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ON THE SUM OF THE POWERS OF DISTANCE SIGNLESS LAPLACIAN EIGENVALUES OF GRAPHS

S. Pirzada*, Hilal A. Ganie*, A. Alhevaz** and M. Baghipur***

**Department of Mathematics, University of Kashmir, Srinagar, India*

***Faculty of Mathematical Sciences, Shahrood University of Technology,*
P.O. Box: 316-3619995161, *Shahrood, Iran*

****Department of Mathematics, University of Hormozgon, PO Box* 3995,
Bandar Abbas, Iran

e-mails: pirzadasd@kashmiruniversity.ac.in; hilahmad1119kt@gmail.com
a.alhevaz@shahroodut.ac.ir; maryamb8989@gmail.com

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Let G be a connected graph with n vertices, m edges and having distance signless Laplacian eigenvalues $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n \geq 0$. For any real number $\alpha \neq 0$, let $m_\alpha(G) = \sum_{i=1}^{n} \rho_i^\alpha$ be the sum of α^{th} powers of the distance signless Laplacian eigenvalues of the graph G. In this paper, we obtain various bounds for the graph invariant $m_\alpha(G)$, which connects it with different parameters associated to the structure of the graph G. We also obtain various bounds for the quantity $DEL(G)$, the distance signless Laplacian-energy-like invariant of the graph G. These bounds improve some previously known bounds. We also pose some extremal problems about $DEL(G)$.

Key words : Graph; distance signless Laplacian matrix; distance signless Laplacian eigenvalues; transmission regular.

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1. Introduction

Throughout the paper, we consider G as a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. We use standard terminology; for concepts not defined here, we refer the reader to any standard graph theory monograph, such as [15].

Given a simple graph G with n vertices, m edges having vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$, the adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j; and equal to 0, otherwise. If $Deg(G) = diag(d_1, d_2, \dots, d_n)$ is the diagonal matrix of the vertex degrees $d_i = d_G(v_i)$, $i = 1, 2, \dots, n$ associated to G, the matrices $L(G) = Deg(G) - A(G)$ and $Q(G) = Deg(G) + A(G)$ are respectively the Laplacian and the signless Laplacian matrices and their spectrum are respectively the Laplacian spectrum and signless Laplacian spectrum of the graph G. These matrices are real symmetric and positive semi-definite. We let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ to be the Laplacian spectrum and signless Laplacian spectrum of G, respectively.

The *distance* $d_G(u,v)$ (or shortly d_{uv}) between the vertices u and v in G is the length of any shortest path in G connecting u and v. When the graph is clear from the context, we will omit the subscript G from the notation. The *diameter* of G is the maximum distance between any two vertices of G. The *transmission* $Tr_G(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, i.e., $Tr_G(v) = \sum\limits_{u \in V(G)} d_{uv}$. A graph G is said to be r-*transmission regular* if $Tr_G(v) = r$, for each $v \in V(G)$. The *transmission* (also called Wiener index) of a graph G, denoted by $\sigma(G)$, is the sum of the distances between all unordered pairs of vertices in G. Clearly, $\sigma(G) = \frac{1}{2} \sum\limits_{v \in V(G)} Tr_G(v)$.

The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. For all $v_i \in V(G)$, the quantity $Tr_i = Tr_G(v_i)$ has also been referred as the *transmission degree* of the vertex v_i and so in this notation the sequence $\{Tr_1, Tr_2, \dots, Tr_n\}$ will be the *transmission degree sequence* of the graph G. The second transmission degree of v_i denoted by T_i is given by $T_i = \sum_{j=1}^{n} d_{ij} Tr_j$.

The distance matrix of graphs is very useful in various areas of science and engineering. It contains more structural information compared to a simple adjacency matrix. The distance matrix contains information on various walks and self-avoiding walks of chemical graphs. It is immensely useful in the computation of topological indices such as the Wiener index and is useful in the computation of thermodynamic properties such as pressure and temperature coefficients. The distance spectral radius, which is the largest eigenvalue of the distance matrix, is a useful molecular descriptor in QSPR (Quantitative Structure-Property Relationship) modelling. Distance matrix has also applications in the design of communication network, graph embedding theory as well as molecular stability. The distance eigenvalues of graphs have been studied by researchers for many years. For more information regarding distance spectrum, we refer to the survey [4] and the references therein.

Almost all results obtained for the distance matrix of trees were extended to the case of weighted trees by Bapat [6] and Bapat *et al*. [7]. Extensions were done not only concerning the class of graphs but also regarding the distance matrix itself. Indeed, Bapat *et al*. [8] generalized the concept of the distance matrix to that of q-analogue of the distance matrix.

Aouchiche and Hansen [3] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. These matrices are the generalizations of the distance matrix, just as Laplacian and signless Laplacian matrices are the generalizations of the adjacency matrix. The matrix $D^L(G) = Tr(G) - D(G)$ is called the *distance Laplacian matrix* of G, while the matrix $D^Q(G) = Tr(G) + D(G)$ is called the *distance signless Laplacian matrix* of G, where $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ is the diagonal matrix of vertex transmissions of G. If G is connected, then $D^Q(G)$ is symmetric, positive definite (for $n \geq 3$), non-negative and irreducible. Hence, the eigenvalues of $D^Q(G)$ can be arranged as: $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n > 0$. The largest eigenvalue ρ_1 is called the *distance signless Laplacian spectral radius* of G. A lot of work has been done on the spectral radius ρ_1, for some recent work, we refer to [14] and the references therein.

From the above discussion, we observe that the distance matrix and its eigenvalues are of importance not only from chemistry point of view, but also they are very useful in other branches of science and social science. Clearly, the information one gets regarding the graph from the distance matrix is also visible from the distance Laplacian and the distance signless Laplacian matrices of a graph. Since these matrices are using more structural properties of a graph than the distance matrix, it may be that these matrices will explore more information about the graph.

The purpose of spectral analysis of matrices $D^L(G)$ and $D^Q(G)$ is motivated by several problems. Spectral determination of matrices is an interesting and difficult problem in Matrix theory. When restricting to a particular graph matrix, this problem has been discussed for various matrices and a lot of literature is devoted to this direction. Extremal problems regarding the eigenvalues, particularly the spectral radius, the second largest eigenvalue and the smallest eigenvalue of a graph with respect to a matrix associated with a graph is one more problem that makes the spectral study of matrices $D^L(G)$ and $D^Q(G)$ interesting. Much research has been done in this direction, see [4] and the references therein. Another problem, which is always of interest, is to estimate the sum of the k largest eigenvalues of a graph matrix and characterizing the extremal graphs. Several research papers can be found on this problem and some conjectures have been proposed (which are still open to this date), see [12]. One more problem, which is worth to mention is the mean deviation of the eigenvalues and the sum of the singular values of a matrix. This problem is studied for various graph matrices and various problems/conjectures have been settled in this area. The research in this field is taking several new directions due to wider applications and connections to different structural properties of a graph. For some recent papers in this direction, we refer to [10, 11] and the references therein.

For a real number $\alpha \neq 0$, let $s_\alpha(G)$ be the sum of α-th powers of the Laplacian eigenvalues of G, that is, $s_\alpha(G) = \sum_{i=1}^{n} \mu_i^\alpha$. The study of this graph invariant was considered by Zhou [18]. Various upper and lower bounds can be found in the literature which relates $s_\alpha(G)$ with the different graph parameters associated to the structure of the graph G, see [9] and the references therein. In particular, when $\alpha = \frac{1}{2}$ and $\alpha = -1$, we obtain

$$s_{\frac{1}{2}}(G) = \sum_{i=1}^{n} \sqrt{\mu_i} = LEL(G) \quad \text{and} \quad ns_{-1}(G) = n\sum_{i=1}^{n} \frac{1}{\mu_i} = Kf(G),$$

called, respectively, the Laplacian-energy-like invariant and the Kirchhoff index of the graph G. A lot of literature is devoted to the invariant $LEL(G)$ and $Kf(G)$, because of their applications in various fields of applied sciences, see [17] and the references therein. Motivated by the definition of $s_\alpha(G)$, the following graph invariant was defined based on the signless Laplacian eigenvalues. For a real number $\alpha \neq 0$, let $s_\alpha^+(G)$ be the sum of α-th powers of the signless Laplacian eigenvalues of a graph G, that is, $s_\alpha^+(G) = \sum_{i=1}^{n} q_i^\alpha$. The study of this graph invariant was considered by Akbari et al. in [1]. Various upper and lower bounds can be found in the literature which relates $s_\alpha^+(G)$ with the different graph parameters associated to a graph G, see [5] and the references therein. In particular, when $\alpha = \frac{1}{2}$, we obtain $s_{\frac{1}{2}}^+(G) = \sum_{i=1}^{n} \sqrt{q_i} = IE(G)$, called incidence energy of the graph G. Considerable work has been done with regard to invariant $IE(G)$, because of its applications in various fields of the applied science, see [16] and the references therein.

Motivated by the definitions of the graph invariants $s_\alpha(G)$ and $s_\alpha^+(G)$, for real number $\alpha \neq 0$, we put forward the graph invariant

$$m_\alpha(G) = \sum_{i=1}^{n} \rho_i^\alpha, \tag{1.1}$$

for the sum of α-th powers of distance signless Laplacian eigenvalues of graph G. In particular, for $\alpha = \frac{1}{2}$, we obtain

$$m_{\frac{1}{2}}(G) = \sum_{i=1}^{n} \sqrt{\rho_i}. \tag{1.2}$$

This quantity denoted by $DEL(G)$ is called the distance signless Laplacian-energy-like invariant of the graph G [2].

The paper is organized as follows. In Section 2, we obtain various bounds for the graph invariant $m_\alpha(G)$, which connects it with different parameters (like maximum transmission, transmission degrees, number of vertices, transmission number, etc.) associated to the structure of the graph G. In Section 3, we obtain various bounds for the distance signless Laplacian-energy-like invariant $DEL(G)$ of the graph G, which improve some previously known bounds. Also we raise some extremal problems about $DEL(G)$.

2. Upper and Lower Bounds for $m_\alpha(G)$

In this section, we obtain various bounds for the invariant $m_\alpha(G)$, in terms of graph parameters associated to the graph G. Since $m_1(G)$ and $m_2(G)$ are respectively the trace of the matrices $D^Q(G)$ and $D^Q(G)^2$, we have the following observation, which can be found in [2].

Lemma 2.1 — [2]. If the transmission degree sequence of G is $\{Tr_1, Tr_2, \ldots, Tr_n\}$, then

$$m_1(G) = \sum_{i=1}^{n} \rho_i = 2\sigma(G) \quad \text{and} \quad m_2(G) = \sum_{i=1}^{n} \rho_i^2 = 2 \sum_{1\le i<j\le n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2.$$

The following observation can be found in [2].

Lemma 2.2 — [2]. A connected graph G has two distinct D^Q-eigenvalues if and only if G is a complete graph.

For non-increasing real sequences $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ of length n, we say that (x) is majorized by (y) or (y) majorizes (x), denoted by $(x) \preceq (y)$ if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{and} \quad \sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i, \quad \text{for all } k = 1, 2, \ldots, n-1.$$

The following observation can be found in [13].

Lemma 2.3 — [13]. Let $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences of real numbers of length n. If $(x) \preceq (y)$, the for any convex function ψ, we have $\sum_{i=1}^{n} \psi(x_i) \le \sum_{i=1}^{n} \psi(y_i)$. Furthermore, if $(x) \prec (y)$ and ψ is strictly convex, then $\sum_{i=1}^{n} \psi(x_i) < \sum_{i=1}^{n} \psi(y_i)$.

The spectrum of a distance signless Laplacian matrix $D^Q(G)$ and majorization are connected by the majorization relation between the spectrum and diagonal elements of Hermitian matrices. The relation given below immediately follows from Schur's theorem.

Lemma 2.4 — Let G be a connected graph of order $n \geq 3$ having distance signless Laplacian eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ and transmission degrees $Tr_1, Tr_2, \ldots, Tr_n$. Then

$$(Tr_1, Tr_2, \ldots, Tr_n) \preceq (\rho_1, \rho_2, \ldots, \rho_n).$$

***Theorem* 2.5** — *Let G be a connected graph of order $n \geq 3$ having transmission degrees $Tr_1, Tr_2, \ldots, Tr_n$.*

(i) If $\alpha < 0$ or $\alpha > 1$, then $m_\alpha \geq \sum_{i=1}^{n} (Tr_i)^\alpha$;

(ii) If $0 < \alpha < 1$, then $m_\alpha \leq \sum_{i=1}^{n} (Tr_i)^\alpha$.

Equality occurs in both parts, if and only if $\rho_i = Tr_i$, for all $i = 1, 2, \ldots, n$.

PROOF : (i) For $x > 0$, it can be seen that the function $f(x) = x^\alpha$ is a convex function if $\alpha < 0$ or $\alpha > 1$. Let $(X) = (Tr_1, Tr_2, \ldots, Tr_n)$ and $(Y) = (\rho_1, \rho_2, \ldots, \rho_n)$. Since by Lemma 2.4, $(X) \preceq (Y)$, so by Lemma 2.3 it follows that $m_\alpha \geq \sum_{i=1}^{n} (Tr_i)^\alpha$. Equality occurs if and only if $(X) = (Y)$. That is, if and only if $\rho_i = Tr_i$, for all $i = 1, 2, \ldots, n$.

(ii) For $x > 0$, it can be seen that the function $f(x) = -x^\alpha$ is a convex function if $0 < \alpha < 1$. Therefore, proceeding similarly as in part (i), we arrive at part (ii). □

The following observation is due to Aouchiche and Hansen [3].

Lemma 2.6 — [3]. Let G be a connected graph on n vertices with $m \geq n$ edges and let G' be the connected graph obtained from G by the deletion of an edge. Then $\rho_i(G) \leq \rho(G')$, for all $i = 1, 2, \ldots, n$.

The following observation follows from Lemma 2.6 and the fact that $a \leq b$ implies that $a^t \leq b^t$, for all $t > 0$; and $a^t \geq b^t$, for all $t < 0$.

***Theorem* 2.7** — *Let G be a connected graph on $n \geq 3$ vertices with $m \geq n$ edges and let $G - e$ be the connected graph obtained from G by the deletion of an edge e, then*

$$m_\alpha(G) \leq m_\alpha(G - e), \quad if \quad \alpha > 0,$$
$$m_\alpha(G) \geq m_\alpha(G - e), \quad if \quad \alpha < 0$$

As a consequence of Theorem 2.7, we have the following observation.

Corollary 2.8 — Let G be a connected graph of order $n \geq 3$ with $m \geq n$ edges. Then

$$(2n-2)^{\alpha} + (n-1)(n-2)^{\alpha} \leq m_{\alpha}(G) < m_{\alpha}(T), \quad \text{if } \alpha > 0,$$
$$m_{\alpha}(T) < m_{\alpha}(G) \leq (2n-2)^{\alpha} + (n-1)(n-2)^{\alpha}, \quad \text{if } \alpha < 0$$

equality occurs on the left if $\alpha > 0$ and on the right if $\alpha < 0$, if and only if $G \cong K_n$.

PROOF : Let G be connected graph of order n with m edges. Then G contains a spanning tree say T and is itself a spanning subgraph of K_n. Now, using Theorem 2.7 and the fact that the distance signless Laplacian spectrum of the complete graph K_n is $\{2n-2, n-2^{[n-1]}\}$, the result follows. □

If G is a connected bipartite graph with partite sets of cardinality a and b, then G is a spanning subgraph of the complete bipartite graph $K_{a,b}$. Therefore, we have the following observation from Theorem 2.7 for bipartite graphs.

Corollary 2.9 — Let G be a connected bipartite graph of order $n \geq 3$ with $m \geq n$ edges. Then

$$(a-1)(2n-b-4)^{\alpha} + (b-1)(2n-a-4)^{\alpha} + x_1^{\alpha} + x_2^{\alpha} \leq m_{\alpha}(G) < m_{\alpha}(T), \text{ if } \alpha > 0$$
$$m_{\alpha}(T) < m_{\alpha}(G) \leq (a-1)(2n-b-4)^{\alpha} + (b-1)(2n-a-4)^{\alpha} + x_1^{\alpha} + x_2^{\alpha}, \text{ if } \alpha < 0$$

where $x_1 = \frac{5n-8+\sqrt{9(a-b)^2+4ab}}{2}$ and $x_2 = \frac{5n-8-\sqrt{9(a-b)^2+4ab}}{2}$. Equality occurs on the left if $\alpha > 0$ and on the right if $\alpha < 0$, if and only if $G \cong K_{a,b}$.

If G is a transmission regular graph, we have the following consequence of Theorem 2.7.

Corollary 2.10 — Let G be a connected r-transmission regular graph of order $n \geq 3$. Then

$$(2n-2)^{\alpha} + (n-1)(n-2)^{\alpha} \leq m_{\alpha}(G) \leq m_{\alpha}(C_n), \quad \text{if } \alpha > 0$$
$$m_{\alpha}(C_n) \leq m_{\alpha}(G) \leq (2n-2)^{\alpha} + (n-1)(n-2)^{\alpha}, \quad \text{if } \alpha < 0,$$

equality occurs on the left if $\alpha > 0$ and on the right if $\alpha < 0$, if and only if $G \cong K_n$; and equality occurs on the right if $\alpha > 0$ and on the left if $\alpha < 0$, if and only if $G \cong C_n$.

PROOF : Let G be a connected r-transmission regular graph of order $n \geq 3$, then C_n is a spanning subgraph of G and so the result follows by Theorem 2.7. □

The following lemma follows from Rayleigh's quotient by taking x to be all ones vector.

Lemma 2.11 — If G is a connected graph of order n, then $\rho_1 \geq \frac{4\sigma(G)}{n}$, with equality holding if and only if G is transmission regular.

The following theorem gives a lower and an upper bound for m_{α} in terms of the number of vertices n and the transmission number $\sigma(G)$ of the graph G.

***Theorem* 2.12** — *Let G be a connected graph of order $n \geq 3$.*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$m_\alpha(G) \geq \left(\frac{4\sigma(G)}{n}\right)^\alpha + \frac{\Big(2\sigma(G)(n-2)\Big)^\alpha}{n^\alpha (n-1)^{\alpha-1}},$$

with equality if and only if $G \cong K_n$.

(ii) If $0 < \alpha < 1$, then

$$m_\alpha(G) \leq \left(\frac{4\sigma(G)}{n}\right)^\alpha + \frac{\Big(2\sigma(G)(n-2)\Big)^\alpha}{n^\alpha (n-1)^{\alpha-1}},$$

with equality if and only if $G \cong K_n$.

PROOF : (i) For $\alpha \neq 0, 1$ and $x > 0$, we observe that x^α is a strictly convex function if and only if $\alpha < 0$ or $\alpha > 1$. Suppose that $\alpha < 0$ or $\alpha > 1$. Then

$$\left(\sum_{i=2}^{n} \frac{1}{n-1}\rho_i\right)^\alpha \leq \sum_{i=2}^{n} \frac{1}{n-1}\rho_i^\alpha, \quad \text{that is} \quad \sum_{i=2}^{n} \rho_i^\alpha \geq \frac{1}{(n-1)^{\alpha-1}} \left(\sum_{i=2}^{n} \rho_i\right)^\alpha,$$

with equality if and only if $\rho_2 = \cdots = \rho_n$. So it follows that

$$m_\alpha(G) \geq \rho_1^\alpha + \frac{1}{(n-1)^{\alpha-1}} \left(\sum_{i=2}^{n} \rho_i\right)^\alpha = \rho_1^\alpha + \frac{(2\sigma(G) - \rho_1)^\alpha}{(n-1)^{\alpha-1}}.$$

Consider the function $g(x) = x^\alpha + \frac{(2\sigma(G)-x)^\alpha}{(n-1)^{\alpha-1}}$. By solving $g'(x) = \alpha\left(x^{\alpha-1} - \frac{(2\sigma(G)-x)^{\alpha-1}}{(n-1)^{\alpha-1}}\right) \geq 0$, one can easily see that $g(x)$ is increasing for $x \geq \frac{2\sigma(G)}{n}$. By Lemma 2.11, $\rho_1 \geq \frac{4\sigma(G)}{n} \geq \frac{2\sigma(G)}{n}$ and then

$$\begin{aligned} m_\alpha(G) &\geq g\left(\frac{4\sigma(G)}{n}\right) = \left(\frac{4\sigma(G)}{n}\right)^\alpha + \frac{\left(2\sigma(G) - \frac{4\sigma(G)}{n}\right)^\alpha}{(n-1)^{\alpha-1}} \\ &= \left(\frac{4\sigma(G)}{n}\right)^\alpha + \frac{\Big(2\sigma(G)(n-2)\Big)^\alpha}{n^\alpha (n-1)^{\alpha-1}}, \end{aligned}$$

with equality if and only if $\rho_2 = \cdots = \rho_n$ and $\rho_1 = \frac{4\sigma(G)}{n}$. Therefore, G has exactly two distinct eigenvalues, and Lemma 2.2 indicates that G is the complete graph K_n.

(ii) Now, suppose that $0 < \alpha < 1$. Then

$$\left(\sum_{i=2}^{n} \frac{1}{n-1}\rho_i\right)^\alpha \geq \sum_{i=2}^{n} \frac{1}{n-1}\rho_i^\alpha,$$

with equality if and only if $\rho_2 = \cdots = \rho_n$, and $g(x)$ is decreasing for $x \geq \frac{2\sigma(G)}{n}$. By similar arguments as above, the second part of the theorem follows. □

The following observation is due to Perron and Frobenius.

Lemma 2.13 — Let $S = (s_{ij})$ be a real matrix, and X be an irreducible matrix of the same order. Let $|S|$ denotes the matrix whose (i, j)-entry is $|s_{ij}|$. If $|S| \leq X$ and S has t as an eigenvalue, then $|t| \leq \lambda_1(X)$. If the equality holds, then $|S| = X$, and there is a diagonal matrix E with diagonal entries of absolute value 1 and a constant c of absolute value 1, such that $S = cEXE^{-1}$.

The following lemma gives a lower bound for the distance spectral radius ρ_1 in terms of the order n and the maximum transmission of the graph G.

Lemma 2.14 — Let G be a connected graph of order $n \geq 3$ having largest distance signless Laplacian eigenvalue ρ_1 and transmission degrees $Tr_1 \geq Tr_2 \geq \cdots \geq Tr_n$. Then

$$\rho_1 > \frac{n}{n-1} Tr_1.$$

PROOF : Let λ be the largest eigenvalue of $D^L(G)$. It is well known that 0 is an eigenvalue of $D^L(G)$ with corresponding eigenvector $e = (1, 1 \ldots, 1)^T$. Using the Rayleigh's quotient, we have

$$\lambda = \max_{x \perp e} \frac{x^T D^L(G) x}{||x||^2}.$$

Taking $x = (1, \frac{-1}{n-1}, \frac{-1}{n-1}, \ldots, \frac{-1}{n-1})^T$ and using the fact $Tr_i = \sum_{j=1,\, i\neq j}^{n} d_{ij}$, we have

$$\frac{x^T D^L(G) x}{||x||^2} = \frac{n-1}{n}\Big(Tr_1 - \frac{2}{n-1} Tr_1 + \frac{1}{(n-1)^2} Tr_1\Big) = \frac{n}{n-1} Tr_1.$$

This shows that

$$\lambda \geq \frac{x^T D^L(G) x}{||x||^2} = \frac{n}{n-1} Tr_1.$$

Now, since $D^Q(G) = |D^L(G)|$ and $D^Q(G)$ is irreducible, from Lemma 2.13, it follows that $\lambda = |\lambda| \leq \rho_1$. Since the smallest eigenvalue of $D^L(G)$ is 0 and that of $D^Q(G)$ is greater or equal to $n-2$, it follows from Lemma 2.13 that $\rho_1 > \lambda \geq \frac{n}{n-1} Tr_1$. This completes the proof. □

The following theorem gives a lower and an upper bound for m_α in terms of the number of vertices n, the maximum transmission Tr_1 and the transmission number $\sigma(G)$ the graph G.

***Theorem* 2.15** — *Let G be a connected graph of order $n \geq 3$ and let $Tr_1 = \max_{1 \leq i \leq n} Tr_i$.*

(i) If $\alpha < 0$ *or* $\alpha > 1$, *then*

$$m_\alpha(G) \geq \left(\frac{nTr_1}{n-1}\right)^\alpha + \frac{\Big(2\sigma(G)(n-1) - nTr_1\Big)^\alpha}{(n-1)^{2\alpha-1}}.$$

(ii) If $0 < \alpha < 1$, *then*

$$m_\alpha(G) \leq \left(\frac{nTr_1}{n-1}\right)^\alpha + \frac{\Big(2\sigma(G)(n-1) - nTr_1\Big)^\alpha}{(n-1)^{2\alpha-1}}.$$

PROOF : The proof follows by proceeding similarly as in Theorem 2.12, and using $\rho_1 > \frac{n}{n-1}Tr_1 > Tr_1 \geq \frac{2\sigma(G)}{n}$. □

The next theorem gives a lower and an upper bound for m_α in terms of the number of vertices n and the transmission number $\sigma(G)$ of the graph G.

***Theorem* 2.16** — *Let G be a graph of order $n \geq 3$ and $1 \leq k \leq n-1$ be a positive integer.*

(i) If $0 < \alpha < 1$, *then*

$$m_\alpha(G) \leq k^{1-\alpha}\left(\frac{2k\sigma(G)}{n}\right)^\alpha + (n-k)^{1-\alpha}\left(2\sigma(G) - \frac{2k\sigma(G)}{n}\right)^\alpha.$$

(ii) If $\alpha > 1$, *then*

$$m_\alpha(G) \geq k^{1-\alpha}\left(\frac{2k\sigma(G)}{n}\right)^\alpha + (n-k)^{1-\alpha}\left(2\sigma(G) - \frac{2k\sigma(G)}{n}\right)^\alpha.$$

(iii) If $\alpha < 0$, *then*

$$m_\alpha(G) \leq \quad \min_{1\leq k\leq n-1}\left\{k^{1-\alpha}\left[\tfrac{2k\sigma(G)+\sqrt{\theta}}{n}\right]^\alpha + (n-k)^{1-\alpha}\left[\tfrac{2\sigma(G)(n-k)-\sqrt{\theta}}{n}\right]^\alpha\right\},$$

where $\theta = k(n-k)\Bigg(2n\sum\limits_{1\leq i<j\leq n}(d_{ij})^2 + n\sum_{i=1}^n Tr_i^2 - 4\sigma^2(G)\Bigg)$.

PROOF : (i) Since $0 < \alpha < 1$, by power mean inequality, we have

$$\left(\frac{\sum_{i=1}^k \rho_i^\alpha}{k}\right)^{\frac{1}{\alpha}} \leq \left(\frac{\sum_{i=1}^k \rho_i}{k}\right), \quad \text{i.e.,} \quad \sum_{i=1}^k \rho_i^\alpha \leq k^{1-\alpha}\left(\sum_{i=1}^k \rho_i\right)^\alpha \tag{2.1}$$

with equality holding in (2.1) if and only if $\rho_1 = \rho_2 = \cdots = \rho_k$.

Similarly, we have

$$\sum_{i=k+1}^{n} \rho_i^{\alpha} \leq (n-k)^{1-\alpha} \left(2\sigma(G) - \sum_{i=1}^{k} \rho_i\right)^{\alpha} \quad \text{as} \quad \sum_{i=1}^{n} \rho_i = 2\sigma(G) \tag{2.2}$$

with equality holding in (2.2) if and only if $\rho_{k+1} = \rho_{k+2} = \cdots = \rho_n$.

Since $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, we have

$$\frac{\sum_{i=1}^{k} \rho_i}{k} \geq \frac{\sum_{i=k+1}^{n} \rho_i}{n-k} = \frac{2\sigma(G) - \sum_{i=1}^{k} \rho_i}{n-k}.$$

This gives

$$\sum_{i=1}^{k} \rho_i \geq \frac{2k\sigma(G)}{n}. \tag{2.3}$$

By (2.1) and (2.2), we have

$$\begin{aligned} m_\alpha(G) &= \sum_{i=1}^{n} \rho_i^{\alpha} = \sum_{i=1}^{k} \rho_i^{\alpha} + \sum_{i=k+1}^{n} \rho_i^{\alpha} \\ &\leq k^{1-\alpha} \left(\sum_{i=1}^{k} \rho_i\right)^{\alpha} + (n-k)^{1-\alpha} \left(2\sigma(G) - \sum_{i=1}^{k} \rho_i\right)^{\alpha}. \end{aligned}$$

Let us consider a function

$$f(x) = k^{1-\alpha} x^{\alpha} + (n-k)^{1-\alpha} \left(2\sigma(G) - x\right)^{\alpha}, \quad x \geq \frac{2k\sigma(G)}{n}.$$

Then we have

$$f'(x) = \alpha \left[\left(\frac{x}{k}\right)^{\alpha-1} - \left(\frac{2\sigma(G) - x}{n-k}\right)^{\alpha-1} \right] \leq 0 \quad \text{as} \quad 0 < \alpha < 1 \quad , \quad x \geq \frac{2k\sigma(G)}{n}.$$

Thus $f(x)$ is a decreasing function on $x \geq \frac{2k\sigma(G)}{n}$. So we get the required result.

(ii) Since $\alpha > 1$, using power mean inequality, from (i), we get

$$m_\alpha(G) \geq k^{1-\alpha} \left(\sum_{i=1}^{k} \rho_i\right)^{\alpha} + (n-k)^{1-\alpha} \left(2\sigma(G) - \sum_{i=1}^{k} \rho_i\right)^{\alpha}.$$

Also $f(x)$ is an increasing function on $x \geq \frac{2k\sigma(G)}{n}$ as $\alpha > 1$. Using the same technique as in (i), we get the required result.

(iii) Using Lemma 2.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(2\sigma(G)-\sum_{i=1}^{k}\rho_i)^2 &= (\rho_{k+1}+\cdots+\rho_n)^2\\ &\leq (n-k)(\rho_{k+1}^2+\cdots+\rho_n^2)\\ &= (n-k)\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}Tr_i^2-(\rho_1^2+\cdots+\rho_k^2)\right)\\ &\leq (n-k)\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}Tr_i^2-\frac{1}{k}(\sum_{i=1}^{k}\rho_i)^2\right).\end{aligned}$$

Then it follows that

$$\sum_{i=1}^{k}\rho_i \leq \frac{2k\sigma(G)+\sqrt{k(n-k)\left(2n\sum_{1\leq i<j\leq n}(d_{ij})^2+n\sum_{i=1}^{n}Tr_i^2-4\sigma^2(G)\right)}}{n}.$$

Since $\alpha<0$, from (i), we observe that $f(x)$ is an increasing function on

$$\frac{2k\sigma(G)}{n}\leq x\leq \frac{2k\sigma(G)+\sqrt{k(n-k)\left(2n\sum_{1\leq i<j\leq n}(d_{ij})^2+n\sum_{i=1}^{n}Tr_i^2-4\sigma^2(G)\right)}}{n}.$$

Thus, for $\theta=k(n-k)\left(2n\sum_{1\leq i<j\leq n}(d_{ij})^2+n\sum_{i=1}^{n}Tr_i^2-4\sigma^2(G)\right)$, it follows that

$$f(x)\leq \quad k^{1-\alpha}\left[\frac{2k\sigma(G)+\sqrt{\theta}}{n}\right]^{\alpha}+(n-k)^{1-\alpha}\left[\frac{2\sigma(G)(n-k)-\sqrt{\theta}}{n}\right]^{\alpha}.$$

The result now follows. □

For a connected graph G of order $n\geq 3$, let $P=\prod_{j=1}^{n}\rho_j$, where $\rho_1\geq\rho_2\geq\cdots\geq\rho_n>0$ are the eigenvalues of $D^Q(G)$. The following result gives a lower and an upper bound for m_α in terms of the number of vertices n, the number P and the transmission number $\sigma(G)$ of the graph G.

***Theorem* 2.17** — *Let G be a connected graph of order $n\geq 3$ having transmission number $\sigma(G)$. If $\alpha<0$ or $\alpha>1$, then*

$$m_\alpha(G)\geq\left(\frac{4\sigma(G)}{n}\right)^{\alpha}+(n-1)P^{\frac{\alpha}{n-1}}\left(\frac{4\sigma(G)}{n}\right)^{\frac{-\alpha}{n-1}},$$

with equality if and only if $G\cong K_n$

PROOF : By arithmetic-geometric mean inequality, we have

$$m_\alpha(G) = \rho_1^\alpha + \sum_{i=2}^{n} \rho_i^\alpha \geq \rho_1^\alpha + (n-1)\Big(\prod_{i=2}^{n} \rho_i^\alpha\Big)^{\frac{1}{n-1}}$$
$$= \rho_1^\alpha + (n-1)\Big(\frac{P}{\rho_1}\Big)^{\frac{\alpha}{n-1}},$$

with equality if and only if $\rho_2 = \rho_3 = \cdots = \rho_n$. Consider the function $f(x) = x^\alpha + (n-1)P^{\frac{\alpha}{n-1}} x^{\frac{-\alpha}{n-1}}$. Differentiating, we have $f'(x) = \alpha x^{\frac{-\alpha}{n-1}-1}(x^{\frac{n\alpha}{n-1}} - P^{\frac{\alpha}{n-1}})$. It is easy to see that whether $\alpha < 0$ or $\alpha > 1$, the function $f(x)$ is increasing for $x \geq P^{\frac{1}{n}}$. By Lemma 2.11, we have

$$\rho_1 \geq \frac{4\sigma(G)}{n} \geq \frac{2\sigma(G)}{n} = \frac{\sum_{i=1}^{n} \rho_i}{n} \geq \Big(\prod_{i=1}^{n} \rho_i\Big)^{\frac{1}{n}} = P^{\frac{1}{n}}.$$

So, it follows that

$$m_\alpha(G) = f(\alpha) \geq f\left(\frac{4\sigma(G)}{n}\right) = \left(\frac{4\sigma(G)}{n}\right)^\alpha + (n-1)P^{\frac{\alpha}{n-1}}\Big(\frac{4\sigma(G)}{n}\Big)^{\frac{-\alpha}{n-1}}.$$

Equality occurs if and only if $\rho_1 = \frac{4\sigma(G)}{n}$ and $\rho_2 = \rho_3 = \cdots = \rho_n$. That is, if and only if G is a transmission regular graph with two distinct distance signless Laplacian eigenvalues. That is, by Lemma 2.2, if and only if $G \cong K_n$. □

The following theorem gives a lower bound and an upper bound for $m_\alpha(G)$ in terms of transmission degrees, sum of squares of distances and the transmission number of the graph G.

***Theorem* 2.18** — *Let G be a connected graph of order $n \geq 3$ having transmission number $\sigma(G)$. Then*

$$m_\alpha(G) \geq \frac{\big(2\sigma(G)\big)^{2-\alpha}}{\Big(2\sum\limits_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\Big)^{1-\alpha}}, \ \ if \ \ \alpha < 0, \ 0 < \alpha < 1$$

$$m_\alpha(G) \leq \frac{\big(2\sigma(G)\big)^{2-\alpha}}{\Big(2\sum\limits_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\Big)^{1-\alpha}}, \ \ if \ \ 1 < \alpha < 2, \ \alpha > 2.$$

PROOF : Let $b_1, b_2, \ldots, b_s$ be positive real numbers and let t be a real number with $t \neq 0, \frac{1}{2}, 1$. If $t < 0$ or $t > 1$, it is clear that $\frac{2t-1}{t} > 0$. By Hölder's inequality, we have

$$\sum_{i=1}^{s} b_i^t = \sum_{i=1}^{s} b_i^{\frac{t}{2t-1}} b_i^{\frac{2t(t-1)}{2t-1}} \leq \left(\sum_{i=1}^{s}\Big(b_i^{\frac{t}{2t-1}}\Big)^{\frac{2t-1}{t}}\right)^{\frac{t}{2t-1}} \left(\sum_{i=1}^{s}\Big(b_i^{\frac{2t(t-1)}{2t-1}}\Big)^{\frac{2t-1}{t-1}}\right)^{\frac{t-1}{2t-1}}$$
$$= \Big(\sum_{i=1}^{s} b_i\Big)^{\frac{t}{2t-1}} \Big(\sum_{i=1}^{s} b_i^{2t}\Big)^{\frac{t-1}{2t-1}},$$

$$\Rightarrow \quad \Big(\sum_{i=1}^{s} b_i\Big)^{\frac{t}{2t-1}} \geq \frac{\sum\limits_{i=1}^{s} b_i^t}{\Big(\sum\limits_{i=1}^{s} b_i^{2t}\Big)^{\frac{t-1}{2t-1}}} \qquad \text{or,} \qquad \sum_{i=1}^{s} b_i \geq \frac{\Big(\sum\limits_{i=1}^{s} b_i^t\Big)^{\frac{2t-1}{t}}}{\Big(\sum\limits_{i=1}^{s} b_i^{2t}\Big)^{\frac{t-1}{t}}},$$

with equality if and only if $b_1 = b_2 = \cdots = b_s$.

Now, taking $s = n, \;\; b_i = \rho_i^\alpha$ and $t = \frac{1}{\alpha}$, it follows that

$$m_\alpha(G) = \sum_{i=1}^{n} \rho_i^\alpha \geq \frac{\Big(\sum\limits_{i=1}^{n} \rho_i\Big)^{2-\alpha}}{\Big(\sum\limits_{i=1}^{n} \rho_i^2\Big)^{1-\alpha}} = \frac{\big(2\sigma(G)\big)^{2-\alpha}}{\Big(2\sum\limits_{1\leq i<j\leq n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\Big)^{1-\alpha}},$$

for all $\alpha < 0$ or $0 < \alpha < 1$. If $1 < \alpha < 2$ or $\alpha > 2$, then $\frac{1}{2} < t < 1$ or $0 < t < \frac{1}{2}$. Let $p = \frac{2t-1}{t}$ and $q = \frac{2t-1}{t-1}$. If $\frac{1}{2} < t < 1$, then $p > 0, \;\; q < 0$ and if $0 < t < \frac{1}{2}$, then $p < 0, \; q > 0$. Therefore, in each of these cases Hölder's inequality gets reversed and so the result follows. □

The sum of the squares of the degrees of the graph G is called its first Zagreb index $M_1(G)$. Several research papers are devoted towards the study of the topological index $M_1(G)$. Based on this information, we put forward

$$\Gamma_1(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij})^2 = \sum_{1\leq i<j\leq n}(d_{ij})^2$$

and call it first distance Zagreb index of the graph G. It will be of interest to develop a theory for the index $\Gamma_1(G)$, similar to the theory as has been developed for the index $M_1(G)$. If the transmission degree Tr_i of the vertices is known, Lemma 2.1 shows that the study of the index $\Gamma_1(G)$ is equivalent to the study of $m_2(G)$. In fact from Lemma 2.1, we have

$$\Gamma_1(G) = \frac{1}{2}\Big(m_2(G) - \sum_{i=1}^{n} Tr_i^2\Big).$$

If G is an r-transmission regular graph, then $Tr_i = r$, for all $i, \; 1 \leq i \leq n$ and so

$$\Gamma_1(G) = \frac{1}{2}\Big(m_2(G) - nr^2\Big).$$

If G is a connected graph of order n having diameter one, then $G \cong K_n$ and so $m_2(G) = (2n-2)^2 + (n-1)(n-2)^2 = (n-1)n^2$. Since K_n is an r-transmission regular graph with $r = n-1$, it follows that

$$\Gamma_1(K_n) = \frac{1}{2}\Big((n-1)n^2 - n(n-1)^2\Big) = \frac{n(n-1)}{2}.$$

If G is a connected graph of order n having diameter two, then $Tr_i = 2n-2-d_i$, for all $i,\ 1 \le i \le n$ and

$$d_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ vertex adjacent to } j^{th} \text{ vertex} \\ 2, & \text{if } i^{th} \text{ vertex not adjacent to } j^{th} \text{ vertex.} \end{cases}$$

Therefore,

$$\begin{aligned} 2\Gamma_1(G) &= \sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij})^2 = \sum_{i=1}^{n}\left(d_{i1}^2 + d_{i2}^2 + \cdots + d_{in}^2\right) \\ &= \sum_{i=1}^{n} d_{i1}^2 + \sum_{i=1}^{n} d_{i2}^2 + \cdots + \sum_{i=1}^{n} d_{in}^2 \\ &= (1^2.d_1 + 2^2(n-1-d_1)) + (1^2.d_2 + 2^2(n-1-d_2)) + \cdots + (1^2.d_n + 2^2(n-1-d_n)) \\ &= 4n(n-1) - 6m, \quad \text{as } \sum_{i=1}^{n} d_i = 2m. \end{aligned}$$

Also

$$\sum_{i=1}^{n} Tr_i^2 = \sum_{i=1}^{n}(2n-2-d_i)^2 = 4n(n-1)^2 + M_1(G) - 8m(n-1).$$

Thus,

$$m_2(G) = 2\Gamma_1(G) + \sum_{i=1}^{n} Tr_i^2 = 4n^2(n-1) + M_1(G) - 2m(4n-1).$$

For a connected graph of order n and diameter two, this shows that the study of index $M_1(G)$ is equivalent to the index $m_2(G)$. From this discussion one gets an insight that like the index $M_1(G)$, the index $m_\alpha(G)$ may be useful when one restricts G to molecular graphs.

3. Bounds for Distance Signless Laplacian-Energy-Like Invariant

Motivated by the definitions of Laplacian-energy-like invariant $LEL(G)$ and the incidence energy $IE(G)$ of the graph G, Alhevaz *et al*. [2] put forward the quantity $DEL(G) = \sum_{i=1}^{n} \sqrt{\rho_i}$ which is like the quantities $LEL(G)$ and $IE(G)$, based on the distance signless Laplacian eigenvalues of the graph G. This quantity is called distance signless Laplacian-energy-like invariant and several lower and upper bounds for $DEL(G)$ can be found in [2].

In this section, we obtain some upper and lower bounds for distance signless Laplacian energy-like invariant $DEL(G)$ in terms of the number of vertices, the maximum transmission Tr_1 and the transmission number $\sigma(G)$ of a connected graph G. These bounds improve some previously known bounds.

***Theorem* 3.1** — *Let G be a connected graph of order $n \geq 3$ having transmission degrees $Tr_i,\ 1 \leq i \leq n$ and distance signless Laplacian eigenvalues $\rho_i,\ 1 \leq i \leq n$. Then*

$$DEL(G) \leq \sum_{i=1}^{n} \sqrt{Tr_i},$$

equality occurs if and only if $\rho_i = Tr_i$, for all $i = 1, 2, \ldots, n$.

PROOF : The proof follows from definition and part (ii) of Theorem 2.5. □

Alhevaz *et al*. [2] obtained the following upper bound for $DEL(G)$.

$$DEL(G) \leq \frac{1}{2}\left(\sqrt{\frac{n}{2}} + \sqrt{\frac{2}{n}}\right)\left(\sum_{i=1}^{n} \sqrt{\rho_i}\right)^2. \tag{3.1}$$

Remark 3.2 : It is easy to see that the upper bound given by Theorem 3.1 is always better than the upper bound (3.1).

The following result follows from Lemma 2.6 and shows that the distance signless Laplacian-energy-like invariant of a graph and its spanning subgraph are related.

***Theorem* 3.3** — *Let G be a connected graph on $n \geq 3$ vertices with $m \geq n$ edges and let $G - e$ be the connected graph obtained from G by the deletion of an edge e, then*

$$DEL(G) \leq DEL(G - e).$$

The following observation is a consequence of Theorem 3.3.

Corollary 3.4 — Let G be a connected graph of order $n \geq 3$ with $m \geq n$ edges. Then

$$\sqrt{2n-2} + (n-1)\sqrt{n-2} \leq DEL(G) < DEL(T),$$

equality occurs on the left if and only if $G \cong K_n$.

PROOF : Let G be connected graph of order n, with m edges. Then G contains a spanning tree say T and is itself a spanning subgraph of K_n. Now, using Theorem 3.3, the result follows. □

From Corollary 3.4, it follows that among all connected graphs complete graph K_n has the minimum distance signless Laplacian-energy-like invariant $DEL(G)$ and tree has the maximum distance signless Laplacian-energy-like invariant $DEL(G)$. For the bipartite graphs, we have the following observation from Theorem 3.3.

Corollary 3.5 — Let G be a connected bipartite graph of order $n \geq 3$ with $m \geq n$ edges. Then

$$(a-1)\sqrt{2n-b-4}+(b-1)\sqrt{2n-a-4}+\sqrt{x_1}+\sqrt{x_2} \leq DEL(G) < DEL(T),$$

where $x_1 = \frac{5n-8+\sqrt{9(a-b)^2+4ab}}{2}$ and $x_2 = \frac{5n-8-\sqrt{9(a-b)^2+4ab}}{2}$. Equality occurs on the left, if and only if $G \cong K_{a,b}$.

Corollary 3.5 implies that among all connected bipartite graphs the complete bipartite graph $K_{a,b}$ has the minimum distance signless Laplacian-energy-like invariant $DEL(G)$ and tree has the maximum distance signless Laplacian-energy-like invariant $DEL(G)$. For transmission regular graphs, we have the following observation from Theorem 3.3.

Corollary 3.6 — Let G be a connected r-transmission regular graph of order $n \geq 3$. Then

$$\sqrt{2n-2}+(n-1)\sqrt{n-2} \leq DEL(G) \leq DEL(C_n),$$

equality occurs on the left, if and only if $G \cong K_n$ and on the right if and only if $G \cong C_n$.

PROOF : If G is a connected r-transmission regular graph of order $n \geq 3$, then C_n is a spanning subgraph of G and so the result follows by Theorem 3.3. □

Corollary 3.6 gives that among all transmission regular graphs the complete graph K_n has the minimum distance signless Laplacian-energy-like invariant $DEL(G)$ and the cycle C_n has the maximum distance signless Laplacian-energy-like invariant $DEL(G)$.

Extremal problems of similar kind have been considered for the quantities $LEL(G)$ and $IE(G)$ and the graphs which attain the extremal value for these quantities for various families like trees, unicyclic graphs, bicyclic graphs, tricyclic graphs, bipartite graphs, etc, are now known. It will be of interest to discuss similar extremal problems for the quantity $DEL(G)$. Therefore, we put forward the following problems.

Problem 1 : Among all trees, connected unicyclic graphs, bicyclic graphs, tricyclic graphs, etc, determine the graphs which attain the minimum and maximum values for the quantity $DEL(G)$.

Problem 2 : Among all connected graphs with fixed matching number/ clique number/ domination number/number of pendent vertices, etc, determine the graphs which attain the minimum and maximum values for the quantity $DEL(G)$.

The following theorem gives an upper bound for $DEL(G)$ in terms of the number of vertices n and the transmission number $\sigma(G)$ of the graph G.

***Theorem* 3.7** — *Let G be a connected graph of order $n \geq 3$. Then*

$$DEL(G) \leq \sqrt{\frac{4\sigma(G)}{n}} + \sqrt{\frac{(n-1)2\sigma(G)(n-2)}{n}},$$

with equality if and only if $G \cong K_n$.

PROOF : The proof follows from the definition and part (ii) of Theorem 2.12. □

Alhevaz *et al*. [2] obtained the following upper bound for $DEL(G)$ in terms of the number of vertices n and the transmission number $\sigma(G)$ of the graph G.

$$DEL(G) \leq \sqrt{2n\sigma(G)}. \tag{3.2}$$

Remark 3.8 : It is easy to see that the upper bound given by Theorem 3.7 is always better than the upper bound (3.2).

The following theorem gives an upper bound for $DEL(G)$ in terms of the number of vertices n and the transmission number $\sigma(G)$ of the graph G.

***Theorem* 3.9** — *Let G be a connected graph of order $n \geq 3$ and transmission number $\sigma(G)$. Then*

$$DEL(G) \geq \sqrt{2\sigma(G) + (n-1)\Big[(n-2)^2 + 2\sqrt{2(n-1)(n-2)}\Big]}, \tag{3.3}$$

with equality if and only if $G \cong K_n$.

PROOF : Let G be a connected graph of order $n \geq 2$ having distance signless Laplacian eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$. We have

$$(DEL(G))^2 = \Big(\sum_{i=1}^{n} \sqrt{\rho_i}\Big)^2 = \sum_{i=1}^{n} \rho_i + 2\sum_{i \neq j} \sqrt{\rho_i}\sqrt{\rho_j}. \tag{3.4}$$

Since G is a connected graph, it is a connected spanning subgraph of K_n, therefore by Lemma 2.6, it follows that

$$\rho_1(G) \geq \rho_1(K_n) = 2n-2, \;\; \rho_i(G) \geq \rho_i(K_n) = n-2, \quad i \geq 2.$$

Now,

$$\begin{aligned}\sum_{i\neq j}\sqrt{\rho_i}\sqrt{\rho_j} &= \sqrt{\rho_1}(\sqrt{\rho_2}+\cdots+\sqrt{\rho_n})+\sqrt{\rho_2}(\sqrt{\rho_3}+\cdots+\sqrt{\rho_n})+\cdots+\sqrt{\rho_{n-1}}\sqrt{\rho_n}\\ &\geq (n-1)\sqrt{2(n-1)(n-2)}+(n-2)(n-2)+(n-3)(n-2)+\cdots+(n-2)\\ &= (n-1)\sqrt{2(n-1)(n-2)}+(n-2)(1+2+\cdots+(n-2))\\ &= (n-1)\sqrt{2(n-1)(n-2)}+\frac{(n-1)(n-2)^2}{2}.\end{aligned}$$

Using this in equation (3.4), we obtain

$$DEL(G)\geq\sqrt{2\sigma(G)+(n-1)\Big[(n-2)^2+2\sqrt{2(n-1)(n-2)}\Big]}.$$

Equality occurs in (3.3), if and only if $\rho_1(G)=\rho_1(K_n)=2n-2$ and $\rho_i(G)=\rho_i(K_n)=n-2$, for all $2\leq i\leq n$. That is, if and only if $G\cong K_n$. □

Alhevaz *et al*. [2] obtained the following lower bound for $DEL(G)$ in terms of the transmission number $\sigma(G)$ of the graph G:

$$DEL(G)\geq\sqrt{2\sigma(G)}. \tag{3.5}$$

Remark 3.10 : It is clear that the lower bound (3.3) is always better than the lower bound (3.5).

Now, we give a relation between $DEL(G)$ and $IE(G)$ for a graph G of diameter two.

***Theorem* 3.11** — *Let G be a connected graph of order $n\geq 3$ having diameter 2 and let $\overline{G}$ be the complement of G. Then*

$$DEL(G)=IE(\overline{G}+(n-2)I+J),$$

where I is the identity matrix and J is the all one matrix of order n.

PROOF : Let G be a connected graph of order $n\geq 4$ having diameter d. Let $Deg(G)=diag(d_1,d_2,\ldots,d_n)$ be the diagonal matrix of vertex degrees of G and $Deg(\overline{G})=diag(n-1-d_1,n-1-d_2,\ldots,n-1-d_n)$ be the diagonal matrix of vertex degrees of $\overline{G}$. Let $Q(G)=Deg(G)+A$ be the signless Laplacian matrix of G. Suppose the diameter d of G is two so that the transmission degree $Tr_i=2n-2-d_i$, for all i. Since diameter of G is two, it implies that any two vertices are either adjacent in G or in $\overline{G}$. It then follows that the distance matrix of G can be written as $D(G)=A+2\overline{A}$, where A and $\overline{A}$ are the adjacency matrices of G and $\overline{G}$, respectively. We have

$$\begin{aligned} D^Q(G) = Tr(G) + D(G) &= (2n-2)I - Deg(G) + (A + 2\overline{A}) \\ &= (n-1)I + ((n-1)I - Deg(G)) + \overline{A} + (A + \overline{A}) \\ &= (n-2)I + J + Deg(\overline{G}) + \overline{A} \\ &= (n-2)I + J + Q(\overline{G}) \end{aligned}$$

where I is the identity matrix and J is the all one matrix of order n. From this the result follows. □

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REFERENCES

1. S. Akbari, E. Ghorbani, J. H. Koolen, and M. R. Oboudi, On sum of powers of the Laplacian and signless Laplacian eigenvalues of graphs, *Elec. J. Comb.*, **17** (2010), R115.
2. A. Alhevaz, M. Baghipur, S. Pirzada, and Y. Shang, Some bounds for distance signless Laplacian energy-like invariant of graphs, Symmetry, to appear.
3. M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.*, **439**(1) (2013), 21-33.
4. M. Aouchiche and P. Hansen, Distance spectra of graphs: A survey, *Linear Algebra Appl.*, **458** (2014), 301-386.
5. F. Ashraf, On two conjectures on sum of the powers of signless Laplacian eigenvalues of a graph, *Linear and Multilinear Algebra*, **64** (2016), 1314-1320.
6. R. B. Bapat, Determinant of the distance matrix of a tree with matrix weights, *Linear Algebra Appl.*, **416** (2006), 2-7.
7. R. B. Bapat, S. J. Kirkland, and M. Neumann, On distance matrices and Laplacians, *Linear Algebra Appl.*, **401** (2005), 193-209.
8. R. B. Bapat, A. K. Lal, and S. Pati, A q-analogue of the distance matrix of a tree, *Linear Algebra Appl.*, **416** (2006), 799-814.
9. K. C. Das, K. Xu, and M. Liu, On sum of powers of the Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **439** (2013), 3561-3575.
10. K. C. Das, M. Aouchiche, and P. Hansen, On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs, *Discrete Applied Math.*, **243** (2018), 172-185.

11. R. C. Diaz and O. Rojo, Sharp upper bounds on the distance energies of a graph, *Linear Algebra Appl.*, **545** (2018), 55-75.

12. H. A. Ganie, A. M. Alghamdi, and S. Pirzada, On the sum of the Laplacian eigenvalues of a graph and Brouwer's conjecture, *Linear Algebra Appl.*, **501** (2016), 376-389.

13. J. Liu and B. Liu, A Laplacian-energy-like invariant of a graph, *MATCH Commun. Math. Comput. Chem.*, **59** (2008), 355-372.

14. L. Lu, Q. Huang, and X. Huang, On graphs whose smallest distance (signless Laplacian) eigenvalue has large multiplicity, *Linear Multilinear Algebra*, **66, 11** (2018), 2218-2231.

15. S. Pirzada, *An introduction to graph theory*, Universities Press, OrientBlackSwan, Hyderabad, (2012).

16. S. Pirzada and H. A. Ganie, On Laplacian-energy-like invariant and incidence energy, *Trans. Comb.*, **4** (2015), 41-51.

17. S. Pirzada, H. A. Ganie, and I. Gutman. On Laplacian-energy-like invariant and Kirchhoff index, *MATCH Commun. Math. Comput. Chem.*, **73** (2015), 41-59.

18. B Zhou. On sum of powers of the Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **429** (2008) 2239-2246.