\}} u_2(y)f(y)v_0(y)d\mu(y)\\ &\le \|u_1\chi_{\{\alpha<\psi(x)<b\}}\|_Y \|u_2\chi_{\{a<\phi(y)<b\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'}\|f\|_{\Lambda^{p_0}_{v_0}(w_0)} \le C_1C_2\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}.\end{aligned}$$

Next prove that T_{ab} is a limit of finite rank operators and thus is compact. Let $\epsilon > 0$. Then for every $\alpha \in [a, b]$ there exist c and d with $c < \alpha < d$, such that

$$\|u_1\chi_{\{\alpha<\psi(x)<d\}}\|_Y < \epsilon \text{ or } \|u_2\chi_{\{\alpha<\phi(y)<d\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} < \epsilon \tag{4.3}$$

and

$$\|u_1\chi_{\{c<\psi(x)<\alpha\}}\|_Y < \epsilon \text{ or } \|u_2\chi_{\{c<\phi(y)<\alpha\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} < \epsilon. \tag{4.4}$$

Thus we obtain an open covering of the segment $[a, b]$ by such intervals (c, d) from which there exists a finite subcovering $\{(c_i, d_i)\}$ having appropriate interior point α_i. The points $c_i,\ \alpha_i,\ d_i$ form a partition of $[a, b]$, and we obtain closed intervals $[\beta_j,\ \beta_{j+1}]$, $j = 0,\ 1, ..., N$ such that $\cup_{j=0}^{N}[\beta_j, \beta_{j+1}] = [a, b]$ with $(\beta_i,\ \beta_{i+1}) \cap (\beta_j,\ \beta_{j+1}) = \emptyset$ for $i \neq j$. Obviously in view of the property of Banach function spaces there holds that

$$\|u_1\chi_{\{\beta_j<\psi(x)<\beta_{j+1}\}}\|_Y < \epsilon \text{ or } \|u_2\chi_{\{\beta_j<\phi(y)<\beta_{j+1}\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} < \epsilon.$$

Let

$$Sf(x) = \sum_{j=0}^{N} \chi_{\{\beta_j<\psi(x)<\beta_{j+1}\}}(x)u_1(x) \int_{\{a<\phi(y)<\beta_j\}} u_2(y)f(y)v_0(y)d\mu(y)$$

Then

$$T_{ab}f(x) - Sf(x) = \sum_{j=0}^{N} \chi_{\{\beta_j<\psi(x)\leq\beta_{j+1}\}}(x)u_1(x) \int_{\{\beta_j<\phi(y)<\psi(x)\}} u_2(y)f(y)v_0(y)d\mu(y).$$

According to the assumptions, $\Lambda^{p_1}_{v_1}(w_1)$ (resp. $\Lambda^{p_1,\infty}_{v_1}(w_1)$) is a Banach function space with a norm $\|\cdot\|_{\Gamma^{p_1}_{v_1}(w)}$ (resp. $\|\cdot\|_{\Gamma^{p_1,\infty}_{v_1}(w)}$) which implies $\Gamma^{p_1}_{v_1}(w)'' = \Gamma^{p_1}_{v_1}(w)$ (resp. $\Gamma^{p_1,\infty}_{v_1}(w)'' = \Gamma^{p_1,\infty}_{v_1}(w)$) [2, Theorem 1.2.7] and then

$$\|f\|_{\Lambda^{p_1}_{v_1}(w)} \leq \|f\|_{\Gamma^{p_1}_{v_1}(w)} = \sup_{\|g\|_{\Gamma^{p_1}_{v_1}(w)'}\leq 1} \int_X f(x)g(x)v_1(x)d\mu(x) \tag{4.5}$$

$$\text{(resp. } \|f\|_{\Lambda^{p_1,\infty}_{v_1}(w)} \leq \|f\|_{\Gamma^{p_1,\infty}_{v_1}(w)} = \sup_{\|g\|_{\Gamma^{p_1,\infty}_{v_1}(w)'}\leq 1} \int_X f(x)g(x)v_1(x)d\mu(x)).$$

But since

$$\|g\|_{\Gamma^{p_1}_{v_1}(w)} \leq C\|g\|_{\Lambda^{p_1}_{v_1}(w)},$$

we get

$$\begin{aligned}
\|g\|_{\Gamma^{p_1}_{v_1}(w)'} &= \sup_{\|g\|_{\Gamma^{p_1}_{v_1}(w)}\le 1} \int_X f(x)g(x)v_1(x)d\mu(x)\\
&\ge \sup_{C\|g\|_{\Lambda^{p_1}_{v_1}(w)}\le 1} \int_X f(x)g(x)v_1(x)d\mu(x)\\
&= \frac{1}{C}\sup_{\|Cg\|_{\Lambda^{p_1}_{v_1}(w)}\le 1} \int_X f(x)(Cg(x))v(x)d\mu(x)\\
&= \frac{1}{C}\sup_{\|g\|_{\Lambda^{p_1}_{v_1}(w)}\le 1} \int_X f(x)g(x)v(x)d\mu(x) = \frac{1}{C}\|g\|_{\Lambda^{p_1}_{v_1}(w)'}. \qquad (4.6)
\end{aligned}$$

Thus by (4.5) and (4.6) it follows that

$$\|f\|_{\Lambda^{p_1}_{v_1}(w)} \le \sup_{\frac{1}{C}\|g\|_{\Lambda^{p_1}_{v_1}(w)'}\le 1} \int_X f(x)g(x)v_1(x)d\mu(x) = C\sup_{\|g\|_{\Lambda^{p_1}_{v_1}(w)'}\le 1} \int_X f(x)g(x)v_1(x)d\mu(x), \qquad (4.7)$$

as does the evaluation of $\|f\|_{\Lambda^{p_1,\infty}_{v_1}(w)}$. This implies that

$$\begin{aligned}
\|T_{ab}-S\|_{\Lambda^{p_0}_{v_0}(w_0)\to Y} &= \sup_{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}\le 1} \|(T_{ab}-S)f\|_Y\\
&\le C \sup_{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}\le 1}\sup_{\|g\|_{Y'}\le 1} \int_X (T_{ab}-S)f(x)g(x)v_1(x)d\mu(x)\\
&= C \sup_{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}\le 1}\sup_{\|g\|_{Y'}\le 1} \sum_{j=0}^{N} \int_{\{\beta_j<\psi(x)\le\beta_{j+1}\}} u_1(x)g(x)v_1(x)\\
&\quad\cdot\left(\int_{\beta_j<\phi(y)\le\psi(x)} u_2(y)f(y)v_0(y)d\mu(y)\right)d\mu(x)\\
&\le C \sup_{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}\le 1}\sup_{\|g\|_{Y'}\le 1} \sum_{j=0}^{N} \int_{\{\beta_j<\psi(x)\le\beta_{j+1}\}} u_1(x)g(x)v_1(x)d\mu(x)\\
&\quad\cdot\int_{\{\beta_j<\phi(y)\le\beta_{j+1}\}} u_2(y)f(y)v_0(y)d\mu(y)\\
&= C \sup_{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}\le 1}\sup_{\|g\|_{Y'}\le 1} \sum_{j=0}^{N} I^j_{\psi\phi}.
\end{aligned}$$

If

$$A_1 = \{j : \|u_1\chi_{\{\beta_j<\psi(x)<\beta_{j+1}\}}\|_Y < \epsilon\}$$

and

$$A_2 = \{j : \|u_2\chi_{\{\beta_j<\phi(y)<\beta_{j+1}\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} < \epsilon\},$$

then we obtain by Hölder inequality in $\Lambda^{p_0}_{v_0}(w_0)$ and Y that

$$\sup_{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}\le 1} \sup_{\|g\|_{Y'}\le 1} \sum_{j\in A_1} I^j_{\psi\phi} \le \epsilon C c_1$$

and

$$\sup_{\|f\|_{\Lambda^{p_0}_{v_0}(w_0)}\le 1} \sup_{\|g\|_{Y'}\le 1} \sum_{j\in A_2} I^j_{\psi\phi} \le \epsilon C c_2.$$

Thus we have

$$\|T_{ab} - S\| \le \epsilon C(c_1 + c_2)$$

which yields that T_{ab} is a limit of finite rank operators.

Necessity : Use contradiction. Let there exist numbers $\alpha \in [a, b)$ and $\epsilon_0 > 0$ and a sequence $\{t_n\}$, $t_n \to \alpha^+$ such that

$$\|u_1\chi_{\{a<\psi(x)<t_n\}}\|_Y \ge \epsilon_0 \text{ or } \|u_2\chi_{\{a<\phi(y)<t_n\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} \ge \epsilon_0. \tag{4.8}$$

For $\gamma \in (0, 1)$ there exist functions f_n, g_n with support in $[\alpha, t_n]$ such that $\|f_n\|_{\Lambda^{p_0}_{v_0}(w_0)} \le 1$, $\|g_n\|_{Y'} \le 1$,

$$\int_{\{\alpha<\phi(y)<t_n\}} u_2(y)f_n(y)v_0(y)d\mu(y) \ge \gamma\|u_2\chi_{\{\alpha<\phi(y)<t_n\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'}$$

and

$$\int_{\{\alpha<\psi(x)<t_n\}} u_1(y)g_n(y)v_1(y)d\mu(y) \ge \frac{\gamma}{C}\|u_1\chi_{\{\alpha<\psi(x)<t_n\}}\|_Y, \text{ by (4.7)}.$$

Furthermore there exist numbers $\beta_n \in (\alpha, t_n)$ such that

$$\int_{\{\beta_n<\phi(y)<t_n\}} u_2(y)f_n(y)v_0(y)d\mu(y) \ge \gamma^2\|u_2\chi_{\{a<\phi(y)<t_n\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'},$$

$$\int_{\{\beta_n<\psi(x)<t_n\}} u_1(y)g_n(y)v_1(y)d\mu(y) \ge \frac{\gamma^2}{C}\|u_1\chi_{\{a<\psi(x)<t_n\}}\|_Y.$$

Let $F_n = f_n\chi_{\{\beta_n<\phi(y)<t_n\}}$, and let $m,\ k$ and n be natural numbers such that the condition

$t_m < \beta_k < t_k < \beta_n$ is fulfilled for them. Then we obtain that

$$\begin{aligned}
\|T_{ab}F_m - T_{ab}F_n\|_Y &\geq \|\chi_{\{\beta_k<\psi(x)<t_k\}}(T_{ab}F_m - T_{ab}F_n)\|_Y \\
&= \left\|\chi_{\{\beta_k<\psi(x)<t_k\}}u_1 \int_{\{\beta_m<\phi(y)<t_m\}} u_2(y)f_m(y)v_0(y)d\mu(y)\right\|_Y \\
&= \int_{\{\beta_m<\phi(y)<t_m\}} u_2(y)f_m(y)v_0(y)d\mu(y)\|u_1\chi_{\{\beta_k<\psi(x)<t_k\}}\|_Y \\
&\geq \int_{\{\beta_m<\phi(y)<t_m\}} u_2(y)f_m(y)v_0(y)d\mu(y)\int_{\{\beta_k<\phi(y)<t_k\}} u_1(y)g_n(y)v_1(y)d\mu(y), \\
&\quad \text{by Holder inequality in the space Y} \\
&\geq \frac{\gamma^4\epsilon^2}{C} > 0.
\end{aligned}$$

Since any subsequence of $\{T_{ab}F_k\}$ does not converge in X_2, T_{ab} is noncompact. Thus (4.2) is proved. Analogously verify (4.2).

***Theorem* 4.1** — *(Characterization of Compactness) Let* $W_0 \in \Delta_2$, $1 \leq p_1 < \infty$ *and* $w_1 \in B_{p_1,\infty}$ *if* $Y = \Lambda_{v_1}^{p_1}(w_1)$; *and* $w_1 \in B_{p_1}$ *if* $Y = \Lambda_{v_1}^{p_1,\infty}(w_1)$. *Then the operator* $T : \Lambda_{v_0}^{p_0}(w_0) \to Y$ *is compact if and only if (3.15) and the following conditions (a) and (b) hold:*

(a)

$$\lim_{a\to 0}\sup_{0<R<a} \|u_2\chi_{\{\phi(y)<R\}}\|_{(\Lambda_{v_0}^{p_0}(w_0))'} \cdot \|u_1\chi_{\{R<\psi(x)<a\}}\|_Y = 0 \tag{4.9}$$

and

$$\lim_{a\to\infty}\sup_{a<R<\infty} \|u_2\chi_{\{a<\phi(y)<R\}}\|_{(\Lambda_{v_0}^{p_0}(w_0))'} \cdot \|u_1\chi_{\{\psi(x)>R\}}\|_Y = 0, \tag{4.10}$$

(b) for every $\alpha \in [0,\infty)$ *the following two alternatives are satisfied*

$$\lim_{R\to\alpha^+} \|u_1\chi_{\{\alpha<\psi(x)<R\}}\|_Y = 0 \text{ or } \lim_{R\to\alpha^+} \|u_2\chi_{\{\alpha<\phi(y)<R\}}\|_{(\Lambda_{v_0}^{p_0}(w_0))'} = 0 \tag{4.11}$$

and

$$\lim_{R\to\alpha^-} \|u_1\chi_{\{R<\psi(x)<\alpha\}}\|_Y = 0 \text{ or } \lim_{R\to\alpha^-} \|u_2\chi_{\{R<\phi(y)<\alpha\}}\|_{(\Lambda_{v_0}^{p_0}(w_0))'} = 0. \tag{4.12}$$

PROOF : *Sufficiency*. We prove it through three steps.

Step 1 : Define the operator T_D by replacing u_1 and u_2 by $u_1\chi_{D_1}$ and $u_2\chi_{D_2}$ in the operator T, where

$$D_1 = \{y : r_2 < \psi(x) < r_1\}, \ \ D_2 = \{y : r_2 < \phi(y) < r_1\}, \ \ 0 \leq r_2 < r_1 \leq \infty.$$

By Theorem 3.1,

$$B_D \leq \|T_D\| \leq KB_D, \tag{4.13}$$

where

$$B_D = \sup_{a>0} \|u_2 \chi_{D_2} \chi_{\{\phi(y)<a\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} \cdot \|u_1 \chi_{D_1} \chi_{\{\psi(x)>a\}}\|_Y$$

and K is a constant independent of $v_0,\ u_1,\ \phi,\ \psi$, D_1 and D_2.

Step 2 : For $0 < a < b < \infty$, the operator T can be decomposed as

$$Tf = T_a f + T_b f + T_{ab} f + (Q_b T P_{ab} f + Q_a T P_a f), \tag{4.14}$$

where $P_a f = \chi_{\phi(y)<a} f$, $Q_b f = \chi_{\psi(x)>b} f$, $P_{ab} f = \chi_{a<\phi(y)<b} f$.

Step 3 : Each of the two operators in parentheses in (4.14) is one-dimensional and so is compact. By using Lemma 4.1 we get that T_{ab} is a compact operator if and only if (4.11) and (4.12) hold. Let $\epsilon > 0$. By (4.13) we get

$$\|T_a\| \leq K \sup_{0<R<a} \|u_2 \chi_{\{\phi(y)<R\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} \cdot \|u_1 \chi_{\{R<\psi(x)<a\}}\|_Y$$

and

$$\|T_b\| \leq K \sup_{b<R<\infty} \|u_2 \chi_{\{b<\phi(y)<R\}}\|_{(\Lambda^{p_0}_{v_0}(w_0))'} \cdot \|u_1 \chi_{\{\psi(x)>R\}}\|_Y,$$

In the light of (4.9) and (4.10), there exist $a,\ b$, with $0 < a < b < \infty$, such that $\|T_a\| < \epsilon,\ \|T_b\| < \epsilon$. Hence H is compact, since it is a limit of compact operators when $a \to 0$, and $b \to \infty$.

Necessity : The proof of necessity is similar to that of Lemma 4.1. We omit the details. □

Remark 4.1 : (1) A function f in a quasi-normed space X is said to have absolutely continuous (AC) norm if $\lim_{n\to\infty} \|f\chi_{A_n}\|_X = 0$, for every decreasing sequence of measurable sets $(A_n)_n$ with $\chi_{A_n} \downarrow \emptyset$. If every function in X has this property, we say that X has an (AC) norm (see [4, Definition 2.3.2]). If $\Lambda^{p_1}_{v_1}(w_1)$ and $(\Lambda^{p_0}_{v_0}(w_0))'$ are spaces with (AC) norms (the conditions which make $\Lambda^{p_1}_{v_1}(w_1)$ and $(\Lambda^{p_0}_{v_0}(w_0))'$ have (AC) norms can be found in [24, Remark 4.2]), then it is obvious that (a) may be replaced by a stronger condition

$$\lim_{a\to 0} I(a) = \lim_{a\to\infty} I(a) = 0, \tag{4.15}$$

when $Y = \Lambda^{p_1}_{v_1}(w_1)$. Furthermore, in this circumstance (b) is superfluous since it is automatically satisfied.

In the case when the underlying space $X = \mathbb{R}^n$, the weighted Lorentz space $\Lambda_{v_0}^{p_0}(w_0)$ or $\Lambda_{v_1}^{p_1}(w_1)$ becomes the Lorentz space $L^{r,s}$ and the operator T has some special forms and then the characterization of the compactness of T from $\Lambda_{v_0}^{p_0}(w_0)$ to $\Lambda_{v_1}^{p_1}(w_1)$ achieves simpler forms. Let the operators $T_1,\ T_2$ be defined by

$$T_1 f(x) = u_1(x) \int_{\{|y| \le \psi(x)\}} \frac{f(y)}{|y|^\alpha} dy, \quad T_2 f(x) = \frac{1}{|x|^\alpha} \int_{\{|y| \le \psi(x)\}} u_2(y) f(y) dy.$$

The next result characterizes compactness of type T_1.

Corollary 3.1 — Let $\min(r,\ s) > 1,\ \max(r,\ s) \le p_1,\ \frac{1}{r'} > \frac{\alpha}{n}$, $w_1 \in B_{p_1}$, $u_1 \in L^1$ or $w_1 \notin L^1$, and there exist $\max(r,\ s) \le \sigma \le p_1$ such that (3.3) hold. Then the operator T_1 is compact from $L^{r,s}$ to $\Lambda_{v_1}^{p_1}(w_1)$ if and only if there exists $C > 0$ such that for every $k \in \mathbb{Z}$,

$$A_k := \|u_1 \psi^{n/r' - \alpha} \chi_{2^k < \psi(x) < 2^{k+1}}\|_{\Lambda_{v_1}^{p_1}(w_1)} \le C \tag{4.16}$$

and

$$\lim_{k \to -\infty} A_k = \lim_{k \to \infty} A_k = 0. \tag{4.17}$$

PROOF : (i) Note that under the assumptions $\Lambda_{v_1}^{p_1}(w_1)$ is a Banach space with (AC) norm [4, Theorem 2.3.4] and $L^{r',s'}$ has (AC) norm. Furthermore, (3.1) holds with $w_0(t) = t^{s/r-1}$ and $p_0 = s$ and

$$\left\| \frac{1}{|\cdot|^\alpha} \chi_{B(0,a)} \right\|_{L^{r',s'}} = a^{\frac{n}{r'} - \alpha}.$$

Thus by Theorem 4.1 and Remark 4.1, for T_1 to be compact from $L^{r,s}$ to $\Lambda_{v_1}^{p_1}(w_1)$, it is sufficient and necessary that there exists $C > 0$ such that for all $a > 0$

$$I(a) := a^{\frac{n}{r'} - \alpha} \cdot \|u_1 \chi_{\{\psi(x) > a\}}\|_{\Lambda_{v_1}^{p_1}(w_1)} \le C, \tag{4.18}$$

and

$$\lim_{a \to 0} I(a) = \lim_{a \to \infty} I(a) = 0. \tag{4.19}$$

So it suffices to prove that (4.16) and (4.17) are equivalent to (4.18) and (4.19). We prove it by four steps.

Let

$$\sup_{a>0} I(a) = I, \quad \sup_k A_k = A.$$

We verify $I \approx A$. Indeed, it is easy to see that $A_k \leq CI(2^k)$, and thus $A \leq CI$. On the other hand, let $2^k < a \leq 2^{k+1},\ k \in \mathbb{Z}$. Then

$$\begin{aligned} I(a) &\leq C2^{k(\frac{n}{r'}-\alpha)} \cdot \|u_1\chi_{\{\psi(x)>2^k\}}\|_{\Lambda_{v_1}^{p_1}(w_1)} \\ &\leq C2^{k(\frac{n}{r'}-\alpha)} \cdot \sum_{j=k}^{\infty} \|u_1\chi_{\{2^j<\psi(x)<2^{j+1}\}}\|_{\Lambda_{v_1}^{p_1}(w_1)}, \text{ by } w_1 \in B_{p_1} \\ &\leq C\sum_{j=k}^{\infty} 2^{(k-j)(\frac{n}{r'}-\alpha)} \cdot \|u_1\psi^{n/r'-\alpha}\chi_{\{2^k<\psi(x)<2^{k+1}\}}\|_{\Lambda_{v_1}^{p_1}(w_1)} \\ &\leq CA, \end{aligned}$$

i.e., $I \leq CA$. Secondly, Taking into account $A_k \leq CI(2^k)$, we get that (4.19) implies (4.17). Third, if $\lim_{k\to\infty} A_k = 0$, it is similar to prove that $\lim_{a\to\infty} I(a) = 0$ to the arguments of the proof above that $I \leq CA$. Finally, letting $\lim_{k\to-\infty} A_k = 0$ and $A < \infty$, we prove $\lim_{t\to 0} I(t) = 0$. Indeed, $\lim_{k\to-\infty} A_k = 0$ yields that given $\epsilon > 0$ there exist $K_1 \in \mathbb{Z}$, $K_2 \in \mathbb{N}$ such that

$$\sum_{m=K_2}^{\infty} (2^{n/r'-\alpha})^{-m} < \epsilon, \text{ and } A_k < \epsilon \text{ for } k < K_1. \tag{4.20}$$

Since for $t < 2^{K_1-K_2}$ there exists $k \in \mathbb{Z},\ k < K_1 - K_2$ with $2^k \leq t \leq 2^{k+1}$, we obtain

$$\begin{aligned} I(t) &\leq C(2^k)^{n/r'-\alpha}\|u_1\chi_{\{\psi(x)>2^k\}}\|_{\Lambda_{v_1}^{p_1}(w_1)} \\ &\leq C(2^k)^{n/r'-\alpha}\sum_{j=k}^{\infty}(2^{-j})^{n/r'-\alpha}\|u_1\psi^{n/r'-\alpha}\chi_{\{2^j<\psi(x)<2^{j+1}\}}\|_{\Lambda_{v_1}^{p_1}(w_1)}, \text{ by } w_1 \in B_{p_1} \\ &= C\sum_{m=0}^{\infty}(2^{-m})^{n/r'-\alpha}\|u_1\psi^{n/r'-\alpha}\chi_{\{2^{k+m}<\psi(x)<2^{k+m+1}\}}\|_{\Lambda_{v_1}^{p_1}(w_1)} \\ &= C\left(\sum_{m=0}^{K_2-1} + \sum_{m=K_2}^{\infty}\right) \leq C\epsilon\sum_{m=0}^{K_2-1}(2^{-m})^{n/r'-\alpha} + \epsilon A, \text{ by } (4.20) \\ &\leq C\epsilon, \text{ by } \frac{1}{r'} > \frac{\alpha}{n}, \end{aligned}$$

which ends the proof. □

Remark 4.2 : Prokhorov [33, Theorem 3] proved that if $1 < p \leq q < \infty$, l is a non-negative function on $\mathbb{R}_+$, $Tf(x) = \frac{l(x)}{x}\int_0^x f(y)dy$, $x > 0$, then T is compact from L^p to L^q if and only if

$$\sup_k D_k < \infty, \quad \lim_{k\to 0} D_k = \lim_{k\to\infty} D_k = 0,$$

where $D_k = (\int_{2^k}^{2^{k+1}} \frac{l(x)}{x^{q/p}} dx)^{1/q}$. Obviously Corollary 4.1 generalizes this result. It also extends [24, Corollary 4.3] corresponding to the case $\psi(x) = |x|$.

Let us show the analogous characterization of compactness of operators T_2, which we leave without proof.

Corollary 4.2 — Let $1 < p_0 \le \min(r,\ s)$, $\frac{\alpha}{n} > \frac{1}{r}$, $W_0 \in \Delta_2$, $v_0 \in L^1$ *or* $\tilde{w}_0 \notin L^1$, and there exists $p_0 \le \sigma \le \min(r,\ s)$ such that (3.1) holds where $\widetilde{w_0}(t) = t^{p_0'} W_0^{-p_0'}(t) w_0(t)$. Then the operator T_2 is compact from $\Lambda_{v_0}^{p_0}(w_0)$ to $L^{r,s}$ if and only if there exists $C > 0$ such that for every $k \in \mathbb{Z}$,

$$B_k := \|u_2 \psi^{n/r-\alpha} \chi_{\{2^k < \psi(x) < 2^{k+1}\}}\|_{(\Lambda_{v_0}^{p_0}(w_0))'} \le C$$

and

$$\lim_{k\to-\infty} B_k = \lim_{k\to\infty} B_k = 0.$$

5. Some Notes on Spaces of Homogeneous and Nonhomogeneous Type

Definition 5.1 — A space of homogeneous type (SHT) $(X; d; \mu)$ is a topological space X endowed with a complete measure μ such that: (a) the space of compactly supported continuous functions is dense in $L^1(X; \mu)$, and (b) there exists a non-negative real-valued function (quasimetric) $d : X \times X \to R^1$ satisfying:

(i) $d(x; x) = 0$ for arbitrary $x \in X$;

(ii) $d(x; y) > 0$ for arbitrary $x; y \in X, x \ne y$;

(iii) there exists a positive constant a_0 such that the inequality $d(x; y) \le a_0 d(y; x)$ holds for all $x; y \in X$;

(iv) there exists a positive constant a_1 such that the inequality

$$d(x; y) \le a_1(d(x; z) + d(z; y))$$

holds for arbitrary $x; y; z \in X$;

(v) for every neighborhood V of any point $x \in X$ there exists a number $r > 0$ such that the ball $B(x; r) = \{y \in X : d(x; y) < r\}$ with center in x and radius r is contained in V ;

(vi) the balls $B(x; r)$ are measurable for all $x \in X, r > 0$ and, moreover, $0 < \mu B(x; r) < \infty$;

(vii) there exists a positive constant b such that the inequality (doubling condition) $\mu B(x; 2r) \le b\mu B(x; r)$ is true for all $x \in X$ and for all positive r.

Definition 5.2 — By a space of nonhomogeneous type we mean a measure space with a quasi-metric (X, d, μ) satisfying conditions (i)-(v) of Definition 5.1, i.e., the doubling condition may fail.

Let u_1, u_2 and w be positive $\mu-$measurable functions on X and $x_0 \in X$. Suppose that there exists a point x_0 such that for all numbers $t_1,\ t_2$ with $0 < t_1 < t_2 < \infty$ we have

$$\mu(B(x_0, t_2)\backslash B(x_0, t_1)) > 0.$$

Assume that

$$R_1 f(x) = u_1(x) \int_{\{d(x_0,y) \le \psi(x)\}} f(y) u_2(y) v_0(y)\, d\mu(y), \quad x \in X,$$

$$R_2 f(x) = u_1(x) \int_{\{d(x_0,y) \ge \psi(x)\}} f(y) u_2(y) v_0(y)\, d\mu(y), \quad x \in X$$

and

$$R_3 f(x) = u_1(x) \int_{\{a\psi(x) < d(x_0,y) < b\psi(x)\}} f(y) u_2(y) v_0(y) d\mu(y),\ \ 0 < a < b < \infty,\ x \in X$$

where $\psi(x) : X \to [0, \infty)$ is a measurable function with the condition

$$0 < \{x \in X : t_1 < \psi(x) < t_2\} < \infty,\ 0 < t_1, t_2 < \infty.$$

Then for the operators R_1-R_3 by using the preceding results in Chapter 3 and Chapter 4 we can get the boundedness and compactness criteria from $\Lambda_{v_0}^{p_0}(w_0)$ to $\Lambda_{v_1}^{p_1}(w_1)$ defined on (X, d, μ). The same method is also applicable to the following operators:

$$R_4 f(x) = u_1(x) \int_{\{d(x_0,y) < \psi(x)\}} f(y) u_2(y) v_0(y)\, d\mu(y), \quad x \in X,$$

and

$$R_5 f(x) = u_1(x) \int_{\{d(x_0,y) > \psi(x)\}} f(y) u_2(y) v_0(y)\, d\mu(y), \quad x \in X.$$

When $(X, \mu) = (\mathbb{R}^n, |\cdot|)$, $x_0 = 0$ and $\psi(x) = d(x_0, x) = |x|$, the operators $R_1 - R_5$ have been studied in [7-9], [24, 25, 28] and so on.

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