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EXACT SOLUTIONS OF GENERALIZED RIEMANN PROBLEM FOR NONHOMOGENEOUS SHALLOW WATER EQUATIONS

Sueet Millon Sahoo, T. Raja Sekhar and G. P. Raja Sekhar

Department of Mathematics, Indian Institute of Technology Kharagpur,
Kharagpur 721302, *India*
e-mails: trajasekhar@maths.iitkgp.ac.in; rajas@iitkgp.ac.in

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In this paper, we consider quasilinear hyperbolic system of balance laws describing one-dimensional nonhomogeneous shallow water equations with generalized Riemann initial data. We obtain exact solutions to the shallow water equations with friction by using differential constraint method. A special case of the obtained solution provides well known rarefaction wave to the homogeneous case of the governing equations. We construct a convenient example for the generalized Riemann problem and study the behavior of the solution profiles.

Key words : Generalized Riemann problem; exact solutions; differential constraint method; nonhomogeneous shallow water equations.

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1. INTRODUCTION

System of quasilinear partial differential equations are generally used to describe various physical phenomena in fields like marine engineering, fluid dynamics, plasma physics, chemistry, biology, physics and many other application areas. Exact solutions of such system of equations are important to understand of nonlinear phenomena in various fields of science, especially in physics. Many powerful and systematic methods have been proposed during the last few decades, by several mathematicians such as symmetry analysis, perturbed method, vanishing viscosity method and so on. Apart from these methods, differential constraint method is one of the most powerful method which gives a systematic procedure to find a class of exact solutions for first order hyperbolic system of balance laws. Janenko [1] pioneered the idea of differential constraint method and a survey of the same can be found in the book by Sirdov *et al.* [2].

The procedure of differential constraint method involves two steps. The first step is to find a compatible system of equations corresponding to the given system via differential constraints. The second step is to construct solutions of this compatible system which is overdetermined. Because the solution has to satisfy the differential constraints, it makes easier to construct particular solutions of governing system [3]. Here our main attention is to determine simple wave solutions to generalized Riemann problem (GRP) for hyperbolic system of first order PDEs involving source-like terms. The GRP is the special case of Cauchy problem for quasilinear hyperbolic system of balance laws in one space dimension which is having single discontinuity. The initial data consists of piecewise smooth functions on both sides of the discontinuity. While classical Riemann problems can be solved exactly for many relevant cases (see [4-10]), but GRPs are much more complicated. In case of quasilinear system of first order differential equations, analytical expressions for the solution of GRPs are usually not available. The key challenge is to determine an exact solution to the GRP as well as rarefaction-like wave solutions connecting the resulting left and right non-constant states. Fusco and Manganaro [11] introduced systematic reduction approach to obtain particular classes of solutions to quasilinear hyperbolic systems. Many mathematicians used this differential constraint method to find exact solution of quasilinear systems depending on various applications like fluid flow of second grade fluid (Zhang *et al.* [12]), fast diffusion equation (Majid [13]), p-system with relaxation condition (Curro and Manganaro [14]), nonlinear diffusion equation (Oleg and Verevkin [15]), nonlinear diffusion-convention equation (Edwards and Broadbridge [16]) and GRP for traffic flow (Curro and Fusco [17, 18]).

Inspired by these facts, we wish to find exact solution of GRP for the qasilinear one dimensional shallow-water equations or Saint-Venent equations with friction. This model is based on the depth averaged thin layer approximation of granular flows over sloping beds and takes into account a Columb type friction law with a constant friction coefficient. Analytical and numerical modeling of granular flows is a key issue for industrial and geophysical applications. Karelsky and Petrosyan [19] obtained particular solutions and discussed Riemann problem for shallow water equations. Fu and Sharma [20] investigated global smooth solutions with Cauchy initial data for both classical and modified shallow water equations. An approximate analytical method for determining the evolutionary behavior of a bore of arbitrary strength as it approaches the shoreline on a sloping beach have been described by Radha *et al.* [21]. Faccanoni and Mangeney [22] derived exact solution to granular flow for Riemann problem. Recently, many researchers have shown their interest to find numerical solutions to GRP for shallow water equations for various physical situations by using different numerical methods. Katsaounis [23] discussed transport and diffusion of a pollutant in shallow water flows with and without source terms based on finite difference relaxation schemes. Shock capturing solutions to GRP for shallow water equations with the help of adaptive, unstructured, moving, triangular meshes are obtained by Zhou *et al.* [24]. Plantie [25] solved the GRP for shallow water equations with a constraint on the height of the flow by using finite volume method.

To the best of our knowledge, so far no one has attempted to find exact solutions to GRP for shallow water equations. Exploiting the differential constraint method we have obtained an exact solution for the one dimensional shallow-water equations over an inclined plane for all possible cases of GRP. The paper is structured as follows: in section 2 we introduce the basic equations. We derive differential constraint equations and consistency conditions for the basic equations in section 3. In section 4 we construct an exact solution to the GRP for the original system and we study the behavior of the solution by considering an example of generalized Riemann initial data. Finally, conclusions are given in section 5.

2. GOVERNING EQUATIONS

Originally, the Saint-Venant equations were derived to model flood propagation on shallow slopes and smooth topography [26], but modern formulations have demonstrated that the equations can be recast to apply rigorously to steep slopes and irregular topography [27]. Nonlinear, one-dimensional shallow-water (Saint-Venant) equations provide a suitable approximation for modeling water surges over a wide, uniformly sloping bed inclined at an angle ϕ with respect to the horizontal direction (Figure 1). Correspondingly, we consider the free surface flow of a granular material along a slowly varying bottom profile. We assume that horizontal length scale is much more greater than the vertical scale, i.e., $h << L$ where h and L are the thickness and length along the slope. It is assumed that x-axis is parallel to the uniform slope with an angle ϕ to the horizontal (see Figure 1). The corresponding governing equations in one-dimension are given by (see [28, 29]).

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0, \\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}\left(hu^2 + g_z\dfrac{h^2}{2}\right) = mh, \quad t > 0, x \in \mathbb{R}. \end{cases} \tag{1}$$

where x is the downstream coordinate and t is the time. The unknowns are $h \equiv h(x,t) > 0$, the flow depth measured perpendicular to the plane, and $u \equiv u(x,t)$, the depth-averaged flow velocity and $g_z \equiv g\cos\phi$, projection of the gravitational acceleration on the vertical axis (perpendicular to the plane). We assume $\beta = g_z$. The other terms are $m \equiv g_x + F = \beta(\tan\phi - \tan\delta)$, the constant

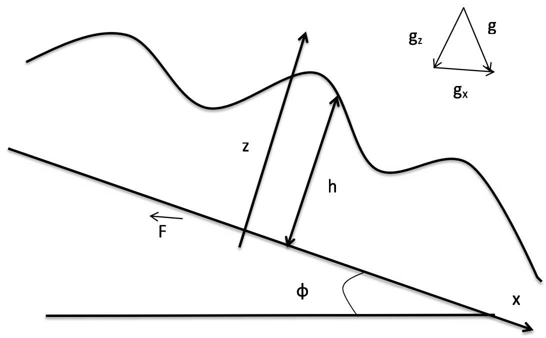

Figure 1: Geometry of the granular flow and reference frame.

x-acceleration resulting from the sum of the forces due to gravity and friction. The friction force is well-known Coulomb-type friction law proposed by Savage and Hutter [15] to describe granular flow behavior, $F = -\beta \tan \delta \leq 0$, where δ is the basal friction angle. Here, we assume that $\tan \phi \geq \tan \delta$, for more inclination and less friction, so that, an initially flowing, granular mass never stops on the plane. If $m \geq 0$ and $u \geq 0$, the system can represent, for example, a granular flow over an inclined plane covered or not by a layer of material made of the same material as the flowing granular mass.

It may be noted that the usual analytical techniques or numerical techniques to solve the non-homogeneous shallow water equations of the form (1) may not lead to non-smooth solutions. Hence, one has to look for alternate methods to obtain such non-smooth solutions, which are indeed all together in a different class. The generalized Riemann data leads to such solutions via GRP. We solve GRP corresponding to system (1).

3. Constraint Equations

In this section we obtain an exact solution to the system (1) by using constraint method [11] for arbitrary initial data. In order to do characteristic analysis, it is convenient to write (1) in quasilinear form as

$$\mathbb{U}_t + A(\mathbb{U})\mathbb{U}_x = B(\mathbb{U}),$$

where

$$\mathbb{U} = \begin{bmatrix} h \\ u \end{bmatrix}, \quad A(\mathbb{U}) = \begin{bmatrix} u & h \\ \beta & u \end{bmatrix} \text{ and } B(\mathbb{U}) = \begin{bmatrix} 0 \\ m \end{bmatrix}. \tag{2}$$

The corresponding eigenvalues of the matrix $A(\mathbb{U})$ are

$$\lambda(\mathbb{U}) = u \pm \sqrt{\beta h}, \tag{3}$$

while the corresponding left and right eigenvectors, are respectively

$$l^1 = \begin{bmatrix} 1 & -\sqrt{\dfrac{h}{\beta}} \end{bmatrix}, \quad l^2 = \begin{bmatrix} 1 & \sqrt{\dfrac{h}{\beta}} \end{bmatrix} \quad \text{and} \quad d^1 = \begin{bmatrix} \dfrac{1}{2} \\ -\dfrac{1}{2}\sqrt{\dfrac{\beta}{h}} \end{bmatrix}, \quad d^2 = \begin{bmatrix} \dfrac{1}{2} \\ \dfrac{1}{2}\sqrt{\dfrac{\beta}{h}} \end{bmatrix}. \tag{4}$$

The differential constraints associated with system (1) are given by (for more details we refer [11, 14])

$$l^i.\mathbb{U}_x = q^i(x, t, \mathbb{U}),$$

where $l^i, i = 1, 2$ are left eigenvectors of $A(\mathbb{U})$, $\mathbb{U}$ is vector of basic variables and q^i is an arbitrary function. For $i = 1$,

$$l^1.U_x = q^1 \Rightarrow h_x - \sqrt{\frac{h}{\beta}} u_x = q^1,$$

and for $i = 2$

$$l^2.U_x = q^2 \Rightarrow h_x + \sqrt{\frac{h}{\beta}} u_x = q^2.$$

Let $q^1 = q^2 = q(x, t, h, u)$. From the above two equations, we have

$$h_x + \phi(h) u_x = q(x, t, h, u) \quad \text{where} \quad \phi(h) = \alpha\sqrt{\frac{h}{\beta}}, \alpha = \pm 1, \tag{5}$$

where $q(x, t, h, u)$ is an arbitrary function. The required differential compatibility between equation (1) and (5) leads to the following pair of consistency conditions

$$\begin{aligned} q_t + (u - \phi(h)\beta)\, q_x + (m - \beta q)\, \phi(h) q_h - \beta q q_u &= 0, \\ \beta\phi(h)^2 q_h + \beta\phi(h) q_u - \frac{3q}{4} &= 0. \end{aligned} \tag{6}$$

Once q is determined from equation (6), the corresponding solution of (1) and (5) is obtained by integrating the following system of partial differential equations

$$\begin{cases} h_t + (u + \beta\phi(h))\, h_x = q\beta\phi(h), \\ u_t + (u + \beta\phi(h))\, u_x = m - q\beta. \end{cases} \tag{7}$$

It may be noted that solvability of (6) is not easy and hence at least one solution of (6) would trigger the corresponding exact solution of (7). We realize that the only solution of the pair of equations given in (6) is $q = 0$. This eases us to integrate (7) with the initial condition $h(x, 0) = h_0(x)$

and $u(x,0) = u_0(x)$, where $h_0(x)$ and $u_0(x)$ are arbitrary functions. Correspondingly, we get the following solution of (7),

$$\begin{aligned} h(x,t) &= h_0(z), \\ u(x,t) &= mt + u_0(z), \end{aligned} \tag{8}$$

where $h_0(z)$ and $u_0(z)$ are supported by the following implicit relations

$$z = x - \left(\frac{mt^2}{2} + \left(u_0(z) - \alpha\sqrt{\beta h_0(z)}\right) t\right), \quad \alpha = \pm 1, \tag{9}$$

are the corresponding characteristic equations. Our solution may not be smooth or it may blow up to infinity for some finite value of t and for any finite values of x. From (9), we can obtain such a value of t, which is known as critical time t_c. A standard analysis leads to the following expression for the critical time

$$t_c = \inf_z \left[\frac{-2\sqrt{h_0(z)}}{2u_0'(z)\sqrt{\beta h_0(z)} - \alpha\beta h_0'(z)}\right]. \tag{10}$$

In view of (9), a critical time t_c exists if

$$2u_0'(z)\sqrt{\beta h_0(z)} - \alpha\beta h_0'(z) \neq 0. \tag{11}$$

In order to avoid the existence of t_c, we require that the initial data $h_0(x)$ and $u_0(x)$ by pass satisfy the condition (11). Further substitution of (8) and (9) into (5) leads to the fully constraint equation in terms of the initial data given by

$$\frac{dh_0(z)}{dz} + \phi(h_0(z))\frac{du_0(z)}{dz} = 0. \tag{12}$$

Next we are interested in introducing the initial conditions known as generalized Riemann data and we shall calculate the exact solution to the given governing equations (1) subject to this generalized data.

4. Generalized Riemann Problem

Consider the following generalized Riemann problem for the governing system (1) with the initial data

$$\mathbb{U}(x,0) = \begin{cases} (h_l(x), u_l(x)), & \text{if} \quad x < 0, \\ (h_r(x), u_r(x)), & \text{if} \quad x > 0, \end{cases} \tag{13}$$

where $h_l(x), u_l(x), h_r(x)$ and $u_r(x)$ are arbitrary functions. We set

$$\begin{aligned}(h_L, u_L) &= \lim_{x\to 0^-}(h_l(x), u_l(x))\\ (h_R, u_R) &= \lim_{x\to 0^+}(h_r(x), u_r(x)),\end{aligned} \tag{14}$$

such that $h_L \neq h_R$ and $u_L \neq u_R$. According to the method of differential constraints, the initial data (13) must satisfy the constraint equation (12). The solution of (12) with the initial value defined in (13) and (14) leads to

$$\begin{cases} u_l(x) = u_L + \dfrac{2\sqrt{\beta}}{\alpha}\left(\sqrt{h_L} - \sqrt{h_l(x)}\right), \\ u_r(x) = u_R + \dfrac{2\sqrt{\beta}}{\alpha}\left(\sqrt{h_R} - \sqrt{h_r(x)}\right). \end{cases} \tag{15}$$

The next step is to rewrite (8) and (9) in order to solve (1) along with (13). In the case $x < 0$, we get

$$\begin{aligned} h(x,t) &= h_l(z),\\ u(x,t) &= mt + u_l(z), \end{aligned} \tag{16}$$

where

$$z = x - \left(\frac{mt^2}{2} + \left(u_l(z) - \alpha\sqrt{\beta h_l(z)}\right)t\right), \quad \alpha = \pm 1, \tag{17}$$

while for $x > 0$, we obtain

$$\begin{aligned} h(x,t) &= h_r(z),\\ u(x,t) &= mt + u_r(z), \end{aligned} \tag{18}$$

where

$$z = x - \left(\frac{mt^2}{2} + \left(u_r(z) - \alpha\sqrt{\beta h_r(z)}\right)t\right), \alpha = \pm 1, \tag{19}$$

where $u_l(x)$ and $u_r(x)$ are given by (15). We normalize the solution at hand in order that when $t = 0$ we have $z = x$. Therefore, the solution (16) and (17) exists for $z < 0$ such that $x < x_l(t)$ where

$$\begin{aligned} x_l(t) &= \lim_{z\to 0^-}\left(\frac{mt^2}{2} + \left(u_l(z) - \alpha\sqrt{\beta h_l(z)}\right)t\right)\\ &= \frac{mt^2}{2} + \left(u_L - \alpha\sqrt{\beta h_L}\right)t. \end{aligned} \tag{20}$$

Similarly, the solution (18) and (19) exist for $x > x_r(t)$ where

$$\begin{aligned} x_r(t) &= \lim_{z\to 0^+}\left(\frac{mt^2}{2} + \left(u_r(z) - \alpha\sqrt{\beta h_r(z)}\right)t\right)\\ &= \frac{mt^2}{2} + \left(u_R - \alpha\sqrt{\beta h_R}\right)t. \end{aligned} \tag{21}$$

It may be noted that in the given generalized Riemann initial data, there is no information for $h(x,t)$ and $u(x,t)$ at $t=0$ and $x=0$. Correspondingly, we define the following initial data for $t=0$ and $x=0$, in terms of a parameter, say a, as

$$\begin{aligned} x(0)=0, u=U(a), h=H(a) \quad \text{with} \quad a\in[0,1], \\ U(0)=u_L, U(1)=u_R, H(0)=h_L, H(1)=h_R, \end{aligned} \tag{22}$$

where H and U are arbitrary functions of a which are to be determined. In order to connect smoothly the left state (16)-(17) to the right state (18)-(19), we integrate the set of equations (7) with the initial data (22). The resulting solution is

$$\begin{aligned} h &= H(a), \\ u &= mt + U(a), \\ x &= \frac{mt^2}{2} + \left(U(a) - \alpha\sqrt{\beta H(a)}\right)t. \end{aligned} \tag{23}$$

On using the above solution (23) in the constraint equation (5), we have

$$H^{'}(a) + \phi(H)U^{'}(a) = 0. \tag{24}$$

By integrating equation (24), taking into account (22) and (23), we get

$$U(a) = u_L + \frac{2\sqrt{\beta}}{\alpha}\left(\sqrt{h_L} - \sqrt{H(a)}\right), \tag{25}$$

along with the condition

$$u_L + \frac{2\sqrt{\beta}}{\alpha}\sqrt{h_L} = u_R + \frac{2\sqrt{\beta}}{\alpha}\sqrt{h_R}. \tag{26}$$

From the third equation of (23), we obtain the value of $H(a)$ as

$$H(a) = \frac{1}{9\beta}\left(\frac{x}{t} - \frac{mt}{2} - u_L - \frac{2}{\alpha}\sqrt{\beta h_L}\right)^2. \tag{27}$$

Substituting the value of $H(a)$ into the equation (25), we get

$$U(a) = \frac{2}{3}\left(\frac{x}{t} - \frac{mt}{2}\right) + \frac{1}{3}u_L + \frac{2}{3\alpha}\sqrt{\beta h_L}. \tag{28}$$

Finally in order to guarantee the existence of the rarefaction-like wave solution (23), we require $\frac{d\lambda}{da} > 0$. Therefore, the central state (23) exists in the region $x_l(t) \le x \le x_r(t)$ and it connects smoothly the left state (16)-(17) with the right state (18)-(19) provided that the conditions (26) are satisfied. Eliminating $H(a)$, $U(a)$ which is given in (27) and (28) respectively from (23), we obtain the solution

$$\begin{aligned} h(x,t) &= \frac{1}{9\beta}\left(\frac{x}{t} - \frac{mt}{2} - u_L - \frac{2}{\alpha}\sqrt{\beta h_L}\right)^2, \\ u(x,t) &= \frac{2}{3}\left(\frac{x}{t} + mt\right) + \frac{1}{3}u_L + \frac{2}{3\alpha}\sqrt{\beta h_L}, \end{aligned} \tag{29}$$

which generalizes the well known rarefaction-like wave. It should be of certain interest to note that when $m \to 0$, then the nonhomogeneous model (1) reduces to the corresponding homogeneous form, while the left state (16)-(17), right state (18)-(19), central state (23) approach, respectively, to *the left state*

$$\begin{aligned} &h(x,t) = h_l(z) \\ &u(x,t) = u_l(z) \\ &x = -\left(u_l(z) - \alpha\sqrt{\beta h_l(z)}\right)t; \qquad z < 0, \end{aligned} \tag{30}$$

the right state

$$\begin{aligned} &h(x,t) = h_r(z) \\ &u(x,t) = u_r(z) \\ &x = -\left(u_r(z) - \alpha\sqrt{\beta h_r(z)}\right)t; \quad z > 0, \end{aligned} \tag{31}$$

the central state

$$\begin{aligned} &h = H(a), \\ &u = U(a), \\ &x = \left(U(a) - \alpha\sqrt{\beta H(a)}\right)t; \qquad z = 0 \end{aligned} \tag{32}$$

along with the condition (25). It is simple to ascertain that relations (30), (31), (32) characterize a rarefaction wave solution of the generalized Riemann data (13), (15) prescribed to the homogeneous system associated with (1). In particular , if we assume that the initial states h_l and h_r are constant, then equations (30), (31), (32) represent the classical rarefaction wave solution of Riemann problem for the homogeneous model obtained from the nonhomogeneous model (1) when $m = 0$. Summarize as what is achieved so far.

Next we construct an example which satisfies the conditions of generalized Riemann data (13) such that $h_L \neq h_R$ and $u_L \neq u_R$. It also satisfies the constraint conditions (15) and the condition for rarefaction-like wave solution (26).

4.1 *Example*

Here we consider a particular example such that

$$\mathbb{U}(x,0) = \begin{cases} (h_l(x), u_l(x)), & \text{if} \quad x < 0 \\ (h_r(x), u_r(x)), & \text{if} \quad x > 0 \end{cases} = \begin{cases} \left(\dfrac{(x^3+2)^2}{4\beta}, x^3+2\right), & \text{if} \quad x < 0, \\ \left(\dfrac{(x^3+3)^2}{4\beta}, x^3+3\right), & \text{if} \quad x > 0. \end{cases} \tag{33}$$

From the above equation (33), we have $u_L = 2 \neq 3 = u_R$ and $h_L = \dfrac{1}{\beta} \neq \dfrac{9}{4\beta} = h_R$. This initial data satisfies the new constraint equations (12), (15) and also the condition (26) for $\alpha = -1$. Further, the critical time t_c does not exist since this quantity is negative for this initial data considered in (33). Then the corresponding solution for the system (1) is given as follows;

The left hand non-constant solution for $x < x_l(t) = \dfrac{mt^2}{2} + 3t$ is given here

$$\begin{cases} h(x,t) = \dfrac{(z^3+2)^2}{4\beta}, \\ u(x,t) = mt + z^3 + 2, \\ z - x + \dfrac{mt^2}{2} + \dfrac{3}{2}(z^3+2)t = 0, \quad z < 0. \end{cases} \tag{34}$$

The central state exists for $x_l(t) < x < x_r(t)$ which is given by

$$\begin{cases} h(x,t) = \dfrac{1}{9\beta}\left(\dfrac{x}{t} - \dfrac{mt}{2}\right)^2, \\ u(x,t) = \dfrac{2}{3}\left(\dfrac{x}{t} + mt\right) \end{cases} \tag{35}$$

The right hand non-constant solution exists for $x > x_r(t) = \dfrac{mt^2}{2} + \dfrac{9}{2}t$ which is provided by

$$\begin{cases} h(x,t) = \dfrac{(z^3+3)^2}{4\beta}, \\ u(x,t) = mt + z^3 + 3, \\ z - x + \dfrac{mt^2}{2} + \dfrac{3}{2}(z^3+3)t = 0, \quad z < 0. \end{cases} \tag{36}$$

In Figure 2, we show the variation of $h(x,t)$ and $u(x,t)$ with x at different values of t. We have considered the expressions given in (34), (35) and (36) to obtain these Figures via Mathematica 9. It is observed that the profiles of $h(x,t)$ and $u(x,t)$ are moving from left to right with time which is a property of rarefactionlike-wave solutions. We can also observe that at $t = 0$, the initial profiles of $h(x,t)$ and $u(x,t)$ are discontinuous.

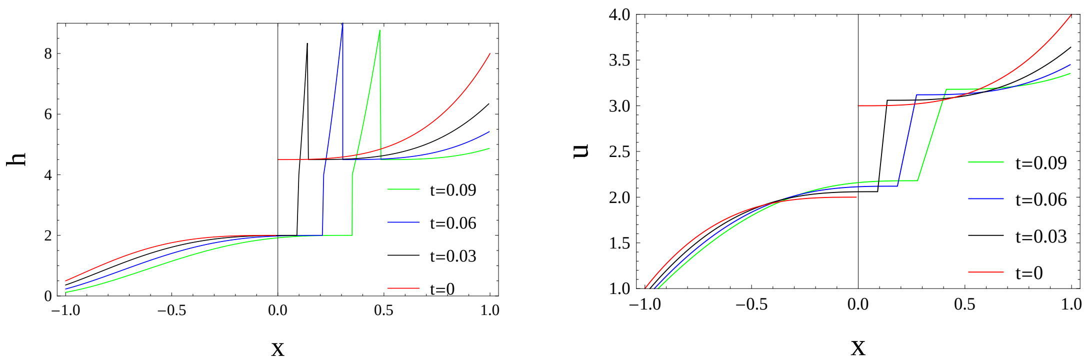

Figure 2: The solution of GRP for (1) and (33) for $t = 0.09, t = 0.06, t = 0.03, t = 0, \beta = 1,\ \alpha = -1, m = 1$.

5. Conclusions

We have obtained an exact solution to nonhomogeneous shallow water equations for GRP by using differential constraint method. The solution obtained consists of a non-constant left state and a non-constant right state that are connected smoothly by the non-constant solution which is a rarefaction-like wave solution. The obtained solution also satisfies the homogeneous shallow water equation when $m \to 0$. Finally, we considered an example of generalized Riemann initial data from which we constructed solution of the GRP for the governing system and discussed the behavior of the solution profiles.

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