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A SIMPLE OBSERVATION ON DECOMPOSITION OF MULTIPLE ZETA VALUES

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A decomposition of Multiple Zeta Values has been derived in terms of Riemann Zeta Values, thereby obtaining a formula of Multiple Zeta Values given by T. Arakawa and M. Kaneko. Further, using the decomposition formula, some of the identities arising from Multiple Zeta Values have been reestablished in a larger domain.

Key words : Multiple zeta values.

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1. Introduction

For $s \in \mathbb{C}$ with $Re(s) > 1$, the Riemann Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This function keeps many mysteries. The most famous of them is the Riemann Hypothesis which states that all the non-trivial zeros of $\zeta(s)$ lie on $Re(s) = 1/2$ and the Hypothesis is still an open problem. Among other mysteries, an interesting and still unsolved problem is the problem of determining polynomial relations over Q for the numbers $\zeta(s)$, $s = 2, 3, 4, ...$ The first breakthrough in this direction is due to Euler, who showed that $\zeta(2k)$ is always a rational multiple of π^{2k}. Euler (1735) proved that

$$2\zeta(2k) \quad = \quad (-1)^{k-1}(2\pi)^{2k}\frac{B_{2k}}{(2k)!}. \tag{1}$$

where B_{2k} are rational numbers, called Bernoulli numbers, defined by the power series expansion

$$\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}.$$

Very less is known about the values $\zeta(s)$ at odd positive integers. The study of such values of Riemann Zeta function has motivated the study of multiple zeta values (MZV's) defined as

$$\zeta(s_1, s_2, ..., s_r) = \sum_{n_1 > n_2 > ... > n_r \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} .... n_r^{s_r}},$$

where s_i are integers for $1 \leq i \leq r$, $s_1 \geq 2$ and $s_i \geq 1$ for $2 \leq i \leq r$. The sum $s_1 + s_2 + ... + s_r$ is called the weight of $\zeta(s_1, s_2, ..., s_r)$ and r is called its length.

The study of multiple zeta values has been a topic of interest to many mathematicians for the last few decades due to its applications in many branches of mathematics and physics, such as knot theory, quantum field theory etc. An overview of multiple zeta values has been provided in [4].

Some of the results related to Multiple Zeta Values with all the arguments same are:

$$\zeta(\underbrace{2, 2, ..., 2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}$$

$$\zeta(\underbrace{4, 4, ..., 4}_{n}) = \frac{4.(2\pi)^{4n}}{(4n+2)!} \left(\frac{1}{2}\right)^{2n+1}$$

$$\zeta(\underbrace{6, 6, ..., 6}_{n}) = \frac{6.(2\pi)^{6n}}{(6n+3)!}$$

$$\zeta(\underbrace{8, 8, ..., 8}_{n}) = \frac{8.(2\pi)^{8n}}{(8n+4)!} \left\{ \left(1 + \frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1 - \frac{1}{\sqrt{2}}\right)^{4n+2} \right\}$$

In fact, a general formula for such Multiple Zeta Values has been derived by T. Arakawa and M. Kaneko [1], which is as follows: For $k, n \in \mathbb{N}$

$$\zeta(\underbrace{2k, 2k, ..., 2k}_{n}) = C_n^{(k)} \frac{(2\pi i)^{2nk}}{(2kn)!} \tag{2}$$

where $C_n^{(k)}$ is defined by the following recurrence relations:

$$C_0^{(k)} = 1, \; C_n^{(k)} = \frac{1}{2n} \sum_{l=1}^{n} (-1)^l \begin{pmatrix} 2nk \\ 2lk \end{pmatrix} B_{2lk} C_{n-l}^{(k)}. \tag{3}$$

The idea to decompose Multiple Zeta Values $\zeta(s, s, ..., s)$ with s any integer greater than 1, has motivated the work of this article.

2. Decomposition of Multiple Zeta Values

The double sum

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{n_1^{s_1}}\frac{1}{n_2^{s_2}}$$

can be written as

$$\sum_{n_1,n_2=1;n_1>n_2}^{\infty}\frac{1}{n_1^{s_1}}\frac{1}{n_2^{s_2}}+\sum_{n_1,n_2=1;n_2>n_1}^{\infty}\frac{1}{n_1^{s_1}}\frac{1}{n_2^{s_2}}+\sum_{n_1,n_2=1;n_1=n_2}^{\infty}\frac{1}{n_1^{s_1}}\frac{1}{n_2^{s_2}},$$

that is,

$$\zeta(s_1)\zeta(s_2)=\zeta(s_1,s_2)+\zeta(s_2,s_1)+\zeta(s_1+s_2). \tag{4}$$

In a similar manner, it is not difficult to see that

$$\begin{aligned}\zeta(s_1,s_2,...,s_r)\zeta(s)&=\zeta(s,s_1,s_2,...,s_r)+\zeta(s_1,s,s_2,...,s_r)+...+\zeta(s_1,s_2,...,s_r,s)+\\ &\zeta(s+s_1,s_2,...,s_r)+\zeta(s_1,s+s_2,...,s_r)+....+\zeta(s_1,s_2,...,s+s_r)\end{aligned} \tag{5}$$

Using this fact, one can easily prove the following lemma.

Lemma 2.1 — For any integer $s>1$ and $n\in\mathbb{N}$,

$$\begin{aligned}\zeta(\underbrace{s,s,....,s}_{n}) &= \frac{1}{n}\{\zeta(\underbrace{s,s,....,s}_{n-1})\zeta(s)-\zeta(\underbrace{s,s,....,s}_{n-2})\zeta(2s)\\ &+....+(-1)^{n-2}\zeta(s)\zeta((n-1)s)+(-1)^{n-1}\zeta(ns)\}\end{aligned}$$

Proof : Using (5), we first obtain the products $\zeta(\underbrace{s,s,....,s}_{n-r})\zeta(rs)$ for $r=1,2,3...,n-1$, as follows:

$$\begin{aligned}\zeta(\underbrace{s,s,....,s}_{n-1})\zeta(s) &= n\zeta(\underbrace{s,s,...,s}_{n})+\zeta(\underbrace{2s,s,....,s}_{n-1})+\zeta(\underbrace{s,2s,...,s}_{n-1})+.....+\zeta(\underbrace{s,s,...,2s}_{n-1})\\ \zeta(\underbrace{s,s,....,s}_{n-2})\zeta(2s) &= \zeta(\underbrace{2s,s,....,s}_{n-1})+\zeta(\underbrace{s,2s,...,s}_{n-1})+.....+\zeta(\underbrace{s,s,...,2s}_{n-1})+\\ &\zeta(\underbrace{3s,s,...,s}_{n-2})+\zeta(\underbrace{s,3s,...,s}_{n-2})+.....+\zeta(\underbrace{s,s,...,3s}_{n-2})\end{aligned}$$

$$\begin{aligned}
\zeta(\underbrace{s,s,....,s}_{n-3})\zeta(3s) &= \zeta(\underbrace{3s,s,...,s}_{n-2})+\zeta(\underbrace{s,3s,...,s}_{n-2})+.....+\zeta(\underbrace{s,s,...,3s}_{n-2})+\\
&\quad \zeta(\underbrace{4s,s,...,s}_{n-3})+\zeta(\underbrace{s,4s,...,s}_{n-3})+....+\zeta(\underbrace{s,s,...,4s}_{n-3})\\
\cdots\cdots\cdots &\quad \cdots\cdots\cdots\\
\zeta(s,s)\zeta((n-2)s) &= \zeta((n-2)s,s,s)+\zeta(s,(n-2)s,s)+\zeta(s,s,(n-2)s)+\\
&\quad \zeta((n-1)s,s)+\zeta(s,(n-1)s)\\
\zeta(s)\zeta((n-1)s) &= \zeta(s,(n-1)s)+\zeta((n-1)s,s)+\zeta(ns)
\end{aligned}$$

Adding all the above equations with alternate signs, the desired result will be obtained. □

In order to derive our decomposition formula, we define: For $n, m \in \mathbb{N}$,

$$\begin{aligned}
&L_0^n = 1\\
&L_1^n = \frac{\zeta(s)}{(n-1)}, \ \ (n>1)\\
&L_2^n = \frac{1}{n-2}\{L_1^n\zeta(s)-\zeta(2s)\}, \ \ (n>2)\\
&L_3^n = \frac{1}{n-3}\{L_2^n\zeta(s)-L_1^n\zeta(2s)+\zeta(3s)\}, \ \ (n>3)
\end{aligned}$$

In general, for $1 \leq m \leq n-1$,

$$L_m^n = \frac{1}{n-m}\{L_{m-1}^n\zeta(s)-L_{m-2}^n\zeta(2s)+L_{m-3}^n\zeta(3s)-....+(-1)^{m-1}\zeta(ms)\}.$$

and for $m = n$,

$$L_n^n = \{L_{n-1}^n\zeta(s)-L_{n-2}^n\zeta(2s)+L_{n-3}^n\zeta(3s)-....+(-1)^{n-1}\zeta(ns)\}.$$

With this, we prove the next lemma.

Lemma 2.2 — Given a positive integer n,

$$\begin{aligned}
L_m^{n+1} &= \frac{L_{m-1}^n}{n}\zeta(s)-\frac{L_{m-2}^{n-1}}{n-1}\zeta(2s)+\frac{L_{m-3}^{n-2}}{n-2}\zeta(3s)-......\\
&\quad +(-1)^{m-2}\frac{L_1^{n-m+2}}{n-m+2}\zeta((m-1)s)+(-1)^{m-1}\frac{\zeta(ms)}{n-m+1}
\end{aligned}$$

holds for all $m \leq n$.

PROOF : The lemma will be proved using induction on m.

$$\text{For } m=1, \ L_1^{n+1} = \frac{\zeta(s)}{n},$$

$$
\begin{aligned}
\text{For } m=2,\ L_2^{n+1} &= \frac{1}{n-1}\{L_1^{n+1}\zeta(s)-\zeta(2s)\} \\
&= \frac{1}{n-1}\left\{\frac{n-1}{n}L_1^n\zeta(s)-\zeta(2s)\right\} \\
&= \frac{L_1^n}{n}\zeta(s)-\frac{1}{n-1}\zeta(2s),
\end{aligned}
$$

Thus lemma holds for $m=1,2$. Assume that $m<n$ and the lemma holds for all positive integers $\leq m$. Now,

$$
\begin{aligned}
(n-m)L_{m+1}^{n+1} &= L_m^{n+1}\zeta(s)-L_{m-1}^{n+1}\zeta(2s)+........+(-1)^{m-1}L_1^{n+1}\zeta(ms) \\
&\quad +(-1)^m\zeta((m+1)s) \\
\\
&= \left\{\frac{L_{m-1}^n}{n}\zeta(s)-\frac{L_{m-2}^{n-1}}{n-1}\zeta(2s)+....+(-1)^{m-1}\frac{\zeta(ms)}{n-m+1}\right\}\zeta(s) \\
&\quad -\left\{\frac{L_{m-2}^n}{n}\zeta(s)-\frac{L_{m-3}^{n-1}}{n-1}\zeta(2s)+....+(-1)^{m-2}\frac{\zeta((m-1)s)}{n-m+2}\right\} \\
&\quad \zeta(2s)+......+(-1)^{m-2}\left\{\frac{L_1^n}{n}\zeta(s)-\frac{\zeta(2s)}{n-1}\right\}\zeta((m-1)s)+ \\
&\quad (-1)^{m-1}\frac{\zeta(s)}{n}\zeta(ms)+(-1)^m\zeta((m+1)s) \\
\\
&= \frac{\zeta(s)}{n}\left\{L_{m-1}^n\zeta(s)-L_{m-2}^n\zeta(2s)+....+(-1)^{m-2}L_1^n\zeta((m-1)s)\right. \\
&\quad \left.+(-1)^{m-1}\zeta(ms)\right\}-\frac{\zeta(2s)}{n-1}\left\{L_{m-2}^{n-1}\zeta(s)-L_{m-3}^{n-1}\zeta(2s)+\right. \\
&\quad \left. ......+(-1)^{m-2}\zeta((m-1)s)\right\}+.......+\frac{(-1)^{m-1}}{n-m+1}\zeta(s)\zeta(ms) \\
&\quad +(-1)^m\zeta((m+1)s) \\
\\
&= \frac{n-m}{n}L_m^n\zeta(s)-\frac{n-m}{n-1}L_{m-1}^{n-1}\zeta(2s)+...+(-1)^{m-1}\frac{n-m}{n-m+1} \\
&\quad L_1^{n-m+1}\zeta(ms)+(-1)^m\zeta((m+1)s).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
L_{m+1}^{n+1} &= \frac{L_m^n}{n}\zeta(s)-\frac{L_{m-1}^{n-1}}{n-1}\zeta(2s)+...+(-1)^{m-1}\frac{L_1^{n-m+1}}{n-m+1}\zeta(ms) \\
&\quad +(-1)^m\frac{\zeta((m+1)s)}{n-m}.
\end{aligned}
$$

Thus lemma holds for $m+1$ and hence for all $m \leq n$. □

Now, we are in a position to prove the following Decomposition Theorem for the Multiple Zeta Values:

***Theorem* 2.1** — *For* $n \geq 1$, $s > 1$,

$$\begin{aligned} \zeta(\underbrace{s, s, ..., s}_{n}) &= \frac{L_n^n}{n}, \\ \text{where } L_n^n &= \sum_{l=1}^{n} (-1)^{l-1} L_{n-l}^n \zeta(ls). \end{aligned}$$

PROOF : The theorem will be proved using induction on n. For $n = 1$, it is clearly true. For $n = 2$,

$$\zeta(s, s) = \frac{1}{2}\{\zeta(s)\zeta(s) - \zeta(2s)\} = \frac{1}{2}\{L_1^2\zeta(s) - \zeta(2s)\} = \frac{L_2^2}{2}.$$

Thus the result holds for $n = 1, 2$. Assume that it holds for all n $(2 \leq n \leq m)$. We will prove that it is true for $m+1$. By Lemma 2.1 and induction hypothesis,

$$\begin{aligned} &(m+1)\zeta(\underbrace{s, s, ..., s}_{m+1}) \\ = \;& \zeta(\underbrace{s, s, ..., s}_{m})\zeta(s) - \zeta(\underbrace{s, s, ..., s}_{m-1})\zeta(2s) + ... + (-1)^{m-1}\zeta(s)\zeta(ms) + (-1)^m\zeta((m+1)s). \\ = \;& \frac{L_m^m\zeta(s)}{m} - \frac{L_{m-1}^{m-1}\zeta(2s)}{m-1} + ... + (-1)^{m-1}\zeta(s)\zeta(ms) + (-1)^m\zeta((m+1)s) \\ = \;& \left\{L_{m-1}^m\zeta(s) - L_{m-2}^m\zeta(2s) + L_{m-3}^m\zeta(3s) - .... + (-1)^{m-1}\zeta(ms)\right\}\frac{\zeta(s)}{m} - \\ & \left\{L_{m-2}^{m-1}\zeta(s) - L_{m-3}^{m-1}\zeta(2s) + L_{m-4}^{m-1}\zeta(3s) - .... + (-1)^{m-2}\zeta((m-1)s)\right\}\frac{\zeta(2s)}{m-1} \\ & + ... + (-1)^{m-1}\zeta(s)\zeta(ms) + (-1)^m\zeta((m+1)s). \end{aligned}$$

Arranging appropriately and using Lemma 2.2,

$$\begin{aligned}
(m+1)\zeta(\underbrace{s,s,....,s}_{m+1}) &= \left\{\frac{L_{m-1}^{m}\zeta(s)}{m} - \frac{L_{m-2}^{m-1}\zeta(2s)}{m-1} + ... + (-1)^{m-1}\zeta(ms)\right\}\zeta(s) - \\
&\quad \left\{\frac{L_{m-2}^{m}\zeta(s)}{m} - \frac{L_{m-3}^{m-1}\zeta(2s)}{m-1} + ... + (-1)^{m-2}\frac{\zeta((m-1)s)}{2}\right\}\zeta(2s) \\
&\quad + ... + (-1)^{m}\zeta((m+1)s) \\
&= L_{m}^{m+1}\zeta(s) - L_{m-1}^{m+1}\zeta(2s) + ... + (-1)^{m}\zeta((m+1)s) \\
&= L_{m+1}^{m+1}.
\end{aligned}$$

Hence the theorem. □

We conclude this section by deriving formula (2) from Theorem 2.1 as follows: Let $n, m \in \mathbb{N}$. For a fixed $k \in \mathbb{N}$, define

$$\begin{aligned}
&\overline{L_0^n} = 1, \\
&\overline{L_1^n} = -\frac{1}{2(n-1)}\frac{B_{2k}}{(2k)!}, \quad (n > 1) \\
&\overline{L_2^n} = -\frac{1}{2(n-2)}\left\{\overline{L_1^n}\frac{B_{2k}}{(2k)!} - \frac{B_{4k}}{(4k)!}\right\}, \quad (n > 2)
\end{aligned}$$

In general, for $1 \le m \le n-1$,

$$\overline{L_m^n} = -\frac{1}{2(n-m)}\left\{\overline{L_{m-1}^n}\frac{B_{2k}}{(2k)!} - \overline{L_{m-2}^n}\frac{B_{4k}}{(4k)!} + ... + (-1)^{m-1}\frac{B_{2mk}}{(2mk)!}\right\}$$

and

$$\overline{L_n^n} = -\frac{1}{2}\left\{\overline{L_{n-1}^n}\frac{B_{2k}}{(2k)!} - \overline{L_{n-2}^n}\frac{B_{4k}}{(4k)!} + ... + (-1)^{n-1}\frac{B_{2nk}}{(2nk)!}\right\}$$

It can be easily verified that, when $s = 2k$,

$$\begin{aligned}
L_m^n &= \overline{L_m^n}\,(2\pi i)^{2mk}, \text{ for } 0 \le m \le n \\
\text{and } n\,C_n^{(k)} &= \overline{L_n^n}\,(2nk)!
\end{aligned}$$

Hence,

$$\zeta(\underbrace{2k, 2k, ..., 2k}_{n}) = \frac{\overline{L_n^n}}{n}(2\pi i)^{2nk}$$
$$= \frac{C_n^{(k)}}{(2nk)!}(2\pi i)^{2nk}$$

3. Series Arising from MZV's

In this section, we reestablish the Propositions 2.1, 2.2, 3.1 and 3.2 of [3] in a larger domain, viz. for any integer greater than 1. At this point, we need to introduce the Hurwitz zeta function and multiple Hurwitz zeta function. The Hurwitz zeta function is a generalization of Riemann zeta function and is defined as

$$\zeta(s; a) = \sum_{i=0}^{\infty} \frac{1}{(i+a)^s}, \text{ where } s > 1 \text{ and } a \in \mathbb{R}\backslash\mathbb{Z}_0^-$$

and the multiple Hurwitz zeta function is defined as

$$\zeta(s_1, s_2, ..., s_r; a_1, a_2, ..., a_r) = \sum_{n_1 > n_2 > ... > n_r \geq 1} \frac{1}{(n_1+a_1)^{s_1}(n_2+a_2)^{s_2}....(n_r+a_r)^{s_r}},$$

where $s_1 > 1$, $s_1 + s_2 > 2$,.....,$s_1 + s_2 + ... + s_r > r$ and $a_1, a_2, ..., a_r \in \mathbb{R}\backslash\mathbb{Z}$.

The i^{th} generalized harmonic number of order $n \in \mathbb{N}$ is defined as

$$H_{i,n} = \sum_{r=1}^{i} \frac{1}{r^n}.$$

For $n \in \mathbb{N}$, we also define

$$H'_{2i-1,n} = \sum_{r=1}^{i} \frac{1}{(2r-1)^n}.$$

Further, consider the sum

$$S_{m,k}^n = \sum_{i=1}^{n} \frac{H_{i,m}}{i^k}$$

(where $m \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{1\}$), which is clearly the n^{th} partial sum of the classical linear Euler sum [2]

$$S_{m,k} = \sum_{i=1}^{\infty} \frac{H_{i,m}}{i^k}$$

As in (4), it is easy to see that

$$\begin{aligned}\zeta(s_1;a)\zeta(s_2;a) &= \zeta(s_1,s_2;a,a)+\zeta(s_2,s_1;a,a)+\zeta(s_1+s_2;a)\\ &\quad+\frac{1}{a^{s_1}}\zeta(s_2;a)+\frac{1}{a^{s_2}}\zeta(s_1;a)-\frac{2}{a^{s_1+s_2}}.\end{aligned}$$

and also

$$\zeta(s_1,s_2,..,s_r;a,a,..,a)\zeta(s;a)=\frac{1}{a^s}\zeta(s_1,s_2,..,s_r;a,a,..,a)+\zeta(s,s_1,s_2,..,s_r;a,a,a,..,a)$$

$$+\zeta(s_1,s,s_2,..,s_r;a,a,..,a)+.....+\zeta(s_1,s_2,..,s_r,s;a,a,..,a)+$$

$$\zeta(s+s_1,s_2,..,s_r;a,a,..,a)+\zeta(s_1,s+s_2,..,s_r;a,a,..,a)+....+$$

$$\zeta(s_1,s_2,..,s+s_r;a,a,..,a)$$

Hence, we can state the following lemma, the proof of which is similar to that of Lemma 2.1.

Lemma 3.1 — For any integer $s>1$, $n\in\mathbb{N}$ and $a\in\mathbb{R}\backslash\mathbb{Z}$,

$$\begin{aligned}\zeta(\underbrace{s,s,..,s}_{n};\underbrace{a,a,..,a}_{n}) &= \frac{1}{n}\Bigg[\zeta(\underbrace{s,s,..,s}_{n-1};\underbrace{a,a,..,a}_{n-1})\left(\zeta(s;a)-\frac{1}{a^s}\right)-\zeta(\underbrace{s,s,....,s}_{n-2};\underbrace{a,a,...,a}_{n-2})\\ &\quad\left(\zeta(2s;a)-\frac{1}{a^{2s}}\right)+...+(-1)^{n-2}\left(\zeta(s;a)-\frac{1}{a^s}\right)\\ &\quad\left(\zeta((n-1)s;a)-\frac{1}{a^{(n-1)s}}\right)+(-1)^{n-1}\left(\zeta(ns;a)-\frac{1}{a^{ns}}\right)\Bigg]\end{aligned}$$

By appropriate arrangements of the terms in the respective series $($each of them being absolutely convergent$)$ the following three lemmas can be proved.

Lemma 3.2 — For $m,n>1$,

(i) $\sum\limits_{i=1}^{\infty}\frac{H_{i,n}}{i^m}=\sum\limits_{i=1}^{\infty}\frac{\zeta(m;i)}{i^n}$

(ii) $\sum\limits_{i=1}^{\infty}\frac{H_{i,n}}{i^m}=\zeta(n,m)+\zeta(m+n).$

(iii) $\sum\limits_{i=1}^{\infty}\frac{H'_{2i-1,n}}{(2i-1)^m}=2^{-m}\sum\limits_{i=1}^{\infty}\frac{\zeta(m;i-\frac{1}{2})}{(2i-1)^n}.$

Lemma 3.3 — For $m,n,k>1$,

(i) $\sum\limits_{i=1}^{\infty}\frac{1}{i^k}\sum\limits_{j=1}^{i}\frac{H_{j,n}}{j^m}=\sum\limits_{i=1}^{\infty}\frac{H_{i,n}}{i^m}\zeta(k;i).$

(ii) $\sum\limits_{i=1}^{\infty}\frac{1}{(2i-1)^k}\sum\limits_{j=1}^{i}\frac{H'_{2j-1,n}}{(2j-1)^m}=2^{-k}\sum\limits_{i=1}^{\infty}\frac{H'_{2i-1,n}}{(2i-1)^m}\zeta(k;i-\frac{1}{2}).$

Lemma 3.4 — For $m, n, k > 1$,
$\sum_{i=1}^{\infty}\frac{1}{i^k}\sum_{j=1}^{i}\frac{1}{j^m}\sum_{t=1}^{j}\frac{H_{t,n}}{t^m}=\sum_{i=1}^{\infty}\frac{S_{n,m}^{i}}{i^m}\zeta(k;i)$.

The next four identities (Propositions 3.1, 3.2, 3.3 and 3.4) are respectively the analogues of Propositions 2.1, 2.2, 3.1 and 3.2 of [3].

Proposition 3.1 — For any integer $s > 1$,

$$\sum_{i=1}^{\infty}\frac{\zeta(s;i)}{i^s}=\frac{1}{2}\{L_1^2\zeta(s)+\zeta(2s)\}.$$

PROOF : By Lemma 3.2(ii) and Theorem 2.1,

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{\zeta(s;i)}{i^s} &= \zeta(s,s)+\zeta(2s) \ \text{(taking } m=n=s)\\ &= \frac{L_2^2}{2}+\zeta(2s)\\ &= \frac{1}{2}\{L_1^2\zeta(s)+\zeta(2s)\}.\square\end{aligned}$$

PROPOSITION 3.2 — For any integer $s > 1$,

$$\sum_{i=1}^{\infty}\frac{H_{i,s}}{i^s}\zeta(s;i)=\frac{1}{3}\{L_2^3\zeta(s)+5L_1^3\zeta(2s)+\zeta(3s)\}.$$

PROOF : We have,

$$\zeta(s,s,s) = \frac{1}{3}\{L_2^3\zeta(s)-L_1^3\zeta(2s)+\zeta(3s)\} \tag{6}$$

Again,

$$\begin{aligned}\zeta(s,s,s) &= \sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=i+1}^{\infty}\frac{1}{j^s}\sum_{k=j+1}^{\infty}\frac{1}{k^s}\\ &= \sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=i+1}^{\infty}\frac{1}{j^s}(\zeta(s)-H_{j,s})\\ &= \sum_{i=1}^{\infty}\frac{1}{i^s}\left(\sum_{j=i+1}^{\infty}\frac{\zeta(s)}{j^s}-\sum_{j=i+1}^{\infty}\frac{H_{j,s}}{j^s}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \frac{1}{i^s} \left[\zeta(s)(\zeta(s) - H_{i,s}) - \left(\sum_{j=1}^{\infty} \frac{H_{j,s}}{j^s} - \sum_{j=1}^{i} \frac{H_{j,s}}{j^s} \right) \right] \\
&= (\zeta(s))^3 - \zeta(s) \sum_{i=1}^{\infty} \frac{\zeta(s;i)}{i^s} - \zeta(s) \sum_{j=1}^{\infty} \frac{\zeta(s;j)}{j^s} + \sum_{i=1}^{\infty} \frac{1}{i^s} \sum_{j=1}^{i} \frac{H_{j,s}}{j^s} \\
&= (\zeta(s))^3 - 2\zeta(s) \sum_{i=1}^{\infty} \frac{\zeta(s;i)}{i^s} + \sum_{i=1}^{\infty} \frac{1}{i^s} \sum_{j=1}^{i} \frac{H_{j,s}}{j^s}
\end{aligned}
$$

Therefore,

$$
\sum_{i=1}^{\infty} \frac{1}{i^s} \sum_{j=1}^{i} \frac{H_{j,s}}{j^s} = \zeta(s,s,s) + 2\zeta(s) \sum_{i=1}^{\infty} \frac{\zeta(s;i)}{i^s} - (\zeta(s))^3
$$

Using Proposition 3.1, Lemma 3.3(i) and equation (6),

$$
\begin{aligned}
\sum_{i=1}^{\infty} \frac{H_{i,s}}{i^s} \zeta(s;i) &= \frac{1}{3}\{L_2^3\zeta(s) - L_1^3\zeta(2s) + \zeta(3s)\} + 2\zeta(s)\frac{1}{2}\{L_1^2\zeta(s) + \zeta(2s)\} \\
&\quad -(\zeta(s))^3 \\
&= \frac{1}{3}\{L_2^3\zeta(s) - L_1^3\zeta(2s) + \zeta(3s) + 3(\zeta(s))^3 + 3\zeta(s)\zeta(2s) - 3(\zeta(s))^3\} \\
&= \frac{1}{3}\{L_2^3\zeta(s) - L_1^3\zeta(2s) + \zeta(3s) + 6L_1^3\zeta(2s)\} \\
&= \frac{1}{3}\{L_2^3\zeta(s) + 5L_1^3\zeta(2s) + \zeta(3s)\}. \square
\end{aligned}
$$

Proposition 3.3 — For any integer $s > 1$,

$$
\sum_{i=1}^{\infty} \frac{\zeta(s; i - \frac{1}{2})}{(2i-1)^s} = 2^{s-1} \left\{ \left(1 - \frac{1}{2^s}\right)^2 L_1^2\zeta(s) + \left(1 - \frac{1}{2^{2s}}\right)\zeta(2s) \right\}.
$$

PROOF :

$$
\begin{aligned}
\sum_{i=1}^{\infty} \frac{\zeta(s; i - \frac{1}{2})}{(2i-1)^s} &= \sum_{i=1}^{\infty} \frac{1}{(2i-1)^s} \sum_{j=0}^{\infty} \frac{1}{(j + i - \frac{1}{2})^s} \\
&= 2^s \sum_{i=1}^{\infty} \frac{1}{(2i-1)^s} \sum_{j=0}^{\infty} \frac{1}{(2j + 2i - 1)^s} \\
&= 2^s \sum_{i=1}^{\infty} \frac{1}{(2i-1)^{2s}} + 2^s \sum_{i=1}^{\infty} \frac{1}{(2i-1)^s} \sum_{j=i+1}^{\infty} \frac{1}{(2j-1)^s}. \\
&= 2^s \left(1 - \frac{1}{2^{2s}}\right)\zeta(2s) + 2^s \left(1 - \frac{1}{2^s}\right)^2 \zeta(s)^2 - 2^s \sum_{i=1}^{\infty} \frac{H'_{2i-1,s}}{(2i-1)^s}.
\end{aligned}
$$

Using Lemma $3.2(iii)$, we have

$$\sum_{i=1}^{\infty}\frac{\zeta(s;i-\frac{1}{2})}{(2i-1)^s} = 2^{s-1}\left\{\left(1-\frac{1}{2^s}\right)^2 L_1^2\zeta(s)+\left(1-\frac{1}{2^{2s}}\right)\zeta(2s)\right\}.$$

Proposition 3.4 — For any integer $s>1$,

$$\sum_{i=1}^{\infty}\frac{H'_{2i-1,s}}{(2i-1)^s}\zeta\left(s;i-\frac{1}{2}\right) = \frac{2^s}{3}\left[\left(1-\frac{1}{2^s}\right)^3 L_2^3\zeta(s)+\left(1-\frac{1}{2^s}\right)^2\left(5+\frac{1}{2^s}\right)L_1^3\zeta(2s)+\right.$$
$$\left.\left(1-\frac{1}{2^{3s}}\right)\zeta(3s)\right]$$

PROOF : We have

$$\sum_{i=1}^{\infty}\frac{1}{(2i-1)^s}\sum_{j=1}^{i}\frac{H'_{2j-1,s}}{(2j-1)^s} = \sum_{i=1}^{\infty}\frac{1}{(2i-1)^s}\left\{\sum_{j=1}^{\infty}\frac{H'_{2j-1,s}}{(2j-1)^s}-\sum_{j=i+1}^{\infty}\frac{H'_{2j-1,s}}{(2j-1)^s}\right\}$$
$$= 2^{-s}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^s}\sum_{j=1}^{\infty}\frac{\zeta(s,j-\frac{1}{2})}{(2j-1)^s}-\sum_{i=1}^{\infty}\frac{1}{(2i-1)^s}\sum_{j=i+1}^{\infty}\frac{1}{(2j-1)^s}$$
$$\left\{\sum_{t=1}^{\infty}\frac{1}{(2t-1)^s}-\sum_{t=j+1}^{\infty}\frac{1}{(2t-1)^s}\right\}$$

$$= 2^{-s}\left(1-\frac{1}{2^s}\right)\zeta(s)\sum_{j=1}^{\infty}\frac{\zeta(s,j-\frac{1}{2})}{(2j-1)^s}-\left(1-\frac{1}{2^s}\right)^3\zeta(s)^3+\left(1-\frac{1}{2^s}\right)\zeta(s)$$
$$\sum_{i=1}^{\infty}\frac{H'_{2i-1,s}}{(2i-1)^s}+\sum_{i=1}^{\infty}\frac{1}{(2i-1)^s}\sum_{j=i+1}^{\infty}\frac{1}{(2j-1)^s}\sum_{t=j+1}^{\infty}\frac{1}{(2t-1)^s}$$
$$= 2^{-s+1}\left(1-\frac{1}{2^s}\right)\zeta(s)\sum_{j=1}^{\infty}\frac{\zeta(s,j-\frac{1}{2})}{(2j-1)^s}-\left(1-\frac{1}{2^s}\right)^3\zeta(s)^3+$$
$$2^{-3s}\zeta(s,s,s;-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$$

Using Proposition 3.3, Lemma $3.3(ii)$ and Lemma 3.1,

$$\sum_{i=1}^{\infty}\frac{H'_{2i-1,s}}{(2i-1)^s}\zeta\left(s;i-\frac{1}{2}\right) = 2^s\left[\left(1-\frac{1}{2^s}\right)\zeta(s)\left\{\left(1-\frac{1}{2^s}\right)^2\zeta(s)^2+\left(1-\frac{1}{2^{2s}}\right)\zeta(2s)\right\}\right.$$
$$-\left(1-\frac{1}{2^s}\right)^3\zeta(s)^3+\frac{1}{6}\left\{\left(1-\frac{1}{2^s}\right)^3\zeta(s)^3-3\left(1-\frac{1}{2^s}\right)\right.$$

$$\begin{aligned}
&\left(1-\frac{1}{2^{2s}}\right)\zeta(s)\zeta(2s)+2\left(1-\frac{1}{2^{3s}}\right)\zeta(3s)\Big\}\Big] \\
=\ & \frac{2^s}{6}\left[\left(1-\frac{1}{2^s}\right)^3\zeta(s)^3+3\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{2^{2s}}\right)\zeta(s)\zeta(2s)\right. \\
&\left.+2\left(1-\frac{1}{2^{3s}}\right)\zeta(3s)\right] \\
=\ & \frac{2^s}{6}\left[2\left(1-\frac{1}{2^s}\right)^3 L_2^3\zeta(s)+2\left(1-\frac{1}{2^s}\right)^3\zeta(s)\zeta(2s)+\right. \\
&\left.3\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{2^{2s}}\right)\zeta(s)\zeta(2s)+2\left(1-\frac{1}{2^{3s}}\right)\zeta(3s)\right] \\
=\ & \frac{2^s}{3}\left[\left(1-\frac{1}{2^s}\right)^3 L_2^3\zeta(s)+\left(1-\frac{1}{2^s}\right)^2\left(5+\frac{1}{2^s}\right)L_1^3\zeta(2s)+\right. \\
&\left.\left(1-\frac{1}{2^{3s}}\right)\zeta(3s)\right].\square
\end{aligned}$$

Before concluding the article, we prove the next identity, followed by a corollary.

Proposition 3.5 — For any integer $s>1$,

$$\sum_{i=1}^{\infty}\frac{S_{s,s}^i}{i^s}\zeta(s;i) = \frac{1}{4}\{L_3^4\zeta(s)+11L_2^4\zeta(2s)+L_1^4\zeta(3s)+\zeta(4s)+6(\zeta(2s))^2\}$$

PROOF : We have,

$$\zeta(s,s,s,s) = \frac{1}{4}\{L_3^4\zeta(s)-L_2^4\zeta(2s)+L_1^4\zeta(3s)-\zeta(4s)\}. \tag{7}$$

Let $A=\frac{1}{2}(L_1^2\zeta(s)+\zeta(2s))$ and $B=\frac{1}{3}\{L_2^3\zeta(s)+5L_1^3\zeta(2s)+\zeta(3s)\}$. Now,

$$\begin{aligned}
\zeta(s,s,s,s) &= \sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=i+1}^{\infty}\frac{1}{j^s}\sum_{t=j+1}^{\infty}\frac{1}{t^s}\sum_{k=t+1}^{\infty}\frac{1}{k^s} \\
&= \sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=i+1}^{\infty}\frac{1}{j^s}\sum_{t=j+1}^{\infty}\frac{1}{t^s}(\zeta(s)-H_{t,s}) \\
&= \sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=i+1}^{\infty}\frac{1}{j^s}\left[\sum_{t=1}^{\infty}\frac{(\zeta(s)-H_{t,s})}{t^s}-\sum_{t=1}^{j}\frac{(\zeta(s)-H_{t,s})}{t^s}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=i+1}^{\infty}\frac{1}{j^s}\left[(\zeta(s))^2 - A + \sum_{t=1}^{j}\frac{H_{t,s}}{t^s} - H_{j,s}\zeta(s)\right]\\
&= \sum_{i=1}^{\infty}\frac{1}{i^s}\left[((\zeta(s))^2 - A)(\zeta(s) - H_{i,s}) + \sum_{j=i+1}^{\infty}\frac{1}{j^s}\sum_{t=1}^{j}\frac{H_{t,s}}{t^s} - \sum_{j=i+1}^{\infty}\frac{H_{j,s}}{j^s}\zeta(s)\right]\\
&= ((\zeta(s))^2 - A)^2 + \sum_{i=1}^{\infty}\frac{1}{i^s}\left[B - \sum_{j=1}^{i}\frac{1}{j^s}\sum_{t=1}^{j}\frac{H_{t,s}}{t^s} - \zeta(s)A + \sum_{j=1}^{i}\frac{H_{j,s}}{j^s}\zeta(s)\right]\\
&= ((\zeta(s))^2 - A)^2 + 2B\zeta(s) - (\zeta(s))^2A - \sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=1}^{i}\frac{1}{j^s}\sum_{t=1}^{j}\frac{H_{t,s}}{t^s}
\end{aligned}
$$

Therefore,

$$
\sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{j=1}^{i}\frac{1}{j^s}\sum_{t=1}^{j}\frac{H_{t,s}}{t^s} = ((\zeta(s))^2 - A)^2 + 2B\zeta(s) - (\zeta(s))^2A - \zeta(s,s,s,s).
$$

Thus, using Propositions 3.1 and 3.2, Lemma 3.4 and equation (7),

$$
\begin{aligned}
\sum_{i=1}^{\infty}\frac{S^i_{s,s}}{i^s}\zeta(s;i) &= ((\zeta(s))^2 - A)^2 + 2B\zeta(s) - (\zeta(s))^2A - \zeta(s,s,s,s)\\
&= (\zeta(s))^4 - 3(\zeta(s))^2A + A^2 + 2B\zeta(s) - \zeta(s,s,s,s)
\end{aligned}
$$

$$
\begin{aligned}
&= (\zeta(s))^4 - \frac{3}{2}(\zeta(s))^2(L_1^2\zeta(s) + \zeta(2s)) + \frac{1}{4}(L_1^2\zeta(s) + \zeta(2s))^2\\
&\quad + \frac{2}{3}\zeta(s)\{L_2^3\zeta(s) + 5L_1^3\zeta(2s) + \zeta(3s)\}\\
&\quad - \frac{1}{4}\{L_3^4\zeta(s) - L_2^4\zeta(2s) + L_1^4\zeta(3s) - \zeta(4s)\}\\
&= \frac{1}{4}\{L_3^4\zeta(s) + 11L_2^4\zeta(2s) + L_1^4\zeta(3s) + \zeta(4s) + 6(\zeta(2s))^2\}.
\end{aligned}
$$

Corollary 3.1 — For $k \in \mathbb{N}$,

$$
\begin{aligned}
\sum_{i=1}^{\infty}\frac{S^i_{2k,2k}}{i^{2k}}\zeta(2k;i) &= \frac{(2\pi)^{8k}}{8}\left[(-1)^{4k-1}\frac{B_{8k}}{(8k)!} + (-1)^{4k-2}\frac{2B_{2k}B_{6k}}{3(2k)!(6k)!} + (-)^{4k-2}\left(\frac{B_{4k}}{2(4k)!}\right)^2\right.\\
&\quad\left. + (-1)^{4k-3}\left(\frac{B_{2k}}{2(2k)!}\right)^2\frac{B_{4k}}{(4k)!} + (-1)^{4k-4}\frac{1}{3}\left(\frac{B_{2k}}{2(2k)!}\right)^4\right].
\end{aligned}
$$

PROOF : Taking $s = 2k$ in Proposition 3.5 and using (1), we get the result.

We conclude by stating the following result which can be easily proved by using the techniques of the proof of Lemma 3.4 and Proposition 3.4:

Proposition 3.6 — If

$$S'^{2n-1}_{m,k} = \sum_{i=1}^{n} \frac{H'_{2i-1,m}}{(2i-1)^k},$$

then

(i) For $m, n, k > 1$

$$\sum_{i=1}^{\infty} \frac{1}{(2i-1)^k} \sum_{j=1}^{i} \frac{1}{(2j-1)^m} \sum_{t=1}^{j} \frac{H'_{2t-1,n}}{(2t-1)^m} = \sum_{i=1}^{\infty} \frac{S'^{2i-1}_{n,m}}{(2i-1)^m} \zeta\left(k; i - \frac{1}{2}\right)$$

(ii) For any integer $s > 1$,

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{S'^{2i-1}_{s,s}}{(2i-1)^s} \zeta\left(s; i - \frac{1}{2}\right) = \frac{2^s}{8} \Bigg[& 2\left(1 - \frac{1}{2^{4s}}\right)\zeta(4s) + \frac{8}{3}\left(1 - \frac{1}{2^s}\right) \\ & \left(1 - \frac{1}{2^{3s}}\right)\zeta(s)\zeta(3s) + \left(1 - \frac{1}{2^{2s}}\right)^2 \zeta(2s)^2 \\ & +2\left(1 - \frac{1}{2^s}\right)^2 \left(1 - \frac{1}{2^{2s}}\right)\zeta(s)^2\zeta(2s) + \\ & \frac{1}{3}\left(1 - \frac{1}{2^s}\right)^4 \zeta(s)^4 \Bigg]\end{aligned}$$

and hence at $s = 2k$,

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{S'^{2i-1}_{2k,2k}}{(2i-1)^{2k}} \zeta\left(2k; i - \frac{1}{2}\right) = \frac{2^{10k}\pi^{8k}}{8} \Bigg[& (-1)^{4k-1}\left(1 - \frac{1}{2^{8k}}\right)\frac{B_{8k}}{(8k)!} + (-1)^{4k-2}\left(1 - \frac{1}{2^{2k}}\right) \\ & \left(1 - \frac{1}{2^{6k}}\right)\frac{2B_{2k}B_{6k}}{3(2k)!(6k)!} + (-)^{4k-2}\left(1 - \frac{1}{2^{4k}}\right)^2 \left(\frac{B_{4k}}{2(4k)!}\right)^2 \\ & +(-1)^{4k-3}\left(1 - \frac{1}{2^{2k}}\right)^2 \left(1 - \frac{1}{2^{4k}}\right)\left(\frac{B_{2k}}{2(2k)!}\right)^2 \frac{B_{4k}}{(4k)!} + \\ & \frac{(-1)^{4k-4}}{3}\left(1 - \frac{1}{2^{2k}}\right)^4 \left(\frac{B_{2k}}{2(2k)!}\right)^4 \Bigg].\end{aligned}$$

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