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# ALMOST EVERYWHERE STRONG *C*,1,0 SUMMABILITY OF 2-DIMENSIONAL TRIGONOMETRIC FOURIER SERIES[1]

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In this paper we study the a. e. exponential strong $(C,1,0)$ summability of of the 2-dimensional trigonometric Fourier series of the functions belonging to $L\left(\log^{+}L\right)^{2}$.

**Key words** : 2-Dimensional Fourier series; strong summability; exponential means.

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## 1. Introduction

We shall denote the set of all non-negative integers by $\mathbb{N}$. Let $\mathbb{T}:=[-\pi,\pi)=\mathbb{R}/2\pi$ and $\mathbb{R}:=(-\infty,\infty)$. We denote by $L_p(\mathbb{T})$ the class of all measurable functions $f$ on $\mathbb{R}$ that are $2\pi$-periodic and satisfy

$$\|f\|_p:=\left(\int\limits_{\mathbb{T}}|f|^p\right)^{1/p}<\infty,1\leq p<\infty.$$

Suppose $f\in L_1(\mathbb{T})$ and put

$$M(x;f):=\sup_I\frac{1}{|I|}\int\limits_I|f(y)|\,dy,$$

where the supremum is taken over the collection $\{I\}$ of those open intervals $I$ which contains $x$ and of lenght $\leq 2\pi$.

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The Fourier series of the function $f \in L_1(\mathbb{T})$ with respect to the trigonometric system is the series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}, \tag{1}$$

where

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

are the Fourier coefficients of $f$.

We shall write $a \lesssim b$, if $a < c \cdot b$ and $c > 0$ is an absolute constant.

Denote by $S_n(x, f)$ the partial sums of the Fourier series of $f$ and let

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x, f)$$

be the $(C, 1)$ means of (1). Fejéz [2] proved that $\sigma_n(f)$ converges to $f$ uniformly for any $2\pi$-periodic continuous function. Lebesgue in [18] established almost everywhere convergence of $(C, 1)$ means if $f \in L_1(\mathbb{T})$. The strong summability problem, i.e. the convergence of the strong means

$$\frac{1}{n+1} \sum_{k=0}^{n} |S_k(x, f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0, \tag{2}$$

was first considered by Hardy and Littlewood in [11]. They showed that for any $f \in L_r(\mathbb{T})$ $(1 < r < \infty)$ the strong means tend to $0$ a.e., if $n \to \infty$. The trigonometric Fourier series of $f \in L_1(\mathbb{T})$ is said to be $(H, p)$-summable at $x \in \mathbb{T}$, if the values (2) converge to $0$ as $n \to \infty$. The $(H, p)$-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [19] for $p = 2$, and later by Zygmund [34] for the general case $1 \le p < \infty$. Oskolkov in [20] proved the following

***Theorem* Os** — *(Oskolkov). Let $f \in L_1(\mathbb{T})$ and let $\Phi$ be a continuous positive convex function on $[0, +\infty)$ with $\Phi(0) = 0$ and*

$$\ln \Phi(t) = O(t / \ln \ln t) \quad (t \to \infty). \tag{3}$$

*Then for almost all $x$*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi(|S_k(x, f) - f(x)|) = 0. \tag{4}$$

It was noted in [20] that Totik announced the conjecture that (4) holds almost everywhere for any $f \in L_1(\mathbb{T})$, provided

$$\ln \Phi(t) = O(t) \quad (t \to \infty). \tag{5}$$

In [21] Rodin proved

***Theorem* R — *(Rodin). Let $f \in L_1(\mathbb{T})$. Then for any $A > 0$***

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(\exp\left(A\left|S_k\left(x,f\right)-f\left(x\right)\right|\right)-1\right)=0$$

*for a. e. $x \in \mathbb{T}$.*

Karagulyan [13] proved that the following is true.

***Theorem* K — *(Karagulyan). Suppose that a continuous increasing function $\Phi : [0,\infty) \to [0,\infty), \Phi\left(0\right)=0$, satisfies the condition***

$$\limsup_{t\to+\infty} \frac{\log\Phi\left(t\right)}{t}=\infty.$$

*Then there exists a function $f \in L_1(\mathbb{T})$ for which the relation*

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(\left|S_k\left(x,f\right)\right|\right)=\infty$$

*holds everywhere on $\mathbb{T}$.*

We denote by $L\left(\log^+ L\right)^\alpha\left(\mathbb{T}^2\right)$ the class of measurable functions $f$, with

$$\iint\limits_{\mathbb{T}^2} |f|\left(\log^+|f|\right)^\alpha<\infty, \alpha>0,$$

where $\log^+ u := \mathbb{I}_{(1,\infty)}\log u$.

Let $f \in L_1(\mathbb{T}^2)$, $\mathbb{T}^2 := [-\pi,\pi]^2$ be a function with Fourier series

$$\sum_{m,n=-\infty}^{\infty} \widehat{f}\left(m,n\right)e^{i(mx_1+nx_2)}, \tag{6}$$

where

$$\widehat{f}\left(m,n\right)=\frac{1}{4\pi^2}\iint\limits_{\mathbb{T}^2} f(x_1,x_2)e^{-i(mx_1+nx_2)}dx_1dx_2$$

are the Fourier coefficients of the function $f$. The rectangular partial sums of (6) are defined as follows:

$$S_{MN}\left(x_1,x_2;f\right)=\sum_{m=-M}^{M}\sum_{n=-N}^{N}\widehat{f}\left(m,n\right)e^{i(mx_1+nx_2)}.$$

Set

$$S_M^{(1)}(x_1,x_2;f):=\sum_{m=-M}^{M}\widehat{f}(m,x_2)\,e^{imx_1}$$

and

$$S_N^{(2)}(x_1,x_2;f):=\sum_{n=-N}^{N}\widehat{f}(x_1,n)\,e^{inx_2},$$

where

$$\widehat{f}(m,x_2):=\frac{1}{2\pi}\int\limits_{\mathbb{T}}f(x_1,x_2)\,e^{-imx_1}dx_1,\text{ for a. e. }x_2\in\mathbb{T}$$

and

$$\widehat{f}(x_1,n):=\frac{1}{2\pi}\int\limits_{\mathbb{T}}f(x_1,x_2)\,e^{-inx_2}dx_2,\text{ for a. e. }x_1\in\mathbb{T}$$

Recall the definition of $BMO\,[0,1]$ space. It is the Banach space of functions $f\in L_1[0,1]$ with the norm

$$\|f\|_{BMO}=R(f)+\left|\int_0^1 f(t)dt\right|$$

where

$$R(f)=\sup_I\left(\frac{1}{|I|}\int\limits_I|f-f_I|^2\right)^{1/2},f_I=\frac{1}{|I|}\int_I f(t)dt$$

and the supremum is taken over all intervals $I\subset[0,1]$.

For $(C,1,1)$ summability of two-dimensional trigonometric Fourier series Jessen, Marcinkiewicz and Zygmund [12] has proved, that if $f\in L\log^+L\left(\mathbb{T}^2\right)$, then

$$\lim_{n,m\to\infty}\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(S_{ij}(x_1,x_2;f)-f(x_1,x_2)\right)=0$$

for a. e. $(x_1,x_2)\in\mathbb{T}^2$. Thay also shoved that for every non-negative function $\omega:[0,\infty)\to[0,\infty),\omega\uparrow\infty,\omega(t)(\log t)^{-1}\to 0$ as $t\to\infty$ there exists the measurable function $f$ such that $|f|\,\omega(|f|)\in L_1\left(\mathbb{T}^2\right)$, but its 2-dimensional Fourier series is a.e. non-summable by the $(C,1,1)$ method.

The 2-dimensional a. e. strong summability, i. e.

$$\lim_{n,m\to\infty}\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left|S_{ij}(x_1,x_2;f)-f(x_1,x_2)\right|^p=0$$

was shown by Gogoladze [10] for $f \in L\log^{+}L\left(\mathbb{T}^{2}\right)$. These results show that in the case of 2-dimensional functions the $(C,1,1)$ summability and $(C,1,1)$ strong summability we have the same maximal summability spaces. That is, in both cases we have $L\log^{+}L\left(\mathbb{T}^{2}\right)$.

For $(C,1,0)$ summability of two-dimensional trigonometric Fourier series Ash and Wellard [1] has proved, that if $f \in L\left(\log^{+}L\right)^{2}\left(\mathbb{T}^{2}\right)$, then

$$\lim_{n,m\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\left(S_{im}\left(x_{1},x_{2};f\right)-f\left(x_{1},x_{2}\right)\right)=0$$

for a. e. $(x_{1},x_{2})\in\mathbb{T}^{2}$.

In this paper we study the a. e. exponential strong $(C,1,0)$ summability of the 2-dimensional trigonometric Fourier series of the functions belonging to $L\left(\log^{+}L\right)^{2}$.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems, see Schipp [22-25], Leindler [15-17, Totik [26-28], Goginava, Gogoladze [7, 8], Goginava, Gogoladze, Karagulyan [9], Gát, Goginava, Karagulyan [5, 6], Weisz [30-33].

## 2. Main Results

***Theorem* 1** — *Suppose that* $f \in L\left(\log^{+}L\right)^{2}\left(\mathbb{T}^{2}\right)$. *Then for any* $A>0$

$$\lim_{n,l\to\infty}\frac{1}{2^{n}}\sum_{k=0}^{2^{n}-1}\left(\exp\left(A\left|S_{kl}\left(x_{1},x_{2};f\right)-f\left(x_{1},x_{2}\right)\right|\right)-1\right)=0$$

*for a. e.,* $(x_{1},x_{2})\in\mathbb{T}^{2}$.

## 3. Auxiliary Propositions

We shall us the following operators comming from Gabisonia [4]

$$G_{N}\left(x;f\right):=\left(\sum_{k=1}^{[N\pi]}\left(\frac{N}{k}\int_{\frac{k-1}{N}}^{\frac{k}{N}}\left|f\left(x+t\right)\right|+\left|f\left(x-t\right)\right|dt\right)^{2}\right)^{1/2}.$$

$$G\left(x;f\right):=\sup_{N\in\mathbb{N}}G_{N}\left(x;f\right),$$

where $p>1, 1/p+1/q=1$. Here $[N\pi]$ denote the integer part of $N\pi$.

The following was proved by Gabisonia [4].

*Lemma* 1 — (Gabisonia). Let $f \in L_1(\mathbb{T})$. Then

$$\lambda \left|\{x \in \mathbb{T} : G(x; f) > \lambda\}\right| \lesssim \|f\|_1 .$$

Set

$$f_n(x; t) := \sum_{k=0}^{2^n-1} S_k(x; f)\, \mathbb{I}_{\delta_k^n}(t) .$$

The next theorem is due to Rodin [4].

*Lemma* 2 — (Rodin). Let $f \in L_1(\mathbb{T})$. The following estimation holds

$$\sup_{n\in\mathbb{N}} \|f_n(x; \cdot)\|_{BMO} \lesssim G(x; f) + M(x; f) + |f(x)| .$$

For a two-dimensional integrable function $f$ we need to introduce the following hybrid functions

$$G_N^{(1)}(x_1, x_2; f) := \left(\sum_{k=1}^{[N\pi]} \left(\frac{N}{k} \int\limits_{\frac{k-1}{N}}^{\frac{k}{N}} |f(x_1 + t_1, x_2)| + |f(x_1 - t_1, x_2)|\, dt_1\right)^q\right)^{1/q},$$

$$G_N^{(2)}(x_1, x_2; f) := \left(\sum_{k=1}^{[N\pi]} \left(\frac{N}{k} \int\limits_{\frac{k-1}{N}}^{\frac{k}{N}} |f(x_1, x_2 + t_2)| + |f(x_1, x_2 - t_2)|\, dt_2\right)^q\right)^{1/q},$$

$$G^{(s)}(x_1, x_2; f) := \sup_{N\in\mathbb{N}} G_N^{(s)}(x_1, x_2; f),\ s = 1, 2,$$

$$M_1(x_1, x_2; f) := \sup_{I\ x_1} \frac{1}{|I|} \int\limits_I |f(s, x_2)|\, ds,$$

$$M_2(x_1, x_2; f) := \sup_{I\ x_2} \frac{1}{|I|} \int\limits_I |f(x, t)|\, dt.$$

*Lemma* 3 — Let $f \in L\left(\log^+ L\right)^2\left(\mathbb{T}^2\right)$. Then

$$\left|\left\{(x_1, x_2) \in \mathbb{T}^2 : \sup_{p>1} \sup_{n,l\in\mathbb{N}} \frac{\left(\frac{1}{2^n} \sum\limits_{k=0}^{2^n-1} |S_{kl}(x_1, x_2; f)|^p\right)^{1/p}}{p} > \lambda\right\}\right|$$

$$\lesssim \int\limits_{\mathbb{T}^2} |f| \left(\log^+ |f|\right)^2 + 1.$$

PROOF : Set

$$f_{nl}(x_1,x_2;t) := \sum_{k=0}^{2^n-1} S_{kl}(x_1,x_2;f)\,\mathbb{I}_{\delta_k^n}(t).$$

Using Lemma 2, we get

$$\begin{aligned} & \|f_{nl}(x_1,x_2;\cdot)\|_{BMO} \qquad (7)\\ = & \left\|\sum_{k=0}^{2^n-1} S_k^{(1)}\left(x_1,x_2;S_l^{(2)}(f)\right)\mathbb{I}_{\delta_k^n}(\cdot)\right\|_{BMO} \\ \lesssim & G^{(1)}\left(x_1,x_2;\left|S_l^{(2)}(f)\right|\right) + M_1\left(x_1,x_2;\left|S_l^{(2)}(f)\right|\right) \\ & + \left|S_l^{(2)}(x_1,x_2;f)\right| \\ \lesssim & G^{(1)}\left(x_1,x_2;\left|S_*^{(2)}(f)\right|\right) + M_1\left(x_1,x_2;\left|S_*^{(2)}(f)\right|\right) \\ & + \left|S_*^{(2)}(f)\right|, \end{aligned}$$

where

$$S_*^{(2)}(f) := \sup_{n\in\mathbb{N}}\left|S_n^{(2)}(f)\right|.$$

It is known, that (see [3])

$$\|f\|_p \le cp\,\|f\|_{BMO} \quad (f\in BMO, 1\le p<\infty), \qquad (8)$$

where the constant $c$ does not depend on $p$ and $f$. This implies

$$\begin{aligned} & \|f_{nl}(x_1,x_2;\cdot)\|_p \qquad (9)\\ \le & cp\,\|f_{nl}(x_1,x_2;\cdot)\|_{BMO} \\ \lesssim & G^{(1)}\left(x_1,x_2;\left|S_*^{(2)}(f)\right|\right) \\ & + M_1\left(x_1,x_2;\left|S_*^{(2)}(f)\right|\right) + \left|S_*^{(2)}(f)\right|. \end{aligned}$$

On the other hand,

$$\|f_{nl}(x_1,x_2;\cdot)\|_p = \left(\frac{1}{2^n}\sum_{k=0}^{2^n-1}|S_{kl}(x_1,x_2;f)|^p\right)^{1/p} \qquad (10)$$

Consequently, from (7)-(10) we have

$$\begin{aligned} & \left(\frac{1}{2^n}\sum_{k=0}^{2^n-1}\left|S_{kl}\left(x_1,x_2;f\right)\right|^p\right)^{1/p} \\ \leq \quad & cp\left\{G^{(1)}\left(x_1,x_2;\left|S_*^{(2)}\left(f\right)\right|\right)\right. \\ & \left.+M_1\left(x_1,x_2;\left|S_*^{(2)}\left(f\right)\right|\right)+\left|S_*^{(2)}\left(f\right)\right|\right\}. \end{aligned} \tag{11}$$

Let $f\in L\left(\log^+L\right)^2\left(\mathbb{T}^2\right)$. Then $f\left(x_1,\cdot\right)\in L\left(\log^+L\right)^2\left(\mathbb{T}\right)$ for a. e. $x_1\in\mathbb{T}$ and from the theorem of Hunt $S_*^{(2)}\left(x_1,\cdot;f\right)\in L_1\left(\mathbb{T}\right)$ for a. e. $x_1\in\mathbb{T}$. Moreover,

$$\int\limits_{\mathbb{T}}S_*^{(2)}\left(x_1,x_2;f\right)dx_2\lesssim 1+\int\limits_{\mathbb{T}}\left|f\left(x_1,x_2\right)\right|\left(\log^+\left|f\left(x_1,x_2\right)\right|\right)^2dx_2$$

for a. e. $x_1\in\mathbb{T}$.

Set

$$\Omega:=\left\{\left(x_1,x_2\right)\in\mathbb{T}^2:G^{(1)}\left(x_1,x_2;\left|S_*^{(2)}\left(f\right)\right|\right)>\lambda\right\}.$$

Using Fubini's Theorem, Lemma 1 and Theorem of Hunt we can write

$$|\Omega|=\int\limits_{\mathbb{T}^2}\mathbb{I}_\Omega\left(x_1,x_2\right)dx_1dx_2=\int\limits_{\mathbb{T}}\left(\int\limits_{\mathbb{T}}\mathbb{I}_\Omega\left(x_1,x_2\right)dx_1\right)dx_2 \tag{12}$$

$$\lesssim\frac{1}{\lambda}\int\limits_{\mathbb{T}}\left(\int\limits_{\mathbb{T}}S_*^{(2)}\left(x_1,x_2;f\right)dx_1\right)dx_2$$

$$=\frac{c}{\lambda}\int\limits_{\mathbb{T}}\left(\int\limits_{\mathbb{T}}S_*^{(2)}\left(x_1,x_2;f\right)dx_2\right)dx_1$$

$$\lesssim\frac{1}{\lambda}\int\limits_{\mathbb{T}}\left(1+\int\limits_{\mathbb{T}}\left|f\left(x_1,x_2\right)\right|\left(\log^+\left|f\left(x_1,x_2\right)\right|\right)^2dx_2\right)dx_1$$

$$\lesssim\frac{1}{\lambda}\left(1+\int\limits_{\mathbb{T}^2}\left|f\left(x_1,x_2\right)\right|\left(\log^+\left|f\left(x_1,x_2\right)\right|\right)^2dx_1dx_2\right).$$

Since

$$\lambda\left|\left\{\left(x_1,x_2\right)\in\mathbb{T}^2:M_s\left(x_1,x_2;f\right)>\lambda\right\}\right|\lesssim\|f\|_1,s=1,2.$$

analogously, we can prove that

$$\left|\left\{(x_1,x_2)\in\mathbb{T}^2 : M_1\left(x_1,x_2;\left|S_*^{(2)}(f)\right|\right)>\lambda\right\}\right| \tag{13}$$
$$\lesssim \frac{1}{\lambda}\left(1+\int_{\mathbb{T}^2}|f(x_1,x_2)|\left(\log^+|f(x_1,x_2)|\right)^2 dx_1dx_2\right).$$

Combining (11)-(13) we complete the proof of Lemma 3. □

## 4. Proof of Theorem 1

Let $X$ be either $[0,1]$ or $\mathbb{T}^2$, and let $L_M := L_M(X)$ be the Orlicz space of functions on $X$ generated by Young function $M$ i. e. $M$ is a convex continuous even function such that $M(0)=0$ and

$$\lim_{t\to 0+}\frac{M(t)}{t}=\lim_{t\to\infty}\frac{t}{M(t)}=0.$$

It is well known that $L_M$ is a Banach space with respect to the Luxemburg norm

$$\|f\|_M := \inf\left\{\lambda : \lambda>0, \int_X M\left(\frac{|f|}{\lambda}\right)\le 1\right\}<\infty.$$

We will deal with two $M-$functions

$$\Phi(t) := t\left(\log^+ t\right)^2$$

and

$$\Psi(t) := \exp(t)-1.$$

Proof : According to a theorem from [14, Chap. 2, Theorem 9.5] we have

$$\|f\|_M\le 1 \Rightarrow \int_{\mathbb{T}^2} M(|f|)\le \|f\|_M . \tag{14}$$

From this fact we may deduce that

$$0,5\left(1+\int_{\mathbb{T}^2}M(|f|)\right)\le\|f\|_M\le 1+\int_{\mathbb{T}^2}M(|f|)$$

provided $\|f\|_M=1$.

Hence, from Lemma 3 we can write

$$\left|\left\{(x_1,x_2)\in\mathbb{T}^2:\sup_{p>1}\sup_{n,l\in\mathbb{N}}\frac{\left(\frac{1}{2^n}\sum\limits_{k=0}^{2^n-1}\left|S_{kl}(x_1,x_2;f)\right|^p\right)^{1/p}}{p}>\lambda\right\}\right|\lesssim\frac{\|f\|_\Phi}{\lambda}\tag{15}$$

where $\Phi(t)=t\left(\log^+t\right)^2$. Indeed, at first we deduce the case when $\|f\|_\Phi=1$, then using a linearity principle, we get the inequality in the general case.

It is easy to see that

$$\|f\|_\Psi\simeq\sup_{p>1}\frac{\|f\|_p}{p}.$$

Then from (10) we can write

$$\sup_{p>1}\frac{\left(\frac{1}{2^n}\sum\limits_{k=0}^{2^n-1}\left|S_{k,l}(x_1,x_2;f)\right|^p\right)^{1/p}}{p}$$
$$=\sup_{p>1}\frac{\|f_{nl}(x_1,x_2;\cdot)\|_p}{p}\simeq\|f_{nl}(x_1,x_2;\cdot)\|_\Psi.$$

Consequently, from (15) we have

$$\left|\left\{(x_1,x_2)\in\mathbb{T}^2:\sup_{n,l\in\mathbb{N}}\|f_{nl}(x_1,x_2;\cdot)\|_\Psi>\lambda\right\}\right|\lesssim\frac{\|f\|_\Phi}{\lambda}.\tag{16}$$

We can write

$$\frac{1}{2^n}\sum_{k=0}^{2^n-1}\left(\exp\left(A\left|S_{k,l}(x_1,x_2;f)-f(x_1,x_2)\right|\right)-1\right)$$
$$=\frac{1}{2^n}\sum_{k=0}^{2^n-1}\Psi\left(A\left|S_{k,l}(x_1,x_2;f)-f(x_1,x_2)\right|\right)$$
$$=\int_{\mathbb{T}^2}\Psi\left(A\left|\sum_{k=0}^{2^n-1}\left(S_{k,l}(x_1,x_2;f)-f(x_1,x_2)\right)\mathbb{I}_{\delta_k^n}(t)\right|\right)dt.$$

From (14) it follows that for any sequence of functionf $f_{nl}$, the condition $\|f_{nl}\|_\Psi\to0$ implies that $\int_{\mathbb{T}^2}\Psi(|f_{nl}|)\to0$. Thus, to prove the Theorem 1 it is sufficient to prove that

$$\left\|\sum_{k=0}^{2^n-1}\left(S_{k,l}(x_1,x_2;f)-f(x_1,x_2)\right)\mathbb{I}_{\delta_k^n}(\cdot)\right\|_\Psi\to0\tag{17}$$

a.e. $(x_1, x_2) \in \mathbb{T}^2$ as $n, l \to \infty$ for any $f \in L\left(\log^+ L\right)^2\left(\mathbb{T}^2\right)$. It is easy to observe that (17) holds if $f$ is trigonometric polynomial in two variable. Indeed, if $T_{s_1,s_2}(x_1, x_2)$ trigonometric polynomial, then we have

$$S_{i,j}\left(x_1, x_2; T_{2^{s_1},2^{s_2}}\right) - T_{2^{s_1},2^{s_2}}\left(x_1, x_2\right) = 0$$

when $i \geq 2^{s_1}, j \geq 2^{s_2}$. Therefore, if $n \geq s_1$ and $l \geq 2^{s_2}$, then we get

$$\left|\sum_{k=0}^{2^n-1}\left(S_{k,l}\left(x_1, x_2; T_{s_1,s_2}\right) - T_{2^{s_1},2^{s_2}}\left(x_1, x_2\right)\right)\mathbb{I}_{\delta_k^n}(t)\right| \tag{18}$$

$$\leq c\mathbb{I}_{\left[0,\frac{2^{s_1}}{2^n}\right.}(t),$$

where $c$ is a constant depending on $T_{2^{s_1},2^{s_2}}$.

From the definition of norm $\|\cdot\|_M$, it immediately follows that $|f| \leq |g|$ implies $\|f\|_M \leq \|g\|_M$. In addition, for any measurable set $E$, we have (see [14], (9.23)) $\|\mathbb{I}_E\|_M = 0(1)$ as $|E| \to 0$. So, from (18) we conclude that (17) holds if $f = T$. To prove general case, we consider the set

$$G_\lambda := \left\{(x_1, x_2) \in \mathbb{T}^2 : \overline{\lim_{n,l\to\infty}}\left\|\sum_{k=0}^{2^n-1} S_{k,l}\left(x_1, x_2; f\right)\right.\right.$$

$$\left.\left. - f\left(x_1, x_2\right)\right)\mathbb{I}_{\delta_i^n}(\cdot)\right\|_\Psi > \lambda\right\}.$$

To complete the proof of the theorem, it is sufficient to prove that $|G_\lambda| = 0$ if $\lambda > 0$. Since $\Phi(t) = t\left(\log^+ t\right)^2$ satisfies the $\Delta_2$ -condition, according property of Orlicz space (see [14]), we may choose a trigonometric polynomial $T$ such that

$$\|f - T\|_\Phi < \varepsilon.$$

Using the definition of norm, we get

$$\int_{\mathbb{T}^2} \Phi\left(\left|\frac{f-T}{\varepsilon}\right|\right) < 1$$

and consequently,

$$\left|\left\{(x_1, x_2) \in \mathbb{T}^2 : \left|f\left(x_1, x_2\right) - T\left(x_1, x_2\right)\right| > \lambda\right\}\right| \Phi\left(\frac{\lambda}{\varepsilon}\right)$$

$$\leq \int_{\{|f-T|>\lambda\}} \Phi\left(\left|\frac{f-T}{\varepsilon}\right|\right) \leq \int_{\mathbb{T}^2} \Phi\left(\left|\frac{f-T}{\varepsilon}\right|\right) < 1,$$

$$\left|\left\{(x_1,x_2)\in\mathbb{T}^2:|f(x_1,x_2)-T(x_1,x_2)|>\lambda\right\}\right|<\frac{1}{\Phi\left(\frac{\lambda}{\varepsilon}\right)}.$$

Thus, using (16) we get

$$\begin{aligned}G_\lambda=&\left\{(x_1,x_2)\in\mathbb{T}^2:\overline{\lim_{n,l\to\infty}}\left\|\sum_{k=0}^{2^n-1}\left(S_{k,l}(x_1,x_2;f-T)\right.\right.\right.\\&\left.\left.\left.+T(x_1,x_2)-f(x_1,x_2)\right)\mathbb{I}_{\delta_k^n}(\cdot)\right\|_\Psi>\lambda\right\}\\&\leq\left\{(x_1,x_2)\in\mathbb{T}^2:\sup_{n,l\in\mathbb{N}}\left\|\sum_{k=0}^{2^n-1}S_{k,l}(x_1,x_2;f-T)\right.\right.\\&\left.\left.\times\mathbb{I}_{\delta_k^n}(\cdot)\right\|_\Psi+|T(x_1,x_2)-f(x_1,x_2)|>\lambda\right\}\\&\lesssim\frac{\|f-T\|_\Phi}{\lambda}+\frac{1}{\Phi\left(\frac{\lambda}{\varepsilon}\right)}\lesssim\frac{\varepsilon}{\lambda}+\frac{1}{\Phi\left(\frac{\lambda}{\varepsilon}\right)}.\end{aligned}$$

Since $\varepsilon>0$ may be taken sufficienly small, we conclude that $|G_\lambda|=0$ if $\lambda>0$.

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