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THE OPERATOR-VALUED PARALLELISM AND NORM-PARALLELISM IN MATRICES

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Let $\mathcal{H}$ be a Hilbert space, and let $K(\mathcal{H})$ be the C^*-algebra of compact operators on $\mathcal{H}$. In this paper, we present some characterizations of the norm-parallelism for elements of a Hilbert $K(\mathcal{H})$-module by employing the Birkhoff-James orthogonality. Among other things, we present a characterization of transitive relation of the norm-parallelism for elements in a certain Hilbert $K(\mathcal{H})$-module. We also give some characterizations of the Schatten p-norms and the operator norm-parallelism for matrices.

Key words : Hilbert C^*-module; orthonormal basis; parallelism; Schatten p-norm.

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1. Introduction and Preliminaries

Throughout the paper, let $(\mathcal{H}, [\cdot,\cdot])$ be a Hilbert space. Let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote the C^*-algebras of all bounded linear operators and compact linear operators on $\mathcal{H}$, respectively. The numerical range of an operator $T \in B(\mathcal{H})$ is defined by $W(T) = \{[T\xi,\xi] : \quad \xi \in \mathcal{H}, \quad \|\xi\| = 1\}$. Using the Gelfand-Naimark theorem, we identify any C^*-algebra $\mathcal{A}$ as a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. An (left) inner product $\mathcal{A}$-module is a left $\mathcal{A}$-module $\mathcal{E}$ equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle$, which is linear in the first variable and conjugate linear in the second and satisfies (i) $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ if and only if $x = 0$, (ii) $\langle ax,y\rangle = a\langle x,y\rangle$, (ii) $\langle x,y\rangle^* = \langle y,x\rangle$ for all $x,y \in \mathcal{E}, \quad a \in \mathcal{A}$. If $\mathcal{E}$ is complete with respect to the norm defined by $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$, then $\mathcal{E}$ is called a Hilbert

$\mathcal{A}$-module. Obviously, every Hilbert space is a Hilbert $\mathbb{C}$-module. Further, every C^*-algebra $\mathcal{A}$ can be regarded as a Hilbert C^*-module over itself under the inner product $\langle a, b\rangle = ab^*$. More details on the theory of Hilbert C^*-modules can be found, for example, in [11].

A projection $E \in B(\mathcal{H})$ is minimal if $EB(\mathcal{H})E = \mathbb{C}E$. For $\xi, \eta \in \mathcal{H}$, let $\xi \otimes \eta$ denote the corresponding elementary operator $(\xi \otimes \eta)(\zeta) = [\zeta, \eta]\xi$. It is easy to see that $\|\xi \otimes \eta\| = \|\xi\|\|\eta\|$. If $\xi \in \mathcal{H}$ is a unit vector, then $\xi \otimes \xi$ is a minimal projection. Conversely, any minimal projection is of this form. The set of all minimal projections in $K(\mathcal{H})$ is denoted by P_m. A system $\{x_\alpha\}_{\alpha\in\Lambda}$ in a Hilbert $K(\mathcal{H})$-module $\mathcal{E}$ is orthonormal if (i) $\|x_\alpha\| = 1$ for all $\alpha \in \Lambda$, (ii) $\langle x_\alpha, x_\beta\rangle = 0$ for all $\alpha \neq \beta$, (iii) $\langle x_\alpha, x_\alpha\rangle$ is a minimal projection for all $\alpha \in \Lambda$. An orthonormal system is said to be an orthonormal basis for $\mathcal{E}$ if it generates a dense submodule of $\mathcal{E}$. The orthogonal dimension of $\mathcal{E}$ is defined as the cardinal number of any of its orthonormal bases; see [2, 6, 13].

Let $(X, \|\cdot\|)$ be a complex normed space, and let $x, y \in X$. We say that x is orthogonal to y in the Birkhoff-James sense, denoted by $x \perp_B y$, if

$$\|x\| \leq \|x + \alpha y\| \quad \text{for} \quad \text{all} \quad \alpha \in \mathbb{C}.$$

It is easy to see that the Birkhoff-James orthogonality is equivalent to the usual orthogonality in the case when X is an inner product space. For other definitions and more properties of the Birkhoff-James orthogonality in Hilbert C^*-modules; see [1, 3-5, 7-9, 15].

Let $x, y \in X$, we say that x is norm-parallel to y, denoted by $x\|y$, if

$$\|x + \lambda y\| = \|x\| + \|y\| \quad \text{for} \quad \text{some} \quad \lambda \in \mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}.$$

For more details one can see [14, 18].

Clearly, two elements of a Hilbert space are norm-parallel if they are linearly dependent. In the case of normed linear spaces, two linearly dependent vectors are norm-parallel, but the converse is false in general. Notice that the norm parallelism is symmetric and $\mathbb{R}$-homogenous, but not transitive (i.e., $x\|y$ and $y\|z$ do not imply $x\|z$ in general; see [19, Example 2.7], unless X is smooth at y; see [16, Theorem 3.1]). Some characterizations of the norm-parallelism for operators and elements of Hilbert C^*-modules were given in [16-19].

Some characterizations of the norm-parallelism for elements of a Hilbert $K(\mathcal{H})$-module are given in Section 2. Let $\mathcal{F}$ be a Hilbert C^*-module, and let $x, y \in \mathcal{F}$. Zamani and Moslehian [19, Theorem 2.3] proved that $x\|y$ if and only if there exist a sequence of unit vectors ξ_n in a Hilbert space $\mathcal{H}$ and

$\lambda \in \mathbb{T}$ such that

$$\lim_{n\to\infty} \mathrm{Re}[\langle x, \lambda y\rangle \xi_n, \xi_n] = \|x\|\,\|y\|\,.$$

Suppose that $\mathcal{E}$ is a Hilbert $K(\mathcal{H})$-module, that $x, y \in \mathcal{E}$, and that $\langle x, x\rangle$ is a minimal projection in P_m. We show that $x\|y$ if and only if there exists a unit vector $\xi \in \mathcal{H}$ such that

$$|[\langle x, y\rangle \xi, \xi]| = \|y\|\,.$$

Bakić and Guljaš [2] proved that every Hilbert $K(\mathcal{H})$-module $\mathcal{E}$ has an orthonormal basis. Moreover there exists an orthonormal basis $\{x_\alpha\}_{\alpha\in\Lambda}$ for $\mathcal{E}$ such that $\langle x_\alpha, x_\alpha\rangle = E$ for all $\alpha \in \Lambda$ and for some minimal projection $E \in P_m$; see [2, Remark 4]. In the rest of paper, $\{x_\alpha\}^{\xi}_{\alpha\in\Lambda}$ denotes the orthonormal basis of $\mathcal{E}$ such that $\langle x_\alpha, x_\alpha\rangle = E = \xi \otimes \xi$ for all $\alpha \in \Lambda$ and for some unit vector $\xi \in \mathcal{H}$. We show that there is no element $0 \neq x \in \mathcal{E}$ such that $x\|x_\alpha$ for all $\alpha \in \Lambda$.

In Section 3, we use some techniques of [12] to characterize matrix pairs which are norm-parallelism with respect to the Schatten p-norms. Let $\|\cdot\|_\nu$ be an operator norm on $\mathcal{M}_{n\times n}$ induced by the vector norm ν on $\mathcal{M}_{n\times n}$. In [3], Bhatia, and $\check{S}$emrl conjectured that for $A, B \in \mathcal{M}_{n\times n}$ if $A \perp_B B$ in $\|\cdot\|_\nu$, then $Ay \perp_B By$ for some unit vector $y \in \mathcal{M}_{n\times 1}$ with $\nu(y) = 1$ such that $\nu(Ay) = \|A\|_\nu$. Li and Schneider in [12] proved that above conclusion is not true in general.

We show that if $Ay\|By$ in $\nu(\cdot)$ for some unit vector $y \in \mathcal{M}_{n\times 1}$ with $\nu(y) = 1$ such that $\nu(Ay) = \|A\|_\nu$, then $A\|B$ in $\|\cdot\|_\nu$, but by [12, Proposition 4.2 and Example 4.3], the converse is not true in general.

2. NORM-PARALLELISM IN HILBERT $\mathbb{K}(\mathcal{H})$-MODULES

Throughout, $\mathcal{E}$ denotes a Hilbert $K(\mathcal{H})$-module. We begin with a characterization of the norm-parallelism for elements of $\mathcal{E}$. Let $\mathcal{F}$ be a Hilbert C^*-module and let $x, y \in \mathcal{F}$. Zamani and Moslehian [19, Theorem 2.3] proved that $x\|y$ if and only if there exist a sequence of unit vectors ξ_n in a Hilbert space $\mathcal{H}$ and $\lambda \in \mathbb{T}$ such that

$$\lim_{n\to\infty} \mathrm{Re}[\langle x, \lambda y\rangle \xi_n, \xi_n] = \|x\|\,\|y\|\,.$$

Now suppose that $x, y \in \mathcal{E}$. Following we show that $x\|y$ if and only if there exists a unit vector $\xi \in \mathcal{H}$ such that $|[\langle x, x\rangle\langle x, y\rangle \xi, \xi]| = \|x\|^3\|y\|$.

***Theorem* 2.1** — *Let $x, y \in \mathcal{E}$. Then the following statements are equivalent:*

(i) $x\|y$.

(ii) There exists a unit vector $\xi \in \mathcal{H}$ *such that*

$$|[\langle x, x\rangle\langle x, y\rangle\xi, \xi]| = \|x\|^3\|y\|.$$

PROOF $(i) \Rightarrow (ii)$: Let $x\|y$. We may assume that $x \neq 0$. From [18, Theorem 4.7] $\langle x, x\rangle\|\langle x, y\rangle$ and $\|\langle x, y\rangle\| = \|x\|\|y\|$. Theorem 2.4 of [19] states that there exists $\lambda \in \mathbb{T}$ such that

$$\langle x, x\rangle \perp_B \|\langle x, y\rangle\|\langle x, x\rangle + \lambda\|x\|^2\langle x, y\rangle.$$

By the proof of [17, Theorem 2.10], there exists a unit vector $\xi \in \mathcal{H}$ such that $\|\langle x, x\rangle\xi\| = \|\langle x, x\rangle\|$ and

$$\left[\Big(\|\langle x, y\rangle\|\langle x, x\rangle + \lambda\|x\|^2\langle x, y\rangle\Big)\xi, \langle x, x\rangle\xi\right] = 0.$$

Thus $|[\langle x, x\rangle\langle x, y\rangle\xi, \xi]| = \|x\|^2\|\langle x, y\rangle\| = \|x\|^3\|y\|$.

$(ii) \Rightarrow (i)$: Suppose that (ii) holds. Then there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\|x\|^3\|y\| = |[\langle x, x\rangle\langle x, y\rangle\xi, \xi]| \leq \|x\|^2\|\langle x, y\rangle\| \leq \|x\|^2\|x\|\|y\|.$$

Hence $\|\langle x, y\rangle\| = \|x\|\|y\|$ and

$$|[\langle x, x\rangle\langle x, y\rangle\xi, \xi]| = \|x\|^2\|\langle x, y\rangle\| = \|\langle x, x\rangle\|\|\langle x, y\rangle\|.$$

By [17, Theorem 2.10], we have $\langle x, x\rangle\|\langle x, y\rangle$. Then [18, Theorem 4.7] implies that $x\|y$.

Lemma 2.2 — [19, Corollary 2.5]. Let $x, y \in \mathcal{E}$. If $x\|y$, then there exists $\lambda \in \mathbb{T}$ such that

$x \perp_B (\|y\|\langle x, x\rangle x + \lambda\|x\|\langle y, x\rangle x)$ and $y \perp_B (\|x\|\langle y, y\rangle y + \lambda\|y\|\langle y, x\rangle y)$.

Now, we apply Lemma 2.2 to obtain the following characterization of the norm-parallelism for elements of a Hilbert $K(\mathcal{H})$-module.

Corollary 2.3 — Let $x, y \in \mathcal{E}$ and $\langle x, x\rangle$ be idempotent. Then the following statements are equivalent:

(i) $x\|y$;

(ii) $x \perp_B (\|y\|\langle x, x\rangle x + \lambda\|x\|\langle y, x\rangle x)$ and $y \perp_B (\|x\|\langle y, y\rangle y + \lambda\|y\|\langle y, x\rangle y)$

for some $\lambda \in \mathbb{T}$.

$(i) \Rightarrow (ii)$: This implication follows immediately from Lemma 2.2.

$(ii) \Rightarrow (i)$: Suppose that (ii) holds. By [1, Lemma 2.1 part(5)], we have

$$\langle x, x\rangle \perp_B \Big(\|y\|\langle x, x\rangle^2 + \bar{\lambda}\|x\|\langle x, x\rangle\langle x, y\rangle\Big).$$

By [17, Theorem 2.10], there exists a vector ξ in $\mathcal{H}$ such that

$$\Big[\Big(\|y\|\langle x, x\rangle^2 + \bar{\lambda}\|x\|\langle x, x\rangle\langle x, y\rangle\Big)\xi, \langle x, x\rangle\xi\Big] = 0$$

and

$$\|\langle x, x\rangle\xi\| = \|x\|^2.$$

Since $\langle x, x\rangle$ is idempotent, we reach

$$|[\langle x, x\rangle\langle x, y\rangle\xi, \xi]| = \|x\|^3\|y\|.$$

From $(ii) \Rightarrow (i)$ of Theorem 2.1, we obtain $x\|y$. □

Recall that if x is an element of $\mathcal{E}$ such that $\langle x, x\rangle$ is a minimal projection, then $\langle x, x\rangle x = x$; see [2]. Let $x, y \in \mathcal{E}$. If there exist a unit vector $\xi \in \mathcal{H}$ and $\lambda \in \mathbb{T}$ such that $\mathrm{Re}[\langle x, \lambda y\rangle\xi, \xi] = \|x\|\|y\|$, then [19, Theorem 2.3] implies that $x\|y$. The question is under which conditions the converse is true. When $\langle x, x\rangle$ is a minimal projection, by using Theorem 2.1, we get $x\|y$, if and only if there exists a unit vector $\xi \in \mathcal{H}$ such that $\Big|[\langle x, y\rangle\xi, \xi]\Big| = \|y\|$. Now we prove it by a different approach.

***Theorem* 2.4** — *Let $x, y \in \mathcal{E}$. If $\langle x, x\rangle$ is a minimal projection, then $x\|y$ if and only if there exists a unit vector $\xi \in \mathcal{H}$ such that*

$$\Big|[\langle x, y\rangle\xi, \xi]\Big| = \|y\|.$$

PROOF : Let $x\|y$. By Lemma 2.2, there exists a scalar μ in $\mathbb{T}$ such that

$$x \perp_B \Big(\|y\|\langle x, x\rangle x + \mu\|x\|\langle y, x\rangle x\Big).$$

Since $\langle x, x\rangle$ is a minimal projection, by [1, Proposition 2.4], we have

$$\langle x, x\rangle\Big\langle\Big(\|y\|\langle x, x\rangle x + \mu\langle y, x\rangle x\Big), x\Big\rangle = 0,$$

and so

$$\|y\|\langle x,x\rangle + \mu\langle x,x\rangle\langle y,x\rangle\langle x,x\rangle = 0.$$

Hence

$$\langle x,x\rangle\langle x,y\rangle\langle x,x\rangle = -\bar{\mu}\|y\|\langle x,x\rangle. \tag{2.1}$$

Now, by knowing the fact that $\langle x,x\rangle = \xi\otimes\xi$ for some unit vector $\xi \in \mathcal{H}$, since $\xi\otimes\xi\langle x,y\rangle\xi\otimes\xi = [\langle x,y\rangle\xi,\xi]\xi\otimes\xi$, we obtain

$$\Big|[\langle x,y\rangle\xi,\xi]\Big|\|\xi\otimes\xi\| = \|y\|\|\xi\otimes\xi\|,$$

whence

$$\Big|[\langle x,y\rangle\xi,\xi]\Big| = \|y\|.$$

$(ii) \Rightarrow (i)$: Let $\Big|[\langle x,y\rangle\xi,\xi]\Big| = \|y\|$. Then there exists $\lambda \in \mathbb{T}$ such that

$$\mathrm{Re}[\langle x,\lambda y\rangle\xi,\xi] = \|x\|\,\|y\|\,.$$

From [19, Theorem 2.3], we get $x\|y$. □

In the following theorem we use some ideas of [19].

***Theorem* 2.5** — *Let $\{x_\alpha\}_{\alpha\in\Lambda}^{\xi}$ be an orthonormal basis for $\mathcal{E}$ such that as a Hilbert module is not of dimension one. Then there is no nonzero element $x \in \mathcal{E}$ such that $x\|x_\alpha$ for all $\alpha \in \Lambda$.*

PROOF : Let there be a nonzero element $x \in \mathcal{E}$ such that $x\|x_\alpha$ for all $\alpha \in \Lambda$. By [19, Corollary 2.5], there exists a sequence $\{\lambda_\alpha\}$ in $\mathbb{T}$ such that

$$x_\alpha \perp_B \Big(\|x\|\langle x_\alpha,x_\alpha\rangle x_\alpha + \lambda_\alpha\langle x_\alpha,x\rangle x_\alpha\Big) \quad (\alpha\in\Lambda). \tag{2.2}$$

Taking $E = \langle x_\alpha,x_\alpha\rangle$, since $Ex_\alpha = E$, equation (2.1) implies that

$$\langle x_\alpha,x\rangle E = E\langle x_\alpha,x\rangle E = -\overline{\lambda_\alpha}\|x\|E \quad (\alpha\in\Lambda).$$

Hence

$$(\langle x_\alpha,x\rangle E)^*(\langle x_\alpha,x\rangle E) = \lambda_\alpha\overline{\lambda_\alpha}\|x\|^2E^2 = \|x\|^2E \tag{2.3}$$

or equivalently

$$E\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle E = \|x\|^2 E \quad (\alpha \in \Lambda).$$

Therefore $[\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle\xi, \xi] = \|x\|^2$ and

$$[\langle x, x\rangle\xi, \xi] = \Big[\sum_{\alpha\in\Lambda}\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle\xi, \xi\Big] = \sum_{\alpha\in\Lambda}\|x\|^2.$$

Thus

$$\sum_{\alpha\in\Lambda}\|x\|^2 = [\langle x, x\rangle\xi, \xi] = \Big|[\langle x, x\rangle\xi, \xi]\Big| \leq \|\langle x, x\rangle\xi\| \cdot \|\xi\| \leq \|x\|^2,$$

which implies that $x = 0$, that is a contradiction. □

Now we have the following characterization of the transitive relation for elements of a certain Hilbert $K(\mathcal{H})$-module.

Corollary 2.6 — Let $x, y, z \in \mathcal{E}$ be such that $\mathcal{E}$ as a Hilbert module is of dimension one with orthonormal basis $\{y\}^{\xi}$. If $x\|y$ and $y\|z$, then $x\|z$.

PROOF : Let $x\|y$ and $y\|z$. By [19, Corollary 2.5], there exist λ, $\mu \in \mathbb{T}$ such that $y \perp_B (\|x\|\langle y, y\rangle y + \lambda\|y\|\langle y, x\rangle y)$ and $y \perp_B (\|z\|\langle y, y\rangle y + \mu\|y\|\langle z, y\rangle y)$. Taking $E = \langle y, y\rangle$. Equation (2.1) ensures that $E\langle x, y\rangle = -\bar{\lambda}\|x\|E$ and $\langle y, z\rangle E = -\bar{\mu}\|z\|E$. Then $E\langle x, y\rangle\langle y, z\rangle E = \bar{\lambda}\bar{\mu}\,\|x\|\,\|z\|\,E$. Therefore

$$\Big|[\langle x, z\rangle\xi, \xi]\Big| = \Big|[\langle x, y\rangle\langle y, z\rangle\xi, \xi]\Big| = \|x\|\,\|z\|\,.$$

That is $x\|z$.

Example 2.7 : Suppose that $\mathcal{H}$ is an infinite dimensional Hilbert space. Then $\mathcal{H}$ equipped with $\langle x, y\rangle = x \otimes y$ is a Hilbert $K(\mathcal{H})$-module and $\|x\|^2 = \|x \otimes x\|$ is the original norm on $\mathcal{H}$. We can assume that each unit vector $y \in \mathcal{H}$ gives an orthonormal basis $\{y\}^{y}$ for $\mathcal{H}$. Therefore the dimension of H as a Hilbert module is equal to one. Finally, from Corollary 2.6, we see that if $x\|y$ and $y\|z$, for some unit vector y, then

$$\Big|[\langle x, z\rangle y, y]\Big| = \|x\|\|z\|.$$

Therefore, $x\|z$.

3. Norm-Parallelism in Matrices

Let $\mathcal{M}_{n\times n}$ be the space of $n\times n$ complex matrices, let $A,B\in\mathcal{M}_{n\times n}$, and let $\|\cdot\|$ be any norm on $\mathcal{M}_{n\times n}$. We say that A is norm-parallel to B if

$$\|A+\lambda B\| = \|A\|+\|B\|$$

for some $\lambda\in\mathbb{T}$. We also consider the inner product $\langle A,B\rangle = \mathrm{tr}(AB^*)$. The dual norm of $\|\cdot\|$ is defined by

$$\|A\|^D = \max\{|\langle A,B\rangle| : \|B\| \ \leq 1\}.$$

The singular values of $A\in\mathcal{M}_{n\times n}$, which are the square roots of the eigenvalues of the matrix A^*A, are denoted by $s_1(A)\geq\cdots\geq s_n(A)$ and the Schatten p-norm of A is defined by

$$\|A\|_p = \left\{\sum_{i=1}^n s_i(A)^p\right\}^{\frac{1}{p}},$$

if $1\leq p<\infty$ and if $p=\infty,\quad \|A\|_p=\max\{s_j(A):\quad 1\leq j\leq n\}$; see [12]. If $A\in\mathcal{M}_{n\times n}$, then for a positive integer $1\leq k\leq n$, the Ky-Fan k-norm is defined by

$$\|A\|_{(k)} = \sum_{j=1}^k s_j(A),$$

where $s_1(A)\geq\cdots\geq s_n(A)\geq 0$, are the singular values of A. The cases when $k=1$ and $k=n$ correspond to the operator norm $\|\cdot\|_\infty$ and the trace norm $\|\cdot\|_1$, respectively. we have the following result.

Proposition 3.1 — Let $A,B\in\mathcal{M}_{n\times n}$. Then the following statement are equivalent:

(i) $A\|B$, in $\|\cdot\|_\infty$.

(ii) There exist unit vectors $x\in\mathcal{M}_{n\times 1}$ and $y\in\mathcal{M}_{n\times 1}$ such that $\|A\|_\infty = x^*Ay$ and $|x^*By| = \|B\|_\infty$.

(iii) For any $U\in\mathcal{M}_{n\times n}$ with orthonormal columns that form a basis for the eigenspace of AA^* corresponding to the largest eigenvalue and $V=\frac{1}{\|A\|_\infty}A^*U\in\mathcal{M}_{n\times n}$, it holds that

$$-\|B\|_\infty \in W(\lambda U^*BV)$$

for some $\lambda\in\mathbb{T}$.

PROOF : $(i) \Rightarrow (ii)$: Let $A\|B$. Thus

$$A \perp_B (\|B\|_\infty A + \lambda\|A\|_\infty B)$$

for some $\lambda \in \mathbb{T}$. By [12, Theorem 3.1], there exist unit vectors $x \in \mathcal{M}_{n\times 1}$ and $y \in \mathcal{M}_{n\times 1}$ such that $\|A\|_\infty = x^*Ay$ and $x^*(\|B\|_\infty A + \lambda\|A\|_\infty B)y = 0$. Thus we get $|x^*By| = \|B\|_\infty$.

$(ii) \Rightarrow (i)$: It is obvious.

$(i) \Rightarrow (iii)$: If $A\|B$, then

$$A \perp_B (\|B\|_\infty A + \lambda\|A\|_\infty B)$$

for some $\lambda \in \mathbb{T}$. By [12, Theorem 3.1], for any $U \in \mathcal{M}_{n\times n}$ with orthonormal columns that form a basis for the eigenspace of AA^* corresponding to the largest eigenvalue and $V = \frac{1}{\|A\|_\infty}A^*U \in \mathcal{M}_{n\times n}$, we have

$$\begin{aligned} 0 \in W(U^*(\|B\|_\infty A + \lambda\|A\|_\infty B)V) &= W(\|B\|_\infty\|A\|_\infty V^*V + \lambda\|A\|_\infty U^*BV) \\ &= W(\|B\|_\infty\|A\|_\infty I + \lambda\|A\|_\infty U^*BV). \end{aligned}$$

Thus there exists a unit vector $\zeta \in \mathcal{M}_{n\times 1}$ such that $\langle(\|B\|_\infty I + \lambda\|A\|_\infty U^*BV)\zeta, \zeta\rangle = 0$ or $\langle\lambda U^*BV\zeta, \zeta\rangle = -\frac{\|B\|_\infty}{\|A\|_\infty}$. Hence

$$-\|B\|_\infty \in W(\lambda U^*BV)$$

for some $\lambda \in \mathbb{T}$.

$(iii) \Rightarrow (i)$: It is obvious. □

Remark 3.1 — By the equivalence (i)⇔(ii) of Proposition 3.1, $A\|B$, in $\|\cdot\|_\infty$ if and only if there exist unit vectors $x \in \mathcal{M}_{m\times 1}$ and $y \in \mathcal{M}_{n\times 1}$ such that $\|A\|_\infty = x^*Ay$ and $|x^*By| = \|B\|_\infty$, this condition is equivalent to the fact that there is a unit vector $y \in \mathcal{M}_{n\times 1}$ such that $\|A\|_\infty = \langle Ay, Ay\rangle^{\frac{1}{2}}$ and $|\langle Ay, By\rangle| = \|A\|_\infty\|B\|_\infty$.

Remark 3.2 : Suppose that $A, B \in \mathcal{M}_{n\times n}$. If $A\|B$ in $\|\cdot\|_{(k)}$, then $A \perp_B (\|B\|_{(k)}A + \lambda\|A\|_{(k)}B)$. So, by [10, Remark 1], there exists a matrix $F \in \mathcal{M}_{n\times n}$ such that $\|F\|_\infty \leq 1, \quad \|F\|_1 \leq k, \quad \mathrm{tr}(F^*A) = \|A\|_{(k)}$ and $\mathrm{tr}\Big(F^*(\|B\|_{(k)}A + \lambda\|A\|_{(k)}B)\Big) = 0$. Hence $A\|B$ in $\|\cdot\|_{(k)}$ if and only if there exists a matrix $F \in \mathcal{M}_{n\times n}$ such that $\|F\|_\infty \leq 1, \quad \|F\|_1 \leq k, \quad \mathrm{tr}(F^*A) = \|A\|_{(k)}$ and $|\mathrm{tr}(F^*B)| = \|A\|_{(k)}$.

Now by [12, Theorem 3.2], we obtain the following result.

Corollary 3.2 — Let $1 < p < \infty$ and $A, B \in \mathcal{M}_{m\times n}$, where $A = DC$ for some positive matrices $D \in \mathcal{M}_{m\times m}$ and $C \in \mathcal{M}_{m\times n}$ with $CC^* = I_m$. Then $A\|B$, in $\|\cdot\|_p$ if and only if $|\mathrm{tr}(D^{p-1}CB^*)| = \frac{\|B\|_p\|D\|_p^p}{\|A\|_p}$.

PROOF : Let $A\|B$. Thus

$$A \perp_B (\|B\|_p A + \lambda\|A\|_p B)$$

for some $\lambda \in \mathbb{T}$. By [12, Theorem 3.2], $\mathrm{tr}(D^{p-1}C(\|B\|_p A + \lambda\|A\|_p B)^*) = 0$.

Thus $|\mathrm{tr}(D^{p-1}CB^*)| = \frac{\|B\|_p\|D\|_p^p}{\|A\|_p}$. The converse is obvious. □

As a consequence of [12, Theorem 3.3], we have the following result.

Remark 3.3 : Let $A, B \in \mathcal{M}_{n\times n}$ and $A\|B$ in $\|\cdot\|_1$. By a similar computation as previous theorem, we deduce that there exists $F \in \mathcal{M}_{n\times n}$ such that $\|F\|_\infty \leq 1$, $\mathrm{tr}(AF^*) = \|A\|_1$ and $|\mathrm{tr}(BF^*)| = \|B\|_1$.

Let ν be a norm on $\mathcal{M}_{n\times 1}$, and let $\|\cdot\|_\nu$ be the operator norm on $\mathcal{M}_{n\times n}$ induced by ν defined by

$$\|A\|_\nu = \max\{\nu(Ax) : x \in \mathcal{M}_{n\times 1}, \nu(x) \leq 1\}.$$

The dual norm of ν is defined by

$$\nu^D(x) = \max\{|\langle x, y\rangle| : y \in \mathcal{M}_{n\times 1}, \nu(y) \leq 1\}$$

and the dual norm of $\|\cdot\|_\nu$ is defined as

$$\|A\|_\nu^D = \max\{|\langle A, B\rangle| : B \in \mathcal{M}_{n\times n}, \|B\|_\nu \leq 1\}.$$

We say $F \in \mathcal{M}_{n\times n}$ is an extreme point of the unit norm ball of $\mathcal{M}_{n\times n}$ if only if

(i) $p = 1$ and $F = xy^*$ for some unit vectors $x \in \mathcal{M}_{n\times 1}$ and $y \in \mathcal{N}_{n\times 1}$,

(ii) $1 < p < \infty$ and $\|F\|_p = 1$,

(iii) $p = \infty$ and $FF^* = I_n$.

We have the following characterization of the norm parallelism for operator norms in $\mathcal{M}_{n\times n}$.

Proposition 3.3 — Let $\|\cdot\|_\nu$ be an operator norm on $\mathcal{M}_{n\times n}$ induced by the vector norm ν on $\mathcal{M}_{n\times n}$. Let $A, B \in \mathcal{M}_{n\times n}$. Denote by $\mathscr{E}$ and $\mathscr{E}^{\mathscr{D}}$ the set of extreme points of the unit norm balls of ν and ν^D, respectively. If

$$V(A) = \{xy^* : x \in \mathscr{E}^D, y \in \mathscr{E}, \langle A, xy^*\rangle = \|A\|_\nu\},$$

then $A\|B$ in $\|\cdot\|_\nu$ if and only if there exist k extreme points $x_1y_1^*,\dots,x_ky_k^*$ with $k\leq 3$ in the complex case and $k\leq 2$ in the real case, and positive numbers $t_1,\dots,t_k$ with $t_1+\cdots+t_k=1$ such that

$$|\sum_{j=1}^{k} t_j\langle B, x_jy_j^*\rangle| = \|B\|_\nu.$$

PROOF : Let $A\|B$. Then

$$A \perp_B (\|B\|_\nu A + \lambda\|A\|_\nu B)$$

for some $\lambda\in\mathbb{T}$. So by [12, Proposition 4.2], there exist k extreme points $x_1y_1^*,\dots,x_ky_k^*$ with $k\leq 3$ in the complex case and $k\leq 2$ in the real case, and positive numbers $t_1,\dots,t_k$ with $t_1+\cdots+t_k=1$ such that

$$\sum_{j=1}^{k} t_j\langle(\|B\|_\nu A+\lambda\|A\|_\nu B), x_jy_j^*\rangle = 0.$$

Thus $|\sum_{j=1}^{k} t_j\langle B, x_jy_j^*\rangle| = \|B\|_\nu$. The converse is obvious. □

Remark 3.4 : Suppose that ν is a norm on $\mathcal{M}_{n\times 1}$ and that $\|\cdot\|_\nu$ is the corresponding operator norm on $\mathcal{M}_{n\times n}$. Given $A,B\in\mathcal{M}_{n\times n}$. If there exists a vector $y\in\mathcal{M}_{n\times 1}$ with $\nu(y)=1$ such that $\nu(Ay)=\|A\|_\nu$ and $\nu\Big(Ay+\mu(\nu(By)Ay+\lambda\nu(Ay)By)\Big)\geq\nu(Ay)$ for all $\mu\in\mathbb{C}$, that is, $Ay\|By$. Then

$$\|A+\mu(\|B\|_\nu A+\lambda\|A\|_\nu B)\|_\nu \geq \nu\Big(Ay+\mu(\nu(By)Ay+\lambda\nu(Ay)By)\Big)\geq\nu(Ay)=\|A\|_\nu$$

for all for all $\mu\in\mathbb{C}$; hence $A\|B$. On the other hand [12, Proposation 4.2 and Example 4.3] show that the converse of the above result is not true, in general.

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