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ON POSITIVE INJECTIVE TENSOR PRODUCTS BEING GROTHENDIECK SPACES

Shaoyong Zhang*, Zhaohui Gu** and Yongjin Li***

**Department of Mathematics, Harbin University of Science and Technology,*
Harbin 150080, *P. R. China*

***School of Mathematics and Statistics, Guangdong University of Foreign Studies,*
Guangzhou 510006, *P. R. China*

****Department of Mathematics, Department of Mathematics, Sun Yat-sen University,*
Guangzhou 510275, *P. R. China*

e-mails: 13904603565@139.com; zhgugz@163.com; stslyj@mail.sysu.edu.cn

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Let λ be a reflexive Banach sequence lattice and X be a Banach lattice. In this paper, we show that the positive injective tensor product $\lambda\check{\otimes}_{|\varepsilon|}X$ is a Grothendieck space if and only if X is a Grothendieck space and every positive linear operator from λ^* to X^{**} is compact.

Key words : Banach lattice; injective tensor product; Grothendieck spaces.

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1. INTRODUCTION

A Banach space Z is called *a Grothendieck space* if every separably valued bounded linear operator on Z is weakly compact, or equivalently, if every weak* convergent sequence in Z^* is weakly convergent (see, e.g., [3, 6]). All reflexive Banach spaces are Grothendieck spaces. There are a few examples of non-reflexive Grothendieck spaces. For instance, if K is a stonean space then $C(K)$, in particular, ℓ_∞, is a Grothendieck space (see, e.g., [6]). Recently, Li and Bu [7] characterized the positive projective tensor product $\lambda\hat{\otimes}_{|\pi|}X$ that is a Grothendieck space, where λ is a Banach sequence lattice and X is a Banach lattice. In this paper, we will characterize the positive injective tensor product $\lambda\check{\otimes}_{|\varepsilon|}X$ that is a Grothendieck space.

2. PRELIMINARIES

For a Banach space Z, Z^* will denote its topological dual and B_Z will denote its closed unit ball. For a vector lattice X, the X-valued sequence space $X^{\mathbb{N}}$ is a vector lattice with the following order

$$\bar{x} \geq 0 \Longleftrightarrow x_i \geq 0 \ \ \forall i \in \mathbb{N}, \quad \bar{x} = (x_i)_i \in X^{\mathbb{N}},$$

and with the following lattice operations

$$\bar{x} \wedge \bar{y} = (x_i \wedge y_i)_i, \quad \bar{x} \vee \bar{y} = (x_i \vee y_i)_i, \quad \bar{x} = (x_i)_i,\ \bar{y} = (y_i)_i \in X^{\mathbb{N}}.$$

X^+ will denote the positive cone of X. For each $n \in \mathbb{N}$ and each $\bar{x} = (x_i)_i \in X^{\mathbb{N}}$, let

$$\bar{x}(\leq n) = (x_1, \cdots, x_n, 0, 0, \cdots), \quad \bar{x}(\geq n) = (0, \cdots, 0, x_n, x_{n+1}, \cdots).$$

For Banach lattices X and Y, $\mathcal{L}^r(X, Y)$ will denote the space of regular linear operators from X to Y with the usual regular operator norm $\|\cdot\|_r$, and $\mathcal{K}^r(X, Y)$ will denote the linear span of compact positive operators from X to Y. It follows from [9, section 1.3] that if Y is Dedekind complete then $(\mathcal{L}^r(X, Y), \|\cdot\|_r)$ is a Banach lattice.

Recall that a Banach lattice X is called σ-order continuous if $0 \leq x_n \downarrow 0$ in X then $x_n \to 0$ in X; and is called a σ-Levi space if $0 \leq x_n \uparrow$ and $\sup_n \|x_n\| < \infty$ then $\sup_n x_n$ exists in X.

Banach Lattice-valued Sequence Spaces

Let λ be a solid sequence space, that is, a subspace of $\mathbb{R}^{\mathbb{N}}$ such that if $|a_i| \leq |b_i|$ for all $i \in \mathbb{N}$ and $(b_i)_i \in \lambda$ then $(a_i)_i \in \lambda$. The *Köthe dual* of λ is defined to be

$$\lambda' = \left\{ (b_i)_i \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |a_i b_i| < +\infty, \ \ \forall\ (a_i)_i \in \lambda \right\}.$$

In addition, if λ is a Banach lattice, then $\lambda' \subseteq \lambda^*$. Thus λ' with the norm induced by λ^* is also a Banach lattice and for every $a = (a_i)_i \in \lambda$ and every $b = (b_i)_i \in \lambda'$,

$$\|a\|_\lambda = \sup \left\{ \Big| \sum_{i=1}^{\infty} a_i b_i \Big| : (b_i)_i \in B_{\lambda'} \right\},$$

and

$$\|b\|_{\lambda'} = \|b\|_{\lambda^*} = \sup \left\{ \Big| \sum_{i=1}^{\infty} a_i b_i \Big| : (a_i)_i \in B_{\lambda} \right\}.$$

From now on, throughout this paper, we always assume that λ is a σ-Levi Banach sequence lattice with $\|e_i\|_\lambda = 1$ for all $i \in \mathbb{N}$, where e_i's are standard unit vectors in the sequence space λ. In this case, $\lambda'' = \lambda$ and hence, λ is solid.

For a Banach lattice X, let

$$\lambda_\varepsilon(X) = \left\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : \Big(x^*(|x_i|)\Big)_i \in \lambda, \ \ \forall\, x^* \in X^{*+}\right\}$$

and

$$\Big\|\bar{x}\Big\|_{\lambda_\varepsilon(X)} = \sup\left\{\Big\|\Big(x^*(|x_i|)\Big)_i\Big\|_\lambda : \ x^* \in B_{X^{*+}}\right\}, \qquad \forall\, \bar{x} = (x_i)_i \in \lambda_\varepsilon(X).$$

Then $\lambda_\varepsilon(X)$ is a Banach lattice (see, e.g., [1, 2]). Let $\lambda_{\varepsilon,0}(X)$ denote the closed sublattice of $\lambda_\varepsilon(X)$ consisting of all such elements of $\lambda_\varepsilon(X)$ whose tails converge to 0, that is,

$$\lambda_{\varepsilon,0}(X) = \left\{\bar{x} \in \lambda_\varepsilon(X) : \ \lim_n \|\bar{x}(\geq n)\|_{\lambda_\varepsilon(X)} = 0\right\}.$$

Then $\lambda_{\varepsilon,0}(X)$ is an ideal of $\lambda_\varepsilon(X)$. Let

$$\lambda_\pi(X) = \left\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} x_i^*(|x_i|) < +\infty, \ \forall (x_i^*)_i \in \lambda'_\varepsilon(X^*)^+\right\}$$

and

$$\Big\|\bar{x}\Big\|_{\lambda_\pi(X)} = \sup\left\{\sum_{i=1}^{\infty} x_i^*(|x_i|) : \ (x_i^*)_i \in B_{\lambda'_\varepsilon(X^*)^+}\right\}, \qquad \forall\, \bar{x} = (x_i)_i \in \lambda_\pi(X).$$

Then $\lambda_\pi(X)$ is a Banach lattice (see, e.g., [1, 2]). Let $\lambda_{\pi,0}(X)$ denote the closed sublattice of $\lambda_\pi(X)$ consisting of all such elements of $\lambda_\pi(X)$ whose tails converge to 0, that is,

$$\lambda_{\pi,0}(X) = \left\{\bar{x} \in \lambda_\pi(X) : \ \lim_n \|\bar{x}(\geq n)\|_{\lambda_\pi(X)} = 0\right\}.$$

Then $\lambda_{\pi,0}(X)$ is an ideal of $\lambda_\pi(X)$. By [2], $\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$ and $\lambda_{\pi,0}(X)^* = \lambda'_\varepsilon(X^*)$. We recall the following proposition in [2].

Proposition 1 — (i) $\lambda_{\varepsilon,0}(X)^*$ is lattice isometric to $\lambda'_\pi(X^*)$ and $\lambda_{\pi,0}(X)^*$ is lattice isometric to $\lambda'_\varepsilon(X^*)$. (ii) If λ is σ-order continuous then $\lambda_{\pi,0}(X) = \lambda_\pi(X)$. (iii) If λ is σ-order continuous and X is Dedekind complete then $\lambda_{\varepsilon,0}(X) = \lambda_\varepsilon(X)$ if and only if every positive linear operator from $(\lambda')_0$ to X is compact.

Positive tensor products

For Banach lattices X and Y, let $X \otimes Y$ denote the algebraic tensor product of X and Y. For each $u = \sum_{k=1}^m x_k \otimes y_k \in X \otimes Y$, define $T_u : X^* \longrightarrow Y$ by $T_u(x^*) = \sum_{k=1}^m x^*(x_k) y_k$ for each $x^* \in X^*$. The *injective cone* on $X \otimes Y$ is defined to be

$$C_i = \left\{u \in X \otimes Y : T_u \geq 0\right\},$$

and the *positive injective tensor norm* on $X \otimes Y$ is defined to be

$$\|u\|_{|\varepsilon|} = \|T_u\|_r.$$

Let $X\check{\otimes}_{|\varepsilon|}Y$ denote the completion of $X \otimes Y$ with respect to $\|\cdot\|_{|\varepsilon|}$. Then $X\check{\otimes}_{|\varepsilon|}Y$ with C_i as its positive cone is a Banach lattice (see, e.g., [8, section 3.8]), called the *positive injective tensor product* of X and Y.

The *projective cone* on $X \otimes Y$ is defined to be

$$C_p = \left\{ \sum_{k=1}^{n} x_k \otimes y_k : n \in \mathbb{N}, x_k \in X^+, y_k \in Y^+ \right\},$$

and the *positive projective tensor norm* on $X \otimes Y$ is defined to be

$$\|u\|_{|\pi|} = \sup \left\{ \Big| \sum_{k=1}^{n} \phi(x_k, y_k) \Big| : u = \sum_{k=1}^{n} x_k \otimes y_k \in X \otimes Y, \phi \in M \right\},$$

where M is the set of all positive bilinear functionals ϕ on $X \times Y$ with $\|\phi\| \leq 1$. Let $X\hat{\otimes}_{|\pi|}Y$ denote the completion of $X \otimes Y$ with respect to $\|\cdot\|_{|\pi|}$. Then $X\hat{\otimes}_{|\pi|}Y$ with C_p as its positive cone is a Banach lattice (see, e.g., [4, 5], or [8, Section 3.8]), called the *positive projective tensor product* of X and Y. The positive projective tensor norm $\|\cdot\|_{|\pi|}$ has another equivalent form:

$$\|u\|_{|\pi|} = \inf \left\{ \sum_{k=1}^{n} \|x_k\| \cdot \|y_k\| : x_k \in X^+, y_k \in Y^+, |u| \leq \sum_{k=1}^{n} x_k \otimes y_k \right\}.$$

We recall the following proposition in [1, 2].

Proposition 2 — If λ is σ-order continuous, then $\lambda\hat{\otimes}_{|\pi|}X$ is isometrically isomorphic and lattice homomorphic to $\lambda_{\pi,0}(X)$, and $\lambda\check{\otimes}_{|\varepsilon|}X$ is isometrically isomorphic and lattice homomorphic to $\lambda_{\varepsilon,0}(X)$.

3. Grothendieck Spaces for $\lambda\check{\otimes}_{|\varepsilon|}X$

In this section, first we will characterize weak convergent sequences in $\lambda_{\varepsilon,0}(X)$ and weak* convergent sequences in its dual $\lambda_{\varepsilon,0}(X)^*$.

Let U and V be vector spaces such that (U, V) forms a dual pair with respect to a bilinear functional $\langle x, y\rangle$ defined for every $x \in U$ and every $y \in V$. A subset B of U is called $\sigma(U, V)$-*bounded* if $\sup\{|\langle x, y\rangle| : x \in B\} < \infty$ for every $y \in V$, and a sequence $\{x_n\}_1^\infty$ in U is called $\sigma(U, V)$-*converges to* x in U if $\lim_n \langle x_n, y\rangle = \langle x, y\rangle$ for every $y \in V$. A subspace $S(U)$ of $U^{\mathbb{N}}$

is called an *U-valued sequence space*, and it is called *normal* if for every $(t_i)_i \in \ell_\infty$ and every $(x_i)_i \in S(U)$, $(t_i x_i)_i \in S(U)$. Let $S(U)$ be a U-valued normal sequence space and $T(V)$ be a V-valued normal sequence space such that $\sum_{i=1}^{\infty} |\langle x_i, y_i\rangle| < \infty$ for every $\bar{x} = (x_i)_i \in S(U)$ and every $\bar{y} = (y_i)_i \in T(V)$. Then $(S(U), T(V))$ forms a dual pair with respect to the bilinear functional defined by

$$\langle \bar{x}, \bar{y}\rangle = \sum_{i=1}^{\infty} \langle x_i, y_i\rangle, \quad \forall\, \bar{x} = (x_i)_i \in S(U),\ \forall\, \bar{y} = (y_i)_i \in T(V).$$

Lemma 3 — Let $\bar{x}^{(n)} = (x_i^{(n)})_i, \bar{x}^{(0)} = (x_i^{(0)})_i \in S(U)$. We consider the following conditions.

(i) $\sigma(S(U), T(V))$-$\lim_n \bar{x}^{(n)} = \bar{x}$ in $S(U)$.

(ii) $\{\bar{x}^{(n)}\}_1^\infty$ is a $\sigma(S(U), T(V))$-bounded sequence in $S(U)$ and for every $i \in \mathbb{N}$, $\sigma(U, V)$-$\lim_n x_i^{(n)} = x_i^{(0)}$ in U.

(iii) For every $\bar{y} = (y_i)_i \in T(V)$ and every $\sigma(S(U), T(V))$-bounded subset B of $S(U)$,

$$\lim_m \sup \left\{ \Big| \sum_{i=m}^{\infty} \langle x_i, y_i\rangle \Big| : (x_i)_i \in B \right\} = 0.$$

Then the condition (i) is equivalent to the condition (ii) if and only if the condition (iii) holds.

PROOF : Obviously (i) implies (ii). Now assume (ii) and (iii). For every fixed $\bar{y} = (y_i)_i \in T(V)$, by (iii), one has

$$\lim_m \sup \left\{ \Big| \sum_{i=m}^{\infty} \langle x_i^{(n)}, y_i\rangle \Big| : n = 1, 2, \cdots \right\} = 0.$$

Then for each $\varepsilon > 0$, there exists an $m_0 \in \mathbb{N}$ such that

$$\Big| \sum_{i=m_0+1}^{\infty} \langle x_i^{(n)}, y_i\rangle \Big| \leq \varepsilon/2, \qquad n = 1, 2, \cdots .$$

By (ii), there exists an $N \in \mathbb{N}$ such that for each $n \geq N$, one has

$$\Big| \langle x_i^{(n)} - x_i^{(0)}, y_i\rangle \Big| \leq \varepsilon/2m_0, \qquad i = 1, 2, \cdots, m_0.$$

Thus for each $n \geq N$, one has

$$\Big| \langle \bar{x}^{(n)} - \bar{x}^{(0)}, \bar{y}\rangle \Big| \leq \sum_{i=1}^{m_0} \Big| \langle x_i^{(n)} - x_i^{(0)}, y_i\rangle \Big| + \Big| \langle \sum_{i=m_0+1}^{\infty} x_i^{(n)} - x_i^{(0)}, y_i\rangle \Big| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence (i) follows.

Now assume that (iii) does not hold. Then there exist a $\bar{y} = (y_i)_i \in T(V)$ and a $\sigma(S(U), T(V))$-bounded subset B of $S(U)$ such that

$$\lim_m \sup \left\{ \Big| \sum_{i=m}^{\infty} \langle x_i, y_i \rangle \Big| : (x_i)_i \in B \right\} \neq 0.$$

Thus there exist $\varepsilon_0 > 0$, $\bar{x}^{(k)} = (x_i^{(k)})_i \in B$ for $k \in \mathbb{N}$, and a subsequence $n_1 < n_2 < \cdots$ such that

$$\Big| \sum_{i=n_k}^{\infty} \langle x_i^{(k)}, y_i \rangle \Big| \geq \varepsilon_0, \qquad k = 1, 2, \cdots . \tag{1}$$

For each $k \in \mathbb{N}$, let

$$\bar{z}^{(k)} = (0, \cdots, 0, x_{n_k}^{(k)}, x_{n_{k+1}}^{(k)}, \cdots).$$

Then $\sigma(U, V)\text{-}\lim_k z_i^{(k)} = 0$ for each $i \in \mathbb{N}$. Since $S(U)$ is normal, $\bar{z}^{(k)} \in S(U)$ for each $k \in \mathbb{N}$. Moreover, $\{\bar{z}^{(k)}\}_1^{\infty}$ is $\sigma(S(U), T(V))$-bounded since $T(V)$ is also normal. Thus the sequence $\{\bar{z}^{(k)}\}_1^{\infty}$ satisfies the condition (ii). However, by (1), one has $\langle \bar{z}^{(k)}, \bar{y} \rangle \geq \varepsilon_0$ for each $k \in \mathbb{N}$ and hence, $\sigma(S(X), T(Y))\text{-}\lim_k z^{(k)} \neq 0$. Thus the sequence $\{\bar{z}^{(k)}\}_1^{\infty}$ does not satisfy the condition (i). Therefore, if (i) is equivalent to (ii), then (iii) must hold.

Note that $\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$. So $(\lambda_{\varepsilon,0}(X), \lambda'_\pi(X^*))$ forms a dual pair. Thus by Lemma 3, we have the following two propositions.

Proposition 4 — Let $\bar{x}^{(n)} = (x_i^{(n)})_i, \bar{x}^{(0)} = (x_i^{(0)})_i \in \lambda_{\varepsilon,0}(X)$ for each $n \in \mathbb{N}$. We consider the following conditions.

(i) $\lim_n \bar{x}^{(n)} = \bar{x}^{(0)}$ weakly in $\lambda_{\varepsilon,0}(X)$.

(ii) $\lim_n x_i^{(n)} = x_i^{(0)}$ weakly in X for all $i \in \mathbb{N}$, and $\sup_n \|\bar{x}^{(n)}\|_{\lambda_\varepsilon(X)} < \infty$.

(iii) $\lambda'_\pi(X^*) = \lambda'_{\pi,0}(X^*)$.

Then the condition (i) is equivalent to the condition (ii) if and only if the condition (iii) holds.

Proposition 5 — Let $\bar{x}^{*(n)} = (x_i^{*(n)})_i, \bar{x}^{*(0)} = (x_i^{*(0)})_i \in \lambda_{\varepsilon,0}(X)^* (= \lambda'_\pi(X^*))$ for each $n \in \mathbb{N}$. Then $\lim_n \bar{x}^{*(n)} = \bar{x}^{*(0)}$ weak* in $\lambda_{\varepsilon,0}(X)^*$ if and only if $\lim_n x_i^{*(n)} = x_i^{*(0)}$ weak* in X^* for all $i \in \mathbb{N}$ and $\sup_n \|\bar{x}^{*(n)}\|_{\lambda'_\pi(X^*)} < \infty$.

Now we are ready to obtain characterizations of $\lambda_{\varepsilon,0}(X)$ and $\lambda \check{\otimes}_{|\varepsilon|} X$ being Grothendieck spaces.

***Theorem* 6** — *Assume that λ' is σ-order continuous. Then $\lambda_{\varepsilon,0}(X)$ is a Grothendieck space if and only if X is a Grothendieck space and $\lambda_\varepsilon(X^{**}) = \lambda_{\varepsilon,0}(X^{**})$.*

PROOF : By Proposition 1, $\lambda'_\pi(X^*) = \lambda'_{\pi,0}(X^*)$. Thus

$$\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*), \qquad \lambda'_\pi(X^*)^* = \lambda'_{\pi,0}(X^*)^* = \lambda_\varepsilon(X^{**}).$$

It follows from Proposition 4 and Proposition 5 that if X is a Grothendieck space and $\lambda_\varepsilon(X^{**}) = \lambda_{\varepsilon,0}(X^{**})$, then $\lambda_{\varepsilon,0}(X)$ is a Grothendieck space. On the other hand, suppose that $\lambda_{\varepsilon,0}(X)$ is a Grothendieck space. Obviously X also is a Grothendieck space. Next we want to show that $\lambda_\varepsilon(X^{**}) = \lambda_{\varepsilon,0}(X^{**})$.

Let $\bar{x}^{*(0)}$ be in $\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$ and $\{\bar{x}^{*(n)}\}_1^\infty$ be a bounded sequence in $\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$ such that $\lim_n x_i^{*(n)} = x_i^{*(0)}$ weakly in X^* for each $i \in \mathbb{N}$. By Proposition 5, $\lim_n \bar{x}^{*(n)} = \bar{x}^{*(0)}$ weak* in $\lambda_{\varepsilon,0}(X)^*$ and hence, weakly in $\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$. It follows from Proposition 4 that $\lambda_\varepsilon(X^{**}) = \lambda_{\varepsilon,0}(X^{**})$.

Note that if λ is σ-order continuous then $\lambda' = \lambda^*$, and if both λ and λ' are σ-order continuous then λ is reflexive. Thus by combining Proposition 1, Proposition 2, and Theorem 6, we have the main result as follows.

***Theorem* 7** — *Let λ be a reflexive Banach sequence lattice and X be a Banach lattice. Then $\lambda\check{\otimes}_{|\varepsilon|}X$ is a Grothendieck space if and only if X is a Grothendieck space and every positive linear operator from λ^* to X^{**} is compact.*

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