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STOKES FLOW PAST POROUS BODIES OF ARBITRARY SHAPE

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In this paper we discuss a new approach to discuss Stokes flow past porous bodies of arbitrary shape using the Darcy [1] model and Saffman [2] boundary conditions.

Key words : Stokes flow; porous bodies; arbitrary shape; Darcy model; Saffman conditions; approximate solution.

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1. INTRODUCTION

The study of Stokes flow past porous bodies has been studied extensively owing to important applications which have left an indelible mark in many areas of science and engineering. Many models using different boundary conditions have been employed in these studies. Among them the Darcy [1] model has been largely employed in irrigation and oil industry as it is found to be favourable for low porosity systems. Many boundary conditions have been suggested for boundary value problems using Darcy model. But the most favoured ones have been the boundary conditions suggested by Saffman [2]. Most often, the subject of investigations has been the flows of viscous, incompressible fluids past porous spheres as they have been commonly used to model hydrodynamic interactions of polymer molecules and to model particles in emulsions and biological particles. But these are idealized shapes and it would be more practical to consider non-spherical shapes in these problems. Several researchers have studied such problems using the techniques like boundary element analysis and boundary integral method [3-6] which are suitable for such geometries. In this paper, we present a new method that is amenable to discuss boundary value problems involving non-spherical geometries which is different from many other previously known methods like finite difference methods as the solution can be computed at every point in the domain using a simple technique like series expansion in spherical harmonics. We illustrate the technique by applying it to the problem of Stokes flow past a porous body of arbitrary shape and finding its solution analytically. The formulation of the appropriate boundary conditions at the surface of a porous medium was always carried out as an important aspect of many investigations. Initially Joseph and Tao [7] employed the conditions of continuity of pressure, normal velocity and the condition of no-slip for the exterior tangential velocity when they discussed the problem of creeping flow of a viscous fluid past a porous spherical shell when Darcy law was assumed in the porous region. However the experiments of Beavers and Joseph [8] indicated that there is some slip at the boundary and they suggested that the appropriate boundary condition for plane boundaries is

$$\frac{du}{dy} = \frac{\alpha}{k}(u - Q),$$

where u is the velocity parallel to the surface, Q is the velocity inside the porous medium and y is the co-ordinate normal to the surface. Here k is the permeability of the medium. α is called as the dimensionless slip-coefficient which depends on the properties of the porous medium and can be determined experimentally. Saffman [2] used a statistical approach and gave a theoretical justification of the Beavers-Joseph condition and showed that the slip-condition can be written as

$$u = \frac{\sqrt{k}}{\alpha}\frac{du}{dy} + O(k),$$

at the boundary in the limit $k \to 0$ while extending Darcy law to non-homogeneous porous medium. Subsequently, after reviewing the different boundary conditions for Darcy model, Neale *et al*. [9] concluded that the Saffman conditions were the most satisfactory ones, which led to many investigations using these conditions. In this context, we recall that the problem of an arbitrary Stokes flow past a porous sphere was discussed and a method of solution was given previously by Padmavathi *et al*. [10] using Darcy law and Saffman boundary conditions. They also assumed the value of the slip co-efficient to be 1 as suggested by Neale *et al*. [9] and Haber and Mauri [11] who pointed out that there was enough evidence to show that most of the data of Beavers and Joseph can be reasonably correlated by such a choice of the slip co-efficient. Padmavathi *et al*. [10] obtained an important result that whenever the velocity of the unperturbed flow is harmonic, the drag on the porous sphere reduces compared to that on a rigid sphere. They also showed that the torque experienced by a porous sphere always reduces compared to that experienced by a rigid sphere. Another interesting result was that the effective viscosity of a dilute suspension of porous spherical particles diminishes compared to that of rigid particles of the same size. However in a more realistic approach, it would be more apt to consider particles of arbitrary shapes which are more naturally prevalent in practical applications.

In this paper, we engage ourselves in the discussion of the problem of Stokes flow past a porous body $\mathcal{B}$ of arbitrary shape when Saffman boundary conditions are employed by adopting the method suggested by Radha *at al.* [12,13] and which was later used by Choudhuri *et al.* [14], where the technique of least squares is used. Regarding the boundary conditons, the continuity of pressure at the interface has been used by many researchers like Neale and Nader [9], Bernardi *et al.* [15]. Davis and Stone [16], Le Bars and Worster [17], Prakash *et al.* [18], Chen *et al.* [19] among many others. Neale and Nader [9] who used this condition remarked that these conventional continuity boundary conditions are considered to be non-controversial. Davis and Stone [16] have also used this condition for Darcy equation in the case of a porous sphere, where they observed that this condition implies a small jump in normal stress at the boundary, since the external flow includes viscous normal stresses in addition to the pressure. Bernardi *et al.* [15] who studied the modelling of porous media with cracks used this condition and argued that it is an approximation of the real yet hypothetical physical condition which does not require any further modelling. Chen *et al.* [19] who studied the differences between the Stokes-Darcy system and the Stokes-Brinkman system used the condition of continuity of pressure across the interface while considering different interface conditions as in the work of Le Bars and Worster [17]. Continuity of pressure across the interface was also used by Prakash *et al.* [18] who considered arbitrary oscillatory flow past a porous sphere employing Darcy's law inside the porous region. They explained that pressure continuity is a valid boundary condition while dealing with viscous flows past spherical porous bodies that are not weakly permeable. Hence we have employed the condition of continuity of pressure in this paper.

2. Mathematical Formulation and Boundary Conditions

The equations governing the motion of a viscous, incompressible fluid is given by Stokes equations as follows.

$$\mu\nabla^2\mathbf{q} = \nabla p, \tag{1}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{2}$$

where **q** is the velocity, p the pressure and μ the co-efficient of dynamic viscosity of the fluid. In the absence of any boundaries, let the unperturbed flow be given by $(\mathbf{q}, p)$=$(\mathbf{q}_0, p_0)$. In all the discussion which follows we assume that the solution $(\mathbf{q}, p)$ of equations (1) and (2) admit a representation in terms of two scalar functions A and B given by [20, 21]

$$\mathbf{q} = \nabla\times\nabla\times(\mathbf{r}A) + \nabla\times(\mathbf{r}B), \tag{3}$$

$$p = \mu\frac{\partial}{\partial r}(r\nabla^2 A), \tag{4}$$

where

$$\nabla^4 A = 0, \;\; \nabla^2 B = 0. \tag{5,6}$$

Hence $(\mathbf{q}_0, p_0)$ is assumed to be represented in terms of two scalar functions A_0 and B_0. In the presence of a porous body $\mathcal{B}$, the governing equations of motion in the region exterior to the porous body $\mathcal{B}$ is given by equations (1)-(4) whose solution $(\mathbf{q}^e, p^e)$ is given in terms of scalar functions A^e and B^e.

The superscript e indicates flow quantities exterior to the body $\mathcal{B}$.

For the region inside the porous body $\mathcal{B}$, we assume that the flow is governed by Darcy's law

$$\mathbf{q}^i = -\frac{k}{\mu}\nabla p^i, \tag{7}$$

$$\nabla \cdot \mathbf{q}^i = 0, \tag{8}$$

where $\mathbf{q}^i$ represents the filter velocity rather than the actual velocity of the fluid. The superscript i indicates flow quantities in the region inside $\mathcal{B}$. From equations (7) and (8), we observe that

$$\nabla^2 p^i = 0. \tag{9}$$

We non-dimensionalize Stokes equations and Darcy equations by considering the following transformations

$$\mathbf{q}' = \frac{\mathbf{q}}{U}, p' = \frac{L}{\mu U}p, k' = \frac{k}{L},$$
$$x' = \frac{x}{L}, \;\; y' = \frac{y}{L}, \;\; z' = \frac{z}{L},$$

where U is a typical velocity and L a typical length scale. The non-dimensional Stokes equations for the region outside the porous body $\mathcal{B}$ (by dropping the primes for convenience) are as follows.

$$\nabla^2 \mathbf{q}^e = \nabla p^e, \tag{10}$$

$$\nabla . \mathbf{q}^e = 0. \tag{11}$$

In the region inside the porous body $\mathcal{B}$, the non-dimensional Darcy equations assume the following form

$$\mathbf{q}^i = -k\nabla p^i, \tag{12}$$

$$\nabla \cdot \mathbf{q}^i = 0, \tag{13}$$

where k is the Darcy number.

We recall that we have assumed that $(\mathbf{q}_0, p_0)$ denotes the unperturbed flow in the absence of any boundary. In the presence of the porous boundary $\partial\mathcal{B}$, if we denote the disturbance in the region outside the porous body $\mathcal{B}$ by $(\mathbf{q}_1, p_1)$ then the perturbed flow denoted by $(\mathbf{q}^e, p^e)$ in the region outside $\partial\mathcal{B}$ is given by

$$\mathbf{q}^e = \mathbf{q}_0 + \mathbf{q}_1,$$

$$p^e = p_0 + p_1.$$

In addition, we let the corresponding pressure inside the porous body $\mathcal{B}$ be p^i such that $(\mathbf{q}_1, p_1) \to (0,0)$ as $r \to \infty$ and $p^i \to 0$ as $r \to 0$. We assume that $(\mathbf{q}^e, p^e)$ expressed in terms of scalar functions (A^e, B^e) and $(\mathbf{q}_j, p_j)$ expressed in terms of scalar functions (A_j, B_j) ($j = 0, 1$) satisfy equations (3)-(6).

We assume that the scalars A_0, B_0, A_1 and B_1 can be expanded as a series in spherical harmonics as follows.

$$\begin{aligned}
A_0 &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\{(\alpha_{nm}r^n + \gamma_{nm}r^{n+2})\cos m\phi + (\beta_{nm}r^n + \delta_{nm}r^{n+2}) \\
&\quad \sin m\phi\}P_n^m(\zeta), \\
B_0 &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\{(\chi_{nm}\cos m\phi + \eta_{nm}\sin m\phi)\, r^n\}\, P_n^m(\zeta), \\
A_1 &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{\left(\frac{A_{nm}}{r^{n+1}} + \frac{C_{nm}}{r^{n-1}}\right)\cos m\phi + \left(\frac{B_{nm}}{r^{n+1}} + \frac{D_{nm}}{r^{n-1}}\right)\right. \\
&\quad \sin m\phi\}\, P_n^m(\zeta), \\
B_1 &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{(E_{nm}\cos m\phi + F_{nm}\sin m\phi)\,\frac{1}{r^{n+1}}\right\}P_n^m(\zeta), \\
p^i &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(\sigma_{nm}\cos m\phi + \xi_{nm}\sin m\phi) r^n P_n^m(\zeta).
\end{aligned}$$

Here $\zeta = \cos\theta$, and α_{nm}, β_{nm}, γ_{nm}, δ_{nm}, χ_{nm}, η_{nm} are known constants corresponding to the given flow $(\mathbf{q}_0, p_0)$ and A_{nm}, B_{nm}, C_{nm}, D_{nm}, E_{nm}, F_{nm}, σ_{nm} and ξ_{nm} are to be determined from the boundary conditions.

Consider the equation of the boundary $\partial\mathcal{B}$ to be given by $r = f(\theta, \varphi)$.
Let $(\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2)$ be a set of local orthonormal basis, where $\mathbf{n}$ is the normal to the surface. We write $\mathbf{n} = n_1\hat{e}_r + n_2\hat{e}_\theta + n_3\hat{e}_\varphi$ for the boundary $\partial\mathcal{B}$ where $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$ are the unit vectors in spherical polar co-ordinates (r, θ, φ). Then

$$\mathbf{n} = \frac{1}{k_1}\left(1, -\frac{f_\theta}{f}, -\frac{f_\varphi}{f\sin\theta}\right),\ \mathbf{t}_1 = \frac{1}{k_2}(f_\theta \hat{e}_r + f\hat{e}_\theta),$$

$$\mathbf{t}_2 = \frac{1}{k_1k_2k_3}[(\frac{f_\varphi}{\sin\theta})\hat{e}_r - (\frac{f_\theta f_\varphi}{f\sin\theta})\hat{e}_\theta + (\frac{f_\theta^2}{f} + f)\hat{e}_\varphi],$$

where

$$k_1 = \sqrt{1 + \left(\frac{f_\theta}{f}\right)^2 + \left(\frac{f_\varphi}{f\sin\theta}\right)^2},\ k_2 = \sqrt{f_\theta^2 + f^2},$$

$$k_3 = \sqrt{\left(\frac{f_\varphi}{k_1k_2\sin\theta}\right)^2 + \left(\frac{f_\theta f_\varphi}{fk_1k_2\sin\theta}\right)^2 + \left(\frac{f_\theta^2}{fk_1k_2} + \frac{f}{k_1k_2}\right)^2}.$$

It is useful to compute the velocities and stress vectors as in [14].

The boundary conditions assumed at $\partial\mathcal{B}$ are as follows.

1. Pressure is continuous

$$p^e = p^i. \tag{14}$$

2. Normal velocity is continuous

$$q_n^e = q_n^i. \tag{15}$$

i.e.,

$$\mathbf{n}\cdot\mathbf{q}^e = \mathbf{n}\cdot\mathbf{q}^i$$

3. Saffman conditions :

(a)

$$q_{t_1}^e = \frac{\sqrt{k}}{\alpha}\frac{\partial q_{t_1}^e}{\partial n} = \frac{\sqrt{k}}{\alpha}\mathbf{n}\cdot\nabla q_{t_1}^e. \tag{16}$$

(b)

$$q_{t_2}^e = \frac{\sqrt{k}}{\alpha}\frac{\partial q_{t_2}^e}{\partial n} = \frac{\sqrt{k}}{\alpha}\mathbf{n}\cdot\nabla q_{t_2}^e. \tag{17}$$

where α is the slip-co-efficient, and

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$

where

$$q_{t_1} = \mathbf{q} \cdot \mathbf{t}_1, \;\; q_{t_2} = \mathbf{q} \cdot \mathbf{t}_2$$

In the presence of the porous body $\mathcal{B}$, the modified flow exterior to the porous body can be represented in terms of the scalar functions as follows.

$$A^e = A_0 + A_1, \tag{18}$$

$$B^e = B_0 + B_1. \tag{19}$$

Since A^e and B^e are in terms of an infinite series, we consider the approximate solution outside $\mathcal{B}$ to be given by $\mathbf{q}_N^e$ where

$$\mathbf{q}_N^e = \nabla \times \nabla \times (\mathbf{r} A_N^e) + \nabla \times (\mathbf{r} B_N^e), \tag{20}$$

where

$$A_N^e = A_0 + A_1^N, \tag{21}$$

$$B_N^e = B_0 + B_1^N, \tag{22}$$

where A_1^N and B_1^N are the approximations to A_1 and B_1 respectively when the infinite series for A_1 and B_1 expressed in terms of spherical harmonics have been truncated up to N terms. In a similar manner, p_N^i denotes the approximation to p^i inside the porous region $\mathcal{B}$ after truncating its series up to N terms. We digress briefly to recall a result for non-axisymmetric flows past rigid bodies of arbitrary shape that we propose to adopt here. Instead of applying the no-slip condition $\mathbf{q} = 0$ on the boundary of the rigid body $\partial\Omega$, Radha *et al*. [12,13] considered instead the fulfillment of the following condition

$$I \;\; = \;\; \int_{\partial\Omega} |\mathbf{q}|^2 dS = 0. \tag{23}$$

In fact, an alternate condition of the fulfillment of the condition that

$$I_N = \int_{\partial\Omega} |\mathbf{q}_N|^2 dS = 0, \tag{24}$$

has been considered where $\mathbf{q}_N$ is an approximation to the velocity $\mathbf{q}$ obtained by truncating the corresponding series for the biharmonic and harmonic scalar functions used to represent the solution up to N terms. Then the integral in (24) is minimized in order to obtain an approximate analytic solution. We try to adopt this technique for Saffman boundary conditions which can be used for porous boundaries of arbitrary shape. This technique makes use of a complete general solution of Stokes equations given in [20, 21]. This new method enables us to verify the solution for the problem of an arbitrary Stokes flow past a sphere generated by a Stokeslet outside the sphere and also consider the problem of a uniform flow past an ellipsoid and the flow generated due to a Stokeslet outside an ellipsoid. By adopting the technique of Radha *et al*. [12, 13], instead of considering the Saffman boundary conditions (14)-(17), we adopt the following condition

$$\begin{aligned} I &= \int_{\partial\mathcal{B}} \left[(p^e - p^i)^2 + (q_n^e - q_n^i)^2 + \left(q_{t_1}^e - \frac{\sqrt{k}}{\alpha}\frac{\partial q_{t_1}^e}{\partial n} \right)^2 \right. \\ &\left. + \left(q_{t_2}^e - \frac{\sqrt{k}}{\alpha}\frac{\partial q_{t_2}^e}{\partial n} \right)^2 \right] dS = 0. \end{aligned} \tag{25}$$

Since we have truncated the series for the scalar functions occurring in the solution, we satisfy an alternate condition that $I_N = 0$ where the superscript N indicates that the flow quantities have been considered by truncating the terms up to $n = N$ in the series representation of the solution and then the integral I_N is minimized in order to obtain an approximate analytic solution. I_N being non-negative, we are assured of the existence of an infimum. Hence we find $\min(I_N)$ which is obtained by solving the resulting system of linear non-homogeneous equations which are solved for the unknown constants. The equations involve $4N(N+3)$ arbitrary constants $A_{nm}, B_{nm}, C_{nm}, D_{nm}, E_{nm}$, F_{nm}, σ_{nm} and ξ_{nm} where $n = 1, 2, \cdots, N$, $m = 0, 1, \cdots, n$ which are evaluated by minimizing the integral I_N. That is, we impose the conditions $\dfrac{\partial I_N}{\partial X_{ij}} = 0$ where X_{ij} are the constants $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}$, σ_{ij}, ξ_{ij} where $i = 1, 2, \cdots, N$, $j = 0, 1, \cdots, i$. These conditions result in a linear system of non-homogeneous equations involving these constants which are solved to determine the constants for different values of N.

3. Examples and Discussion

Example 1 : Stokeslet external to a sphere.

In this case, we test the method by considering a Stokes flow in the presence of a porous sphere $r = 2$ in a viscous, incompressible fluid due to a Stokeslet of unit strength at $(0, 0, 4)$, whose axis is in the positive direction of x-axis. Here

$$A_0 = \frac{\cos\varphi}{rc\sin\theta}\left[(r\cos\theta - c)R_1 + R_1^2\right],$$

$$B_0 = \frac{2\sin\varphi}{rc\sin\theta}\left[r\cos\theta - c + R_1\right],$$

where $R_1^2 = r^2 + c^2 - 2rc\cos\theta, c = 4.$ It is observed that the solution obtained using this method by truncating at $n = N$ fully agrees with the exact solution given in [10].

Example 2 : Flow generated by a Stokeslet - rigid case [13].

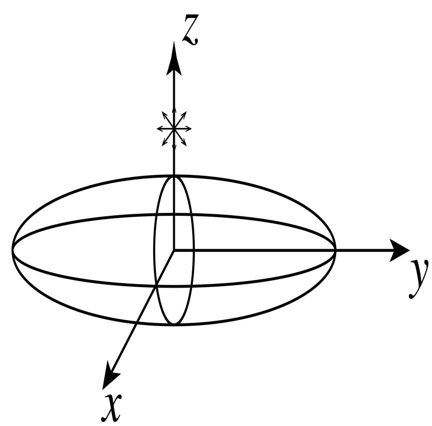

Figure 1 : Stokeslet at $(0, 0, 3)$ located exterior to the ellipsoid

N	Drag	Torque	$\min(I_N)$
1	18.1372	38.9235	1.02971
2	17.5630	38.5130	0.48675
3	17.7958	37.6646	0.269785
4	17.8641	38.3611	0.132845
5	17.8488	38.5155	0.03412
6	17.8448	37.6336	0.0111592
7	17.8528	37.6354	0.00854616
8	17.8550	37.7175	0.00735529
9	17.8667	37.7268	0.00296946
10	17.8671	37.6194	0.00199187
11	17.8712	37.6223	0.00136611
12	17.8715	37.5952	0.001252
13	17.8749	37.5971	0.0007825
14	17.8751	37.5715	0.0006711
15	17.8771	37.5724	0.00052

Table 1 : Drag, Torque and $\min(I_N)$ for different values of N

Consider the Stokes flow of a viscous, incompressible fluid past a rigid ellipsoid given by $\frac{x^2}{4^2} +$

$\frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$ induced due to a Stokeslet of unit strength, located at $(0,0,3)$ on the z-axis and whose axis lies along the positive x-axis.

The unperturbed flow is given by the scalars (A_0, B_0) as in Example 1.

The values of min(I_N) obtained after minimizing the integral are given in Table 1. Observe that I_N decreases as N increases. The drag experienced by the ellipsoid is found to be along the $\hat{i}$ direction and the corresponding torque is along the $\hat{j}$ direction and their magnitudes are given in Table 1.

This method has an advantage over other methods as it is easy to use since it only involves series expansions using spherical harmonics, and the solution is known analytically throughout the domain although we may truncate the series. The calculation of physical quantities like drag require computation of derivatives of velocities to compute stresses which is easy to obtain from these analytical expressions.

When numerical methods are used, the velocities and pressure are known approximately at certain predetermined points. Hence when stresses and drag etc. are calculated from them approximately using numerical differentiation and integration, there may be further increase in error. However in the present method, the derivatives of the velocities can be computed from the analytical expressions in order to compute stresses and drag etc.

Example 3 : Flow past a porous ellipsoid generated by a Stokeslet.

We consider here the Stokes flow in the presence of a porous ellipsoid $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$, due to a Stokeslet of unit strength at $(0,0,3)$ whose axis is in the positive direction of x-axis.

By using the method suggested, (A_1^N, B_1^N) and p_N^i have been calculated for different values of N and the convergence of the minimum value of I_N to zero is observed as N increases.

In Table 2, we compute the minimum value of I_N, the drag and torque experienced by the porous ellipsoid at different stages of N for $k = 10^{-6}$ and for $\alpha = 0.3$. Here the drag experienced by the porous ellipsoid is along $\hat{i}$ direction and torque is along $\hat{j}$ direction. Their magnitudes are given in the table.

N	$\min(I_N)$	Drag	Torque
1.	2.1656	18.5416	39.1986
2.	0.93801	17.9865	36.7346
3.	0.73221	17.7754	35.8494
4.	0.59561	17.817	38.2249
5.	0.40899	18.0615	38.3136
6.	0.33314	18.0604	38.8619
7.	0.23632	18.1656	38.8492
8.	0.15056	18.6393	38.2586
9.	0.10128	18.1406	38.2673
10.	0.07066	18.1415	37.84

Table 2 : Drag, Torque and $\min(I_N)$ for different values of N
$k = 10^{-6}$ and $\alpha = 0.3$

The expressions for A_N^e, B_N^e and p_N^i enable us to write the solution in the regions exterior and interior to the porous boundary ∂B. Thus we have an analytic solution which can be used to compute physical quantities like drag and torque at every point in the boundary even though it is an approximate solution. This clearly would be preferable to other previously known numerical methods where the solution is not only known approximately but is also known only at some predetermined points in the domain. Hence the computed physical quantities would be more accurate when the present method is used.

By computing the expressions for A_N^e, B_N^e and p_N^i for $N = 10$ for different values of k and α, we have presented the magnitudes of the corresponding drag and torque which are along the $\hat{i}$ and $\hat{j}$ directions respectively in Table 3.

S.No.	k	α	Drag	Torque
1.	0		17.867	38.14
2.	10^{-6}	0.3	18.1415	37.84
3.	0.009	0.3	16.2239	29.31
4.	10^{-6}	0.5	18.16	37.88
5.	0.007	0.5	17.069	33.36
6.	0.009	0.5	16.935	32.51

Table 3 : Drag and Torque when $N = 10$ for different values of k and α.

N	$\min(I_N)$	Drag	N	$\min(I_N)$	Drag
1	4.13399	$56.4622U$	9	0.230803	$59.9795U$
2	3.83326	$56.6907U$	10	0.230368	$59.9803U$
3	1.80243	$58.3698U$	11	0.129523	$60.1232U$
4	1.79167	$58.3818U$	12	0.129284	$60.1236U$
5	0.877849	$59.2384U$	13	0.0771365	$60.2061U$
6	0.875834	$59.2411U$	14	0.0771365	$60.2064U$
7	0.437992	$59.7194U$	15	0.0491195	$60.2583U$
8	0.43716	$59.7207U$			

Table 4: $\min(I_N)$ and drag for different values of N

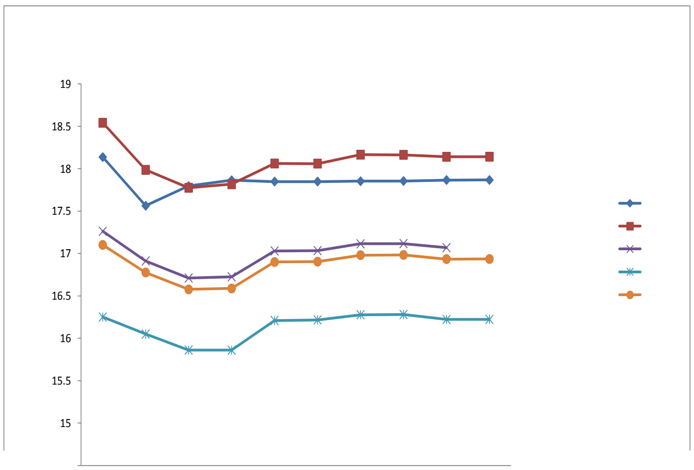

Figure 2: Drag versus N

In Figure 2, the plotted curves 'Drag versus N' depict the variation of drag experienced by an ellipsoid with N in a flow generated by a Stokeslet.

Graph 1 represents the drag experienced by a rigid ellipsoid in a flow generated by a Stokeslet given in Example 2. Graphs 2 to 5 represent the drag experienced by a porous ellipsoid in a flow generated by a Stokeslet in Example 3 when $k = 10^{-6}$ and $\alpha = 0.5$; $k = 0.007$ and $\alpha = 0.5$; $k = 0.009$ and $\alpha = 0.3$; $k = 0.009$ and $\alpha = 0.5$ respectively.

In Figure 3, the plotted curves 'Torque versus N' depict the variation of torque experienced by an ellipsoid with N in a flow generated by a Stokeslet.

Here Graph 1 represents the torque experienced by a rigid ellipsoid in a flow generated by a Stokeslet given in Example 2. Graphs-2 to 5 represent the torque experienced by a porous ellipsoid

is in the direction of $-\hat{k}$) in the rigid case in Table 4, in the case of a uniform flow external to a rigid ellipsoid [13].

We also present in Tables 5 and 6, the minimum value of I_N and drag in the case of a uniform flow past a porous ellipsoid given in this example for $(k = 10^{-6}, \alpha = 0.5)$ and $(k = 10^{-6}, \alpha = 0.3)$ respectively. The drag experienced by the porous ellipsoid is only in the $-\hat{k}$ direction and we give its magnitude in Tables 5 and 6.

N	$\min(I_N)$	$Drag$
1.	6.8112	$55.0215U$
2.	6.5308	$55.237U$
3.	1.95439	$58.1655U$
4.	1.94132	$58.179U$
5.	1.088653	$58.9995U$
6.	1.08719	$59.0015U$
7.	0.54436	$59.5755U$
8.	0.54428	$59.576U$
9.	0.3383	$59.812U$
10.	0.3383	$59.8125U$

Table 5: $k = 10^{-6}$ and $\alpha = 0.5$

N	min(I_N)	$Drag$
1.	6.81315	$55.0035U$
2.	6.53215	$55.2195U$
3.	1.95658	$58.1435U$
4.	1.9434	$58.1575U$
5.	1.08963	$58.977U$
6.	1.08815	$58.979U$
7.	0.54507	$59.552U$
8.	0.54499	$59.552U$
9.	0.338905	$59.7885U$
10.	0.338905	$59.7885U$

Table 6 : $k = 10^{-6}$ and $\alpha = 0.3$.

We observe that in the case of a flow generated by a Stokeslet past a porous ellipsoid, the drag and torque experienced by the ellipsoid decreases as k increases, for a fixed α. Moreover, the drag and torque increase as the slip-co-efficient α increases. In the case of a uniform flow past a porous ellipsoid, the drag decreases as k increases for a fixed α. When $k = 0$, we retrieve the classical result due to Oberbeck [22] for the flow past a rigid ellipsoid and obtain the drag and torque for $N = 15$ in Example 4 to be $-60.2583\hat{k}$ and 0 [13], which are found to be $-60.348\hat{k}$ and 0 respectively by Oberbeck's formula which uses improper integrals. In this case also, we find that for a given k, the drag decreases as the slip co-efficient α decreases.

4. Conclusions

A new method has been proposed to obtain an approximate, analytical solution for the problem of Stokes flow past a porous body of arbitrary shape by modifying a previous method suggested by Radha *et al*. [12, 13]. This method employs a complete general solution of Stokes equations expressed in terms of two scalar functions which can be determined easily. The equations of motion are satisfied exactly and the boundary conditions are satisfied approximately by the solution obtained. The method is easier to use compared to previous methods where improper integrals or ellipsoidal harmonics were used in the case of rigid boundary conditions, as only series expansions using spherical harmonics are used.

The method has been verified for a porous sphere and it has been observed that the solution matches with the exact solution given in [10] for all values of $N = 1 \dots 15$. We have also used this method to discuss the problems of uniform flow and a singularity driven Stokes flow past a porous ellipsoid using Saffman boundary conditions whose solutions are hitherto unknown in literature. In order to increase the accuracy, we can increase the number of terms N in the series.

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