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# CONSERVATIVE DIFFERENCE SCHEME OF SOLITARY WAVE SOLUTIONS OF THE GENERALIZED REGULARIZED LONG-WAVE EQUATION

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Conservative difference scheme for the nonlinear dispersive generalized regularized long-wave (GRLW) equation is proposed. Existence of its difference solutions has been shown. It is proved by the discrete energy method that the difference scheme is uniquely solvable, unconditionally stable and the convergence is of second-order in the maximum norm. The particular case known as the modified regularized long-wave (MRLW) equation is also discussed numerically in details. Furthemore, three invariants of motion are evaluated to determine the conservation properties of the problem. Interaction of two and three solitary waves are shown. Some numerical examples are given in order to validate the theoretical results.

**Key words** : GRLW equation; difference scheme; conservation; existence; uniqueness; stability; convergence; MRLW equation; solitary waves.

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## 1. INTRODUCTION

One of the most famous model for the nonlinear partial differential equations which is called the regularized long-wave (RLW) equation

$$u_t - \mu u_{xxt} + u_x + \delta u u_x = 0, \tag{1.1}$$

where $\mu$ and $\delta$ are positive constants, was originally introduced to describe the behavior of the undular bore by Peregrine [1], and later by Benjamin *et al*. [2]. This equation is very important in physics media since it describes a phenomenon with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ionacoustic and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals. The RLW equation is solved by various methods such as finite difference methods, finite element methods including collocation method with quadratic B-splines, cubic B-splines and recently septic splines, meshless method, (see [3-20]). In addition, the RLW equation is a special case of the generalized long-wave (GRLW) equation

$$u_t - \mu u_{xxt} + u_x + \delta u^m u_x = 0, \tag{1.2}$$

where $m$ is a positive integer. The GRLW Eq. $(1.2)$ was first put forward as a model for small-amplitude long-waves on the surface of water in a channel by Peregrine [2], and later by Benjamin *et al*. [2]. In physical situations such as unidirectional waves propagating in a water channel, long crested waves in near-shore zones, and many other, the generalized regularized long-wave (GRLW) equation serves an alternative model to the KdV equation. The GRLW equation is studied by few authors, Zhang [21] with linearized difference scheme. Mokhtari and Mohammadi used Sinc-collocation [22], Kaya used a numerical simulation of solitary wave solutions [23], El-Danaf *et al*. used Adomian decomposition method (ADM) [24], Roshan used a Petrov-Galerkin method [25], Mohammadi and Mokhtari used the basis of a reproducing kernel space [26].

In this paper, we establish a conservative difference scheme for a model of nonlinear dispersive GRLW equation and prove that the scheme is unconditionally stable and convergent with the convergence of $O(h^2 + k^2)$ in the discrete $L^\infty$-norm. We assume that we are interested in solutions of $(1.2)$ that, for the range of time under consideration, are negligible outside an interval $0 \leq x \leq L$. Such solutions include the solitary waves of GRLW equation on the real line and by products of their interactions. Hence, by taking the interval $\Omega = [0, L]$ large enough in each numerical experiment, so that the solution remains sufficiently small at the endpoints, we can approximate in a satisfactory manner the generation and interaction of real line solitary waves for finite time intervals.

We consider the initial-value and $L$-periodic boundary-value problems for the GRLW equation, thus we seek a real-valued function $u(x,t)$, $L$-periodic in $x$, that satisfies

$$u_t - \mu u_{xxt} + u_x + \delta u^m u_x = 0, \quad (x,t) \in \Omega \times [0,T], \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

$$u(x,t) = u(x+L,t), \quad t \in [0,T], \tag{1.5}$$

where $u_0$ is a given $L$-periodic real function, $L$ is the period and $0 < T < \infty$.

In this article, we establish a nonlinear conservative difference scheme for the GRLW equation and prove that the scheme is unconditionally stable and convergent with the convergence of $O(h^2 + k^2)$ in the discrete $L^\infty$-norm. The remainder of the article is arranged as follows: In section 2, a nonlinear conservative difference scheme is derived. Conservations of discrete mass and discrete energy are discussed in section 3, the priori estimations for numerical solutions are studied in section 4. Existence of approximate solution is derived in section 5. Convergence, stability and uniqueness are proved in section 6. In the last section, some numerical experiments are presented to support our theoretical results. In addition an application for the MRLW equation is presented to test our scheme.

Throughout this article, $C$ denote a generic positive constant which is independent of the discretization parameters $h$ and $k$, but may have different values at different places.

## 2. A Nonlinear Conservative Difference Scheme

It is easy to check that the problem $(1.3) - (1.5)$ satisfies the following energy conservative property.

***Theorem* 1** — *Suppose that $u_0 \in H^1_{per}[0, L]$, then the solution of $(1.3) - (1.5)$ satisfies the energy conservation law:*

$$E(t) = \int_0^L u^2(x,t)dx + \mu \int_0^L u_x^2(x,t)dx = \int_0^L u^2(x,0)dx + \mu \int_0^L u_x^2(x,0)dx = E(0),$$

*for any $t \in [0, T]$.*

PROOF : Using the integration par part, and $(1.5)$ we obtain,

$$\int_0^L u_x u_{xt} dx = \Big[u u_{xt}\Big]_0^L - \int_0^L u u_{xxt} dx = -\int_0^L u u_{xxt} dx.$$

It follows from the above equality and (1.5) that

$$\frac{dE(t)}{dt} = 2\int_0^L u u_t dx + 2\mu \int_0^L u_x u_{xt} dx = 2\int_0^L u(u_t - \mu u_{xxt})dx.$$

From $(1.3)$ we get

$$u_t - \mu u_{xxt} = -u_x - \delta u^m u_x.$$

Therefore

$$\begin{aligned}\frac{dE(t)}{dt} &= -2\int_0^L u(u_x+\delta u^m u_x)dx\\ &= -2\int_0^L uu_x dx - 2\delta\int_0^L u^{m+1}u_x dx.\\ &= -\Big[u^2\Big]_0^L - \frac{2\delta}{m+2}\Big[u^{m+2}\Big]_0^L\\ &= 0.\end{aligned}$$

Therefore, $E(t)$ is a constant function, that is

$$E(t) = \int_0^L u^2(x,t)dx + \mu\int_0^L u_x^2(x,t)dx = E(0). \qquad \square$$

In this section, we propose a two-level nonlinear Crank-Nicolson finite difference scheme for the problem (1.3)-(1.5). For convenience, the following notations are used. For a positive integer $N$, let time-step $k=\frac{T}{N}$ , $t_n = nk$, $n = 0, 1, \dots, N$. Let space-step $h = \frac{L}{J}$, $x_j = jh$, $j = 0, \dots, J$, and let

$$\mathbb{R}^J_{per} = \{V = (V_j)_{j\in\mathbb{Z}} \,/\, V_j \in \mathbb{R} \text{ and } V_{j+J} = V_j, \quad j \in \mathbb{Z}\}.$$

For a function $V^n \in \mathbb{R}^J_{per}$ define the difference operators as:

$$(V_j^n)_x = \frac{V_{j+1}^n - V_j^n}{h}, \quad (V_j^n)_{\bar{x}} = \frac{V_j^n - V_{j-1}^n}{h}, \quad (V_j^n)_{\hat{x}} = \frac{V_{j+1}^n - V_{j-1}^n}{2h},$$

$$V_j^{n+\frac12} = \frac{V_j^{n+1}+V_j^n}{2}, \quad \left(V_j^{n+\frac12}\right)_{\hat{x}} = \frac{V_{j+1}^{n+1}+V_{j+1}^n - V_{j-1}^{n+1} - V_{j-1}^n}{4h}, \quad (V_j^n)_t = \frac{V_j^{n+1}-V_j^n}{k}.$$

For any function $V^n, W^n \in \mathbb{R}^J_{per}$, we introduce the discrete $L^2$ inner product in $\mathbb{R}^J_{per}$.

$$\langle V^n, W^n\rangle = h\sum_{j=1}^J V_j^n W_j^n,$$

and Sobolev norms (or seminorms)

$$\begin{aligned}||V^n|| &= \sqrt{h\sum_{j=1}^J V_j^2}\,, \quad ||V_x^n|| = \sqrt{h\sum_{j=1}^J (V_j^n)_x^2}\,, \quad ||V_{\hat{x}}^n|| = \sqrt{h\sum_{j=1}^J (V_j^n)_{\hat{x}}^2}\,,\\ &||V^n||_\infty = \max_{1\le j\le J}|V_j^n|.\end{aligned}$$

Denote as $H^m_{per}(\Omega)$ the periodic Sobolev space of order $m$. The essential of our difference scheme is that the nonlinear term of (1.3) is rewritten and discretized as

$$\delta u^m u_x = \frac{\delta}{m+2}\Big[u^m u_x + (u^{m+1})_x\Big] \approx \frac{\delta}{m+2}\Big[(U_j^{n+\frac12})^m(U_j^{n+\frac12})_{\hat{x}} + \big[(U_j^{n+\frac12})^{m+1}\big]_{\hat{x}}\Big],$$

and which is a second-order approximation around $(x_j = jh, t_n = nk)$. Therefore, we discretize the problem $(1.1) - (1.3)$ by the following Cranck-Nicolson type finite difference scheme. We approximate $u_j^n \in \mathbb{R}_{per}^J$, $u_j^n := u(x_j, t^n)$, by $U_j^n \in \mathbb{R}_{per}^J$.

$$(U_j^n)_t - \mu(U_j^n)_{x\bar{x}t} + (U_j^{n+\frac{1}{2}})_{\hat{x}} + \varphi(U_j^{n+\frac{1}{2}}, U_j^{n+\frac{1}{2}}) = 0\,, \quad j = 1, \ldots, J, \quad n = 0, \ldots, N-1, \tag{2.1}$$

$$U_j^n = U_{j+J}^n, \quad j = 1, \ldots, J, \quad n = 0, \ldots, N, \tag{2.2}$$

$$U_j^0 = u_0(x_j), \quad j = 1, \ldots, J, \tag{2.3}$$

where

$$\varphi(U_j^{n+\frac{1}{2}}, U_j^{n+\frac{1}{2}}) = \frac{\delta}{m+2}\Big[(U_j^{n+\frac{1}{2}})^m (U_j^{n+\frac{1}{2}})_{\hat{x}} + \big[\big(U_j^{n+\frac{1}{2}}\big)^{m+1}\big]_{\hat{x}}\Big].$$

Next, we need the following lemma, which is a consequence of the well-known formulas of summations by parts and related periodicity conditions.

*Lemma* 1 — For any grid functions $V, W \in \mathbb{R}_{per}^J$ we have

$$\big\langle V_x, W \big\rangle = -\big\langle V, W_{\bar{x}} \big\rangle, \tag{2.4}$$

$$\big\langle V_{\hat{x}}, W \big\rangle = -\big\langle V, W_{\hat{x}} \big\rangle. \tag{2.5}$$

Then, one has,

$$\big\langle V_{x\bar{x}}, V \big\rangle = -||V_x||^2, \tag{2.6}$$

$$\big\langle V_{\hat{x}}, V \big\rangle = 0, \tag{2.7}$$

$$\big\langle \varphi(V, V), V \big\rangle = 0. \tag{2.8}$$

*Lemma* 2 — For any discrete function $V \in \mathbb{R}_{per}^J$, we have

$$||V_{\hat{x}}||^2 \leq ||V_x||^2. \tag{2.9}$$

PROOF : For $V \in \mathbb{R}_{per}^J$, we have

$$||V_x||^2 = ||V_{\bar{x}}||^2. \tag{2.10}$$

By Cauchy Schwartz inequality, we obtain for $V \in \mathbb{R}^J_{per}$

$$\begin{aligned}
||V_{\hat{x}}||^2 &= h\sum_{j=1}^{J}\Big(\frac{V_{j+1}-V_{j-1}}{2h}\Big)^2\\
&= h\sum_{j=1}^{J}\Big(\frac{V_{j+1}-V_{j}}{2h}+\frac{V_{j}-V_{j-1}}{2h}\Big)^2\\
&\leq 2h\sum_{j=1}^{J}\Big(\frac{V_{j+1}-V_{j}}{2h}\Big)^2+2h\sum_{j=1}^{J}\Big(\frac{V_{j}-V_{j-1}}{2h}\Big)^2\\
&= \frac{1}{2}||V_x||^2+\frac{1}{2}||V_{\bar{x}}||^2.
\end{aligned}$$

Therefore, the inequality $(2.9)$ can be easily proved from (2.10). □

## 3. Discrete Conservative Laws

We show here that the difference scheme (2.1)-(2.3) has the property of conservation of mass and energy.

***Theorem* 2** — *The difference scheme* $(2.1)-(2.3)$ *is conservative in the sense*

$$Q^n = Q^{n-1} = \cdots = Q^0, \tag{3.1}$$

$$E^n = E^{n-1} = \cdots = E^0, \tag{3.2}$$

*where* $Q^n = h\sum_{j=1}^{J} U_j^n$ *and* $E^n = ||U^n||^2 + \mu||U_x^n||^2$ *for* $n = 0,1,\dots,N$, *and* $Q^n$ *is the discrete mass and* $E^n$ *is the discrete energy.*

Proof : Multiplying (2.1) by $h$ and summing up from $j=1$ to $J$ we obtain

$$h\sum_{j=1}^{J}\Big(U_j^n\Big)_t - \mu h\sum_{j=1}^{J}\Big(U_j^n\Big)_{x\bar{x}t} + h\sum_{j=1}^{J}\Big(U_j^{n+\frac{1}{2}}\Big)_{\hat{x}} + h\sum_{j=1}^{J}\varphi(U_j^{n+\frac{1}{2}},U_j^{n+\frac{1}{2}}) = 0. \tag{3.3}$$

In view of difference proprieties and the boundary conditions

$$h\sum_{j=1}^{J}\Big(U_j^{n+\frac{1}{2}}\Big)_{\hat{x}} = \frac{1}{2}\sum_{j=1}^{J}(U_{j+1}^{n+\frac{1}{2}}-U_{j-1}^{n+\frac{1}{2}}) = \frac{1}{2}\sum_{j=1}^{J}U_{j+1}^{n+\frac{1}{2}} - \frac{1}{2}\sum_{j=1}^{J}U_{j-1}^{n+\frac{1}{2}} = 0. \tag{3.4}$$

Similarly to the above equation, we have

$$h\sum_{j=1}^{J}\varphi(U_j^{n+\frac{1}{2}},U_j^{n+\frac{1}{2}}) = \frac{\delta h}{m+2}\Big\{\sum_{j=1}^{J}\Big(U_j^{n+\frac{1}{2}}\Big)^m\Big(U_j^{n+\frac{1}{2}}\Big)_{\hat{x}} + \sum_{j=1}^{J}[(U_j^{n+\frac{1}{2}})^{m+1}]_{\hat{x}}\Big\} = 0. \tag{3.5}$$

Noting that

$$h\sum_{j=1}^{J}\Big(U_j^n\Big)_{x\bar{x}t} = \frac{h}{k}\sum_{j=1}^{J}\Big\{(U_j^{n+1})_{x\bar{x}} - (U_j^n)_{x\bar{x}}\Big\}.$$

It follows from the boundary condition (2.2) that

$$h\sum_{j=1}^{J}\Big(U_j^n\Big)_{x\bar{x}t} = \frac{1}{hk}\Big[\sum_{j=1}^{J}(U_{j+1}^{n+1}-U_j^{n+1})+\sum_{j=1}^{J}(U_{j-1}^{n+1}-U_j^{n+1})+\sum_{j=1}^{J}(U_j^n-U_{j+1}^n)+\sum_{j=1}^{J}(U_j^n-U_{j-1}^n)\Big] = 0. \tag{3.6}$$

In addition, we have

$$h\sum_{j=1}^{J}\Big(U_j^n\Big)_t = \frac{h}{k}\sum_{j=1}^{J}(U_j^{n+1} - U_j^n). \tag{3.7}$$

Substituting $(3.4) - (3.7)$ into $(3.3)$, we have

$$\frac{h}{k}\sum_{j=1}^{J}(U_j^{n+1} - U_j^n) = 0. \tag{3.8}$$

Multiplying (3.8) by $k$, the conservation of mass (3.1) can be easily proved.
Taking the inner product of $(2.1)$ with $2U^{n+\frac{1}{2}}$ we have

$$2\Big\langle U_t^n, U^{n+\frac{1}{2}}\Big\rangle - 2\mu\Big\langle (U^n)_{x\bar{x}t}, U^{n+\frac{1}{2}}\Big\rangle + 2\Big\langle (U^{n+\frac{1}{2}})_{\hat{x}}, U^{n+\frac{1}{2}}\Big\rangle + 2\Big\langle \varphi(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}), U^{n+\frac{1}{2}}\Big\rangle = 0. \tag{3.9}$$

By $(2.7)$ and $(2.8)$ we get

$$\Big\langle (U^{n+\frac{1}{2}})_{\hat{x}}, U^{n+\frac{1}{2}}\Big\rangle = 0 \qquad \Big\langle \varphi(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}), U^{n+\frac{1}{2}}\Big\rangle = 0. \tag{3.10}$$

It follows from (3.9), (3.10) and $(2.6)$ that

$$2\Big\langle U_t^n, U^{n+\frac{1}{2}}\Big\rangle + 2\mu\Big\langle (U^n)_{xt}, U_x^{n+\frac{1}{2}}\Big\rangle = 0.$$

This yields to

$$\frac{1}{k}\Big\langle U^{n+1} - U^n, U^{n+1} + U^n\Big\rangle + \frac{\mu}{k}\Big\langle U_x^{n+1} - U_x^n, U_x^{n+1} + U_x^n\Big\rangle = 0,$$

then

$$\frac{1}{k}\Big(||U^{n+1}||^2 - ||U^n||^2\Big) + \frac{\mu}{k}\Big(||U_x^{n+1}||^2 - ||U_x^n||^2\Big) = 0.$$

Finally we obtain

$$||U^{n+1}||^2 + \mu||U_x^{n+1}||^2 = ||U^n||^2 + \mu||U_x^n||^2.$$

Therefore $(3.2)$ holds.

## 4. A Priori Estimates

***Theorem* 3** — *Suppose that $u_0 \in H^1_{per}(\Omega)$, then the solution of the intial-value problem (1.3)-(1.5) satisfies*

$$\|u\|_{L^2(\Omega)} \leq C, \quad \|u_x\|_{L^2(\Omega)} \leq C, \quad \|u\|_{L^\infty(\Omega)} \leq C. \tag{4.1}$$

Proof : By assumption $\mu$ is a positive constant, it follows from Theorem 1 that

$$\|u\|_{L^2(\Omega)} \leq C, \quad \|u_x\|_{L^2(\Omega)} \leq C. \tag{4.2}$$

According to Sobolev inequality, we have

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

*Lemma* 2 — (Discrete Sobolev's inequality [27]). There exist two constants $C_1$ and $C_2$ such that

$$||U^n||_\infty \leq C_1||U^n|| + C_2||U_x^n||.$$

***Theorem* 4** — *Suppose that $u_0 \in H^1_{per}(\Omega)$. Then, the solution of the difference scheme (2.1)-(2.3) is estimated as follows*

$$||U^n||_\infty \leq C. \tag{4.3}$$

Proof : By assumption $\mu$ is a positive constant, it follows from (3.2) that

$$||U^n|| \leq C, \quad ||U_x^n|| \leq C. \tag{4.4}$$

According to the Lemma 3 and $(4.4)$, the Theorem follows. □

## 5. Existence

In order to show the existence of the approximate solution $U^1, U^2, \ldots, U^n$ for the difference scheme $(2.1) - (2.3)$, we shall use the following Brower fixed point Theorem [28].

*Lemma* 4 — Let $H$ be a finite dimensional Hilbert space with inner product $\langle . \,, . \rangle$ and induced norm $||.||_H$. Further, let $g$ be a continuous mapping from $H$ to it self and be such that $\langle g(\chi), \chi \rangle > 0$

for all $\chi \in H$ with $||\chi||_H = \alpha > 0$. Then, there exists an element $\chi^* \in H$ such that $g(\chi^*) = 0$ and $||\chi^*||_H \leq \alpha$.

***Theorem* 5** — *The approximate solution of the difference scheme* $(2.1) - (2.3)$ *exists.*

PROOF : For a fixed $n$, we rewrite $(2.1)$ in the form of

$$2(U_j^{n+\frac{1}{2}} - U_j^n) - 2\mu(U_j^{n+\frac{1}{2}} - U_j^n)_{x\bar{x}} + k(U_j^{n+\frac{1}{2}})_{\hat{x}} + k\varphi(U_j^{n+\frac{1}{2}}, U_j^{n+\frac{1}{2}}) = 0.$$

Let $g : \mathbb{R}^J_{per} \longrightarrow \mathbb{R}^J_{per}$ defined by:

$$g(V) = 2(V - U^n) - 2\mu(V - U^n)_{x\bar{x}} + kV_{\hat{x}} + k\varphi(V, V). \tag{5.1}$$

Therefore, the map $g$ acts on a finite dimensional space and each of its components is quadratic in the components of V ($h$ is fixed). Thus, $g$ is continuous.

Taking the inner product of $(5.1)$ with $V$ and using the Lemma 1 we obtain:

$$\begin{aligned}
\big\langle g(V), V \big\rangle &= 2||V||^2 + 2\mu||V_x||^2 - 2\big\langle V, U^n \big\rangle - 2\mu\big\langle V_x, U_x^n \big\rangle \\
&\geq 2||V||^2 + 2\mu||V_x||^2 - (||U^n||^2 + ||V||^2) - \mu(||U_x^n||^2 + ||V_x||^2) \\
&\geq ||V||^2 + \mu||V_x||^2 - \big(||U^n||^2 + \mu||U_x^n||^2\big) \\
&\geq ||V||^2 - \big(||U^n||^2 + \mu||U_x^n||^2\big).
\end{aligned}$$

Hence for $||V||^2 = ||U^n||^2 + \mu||U_x^n||^2 + 1$, then we have $\big\langle g(V), V \big\rangle > 0$ and from Lemma 4 we deduce the existence of $V^* \in \mathbb{R}^J_{per}$ such that $g(V^*) = 0$. It is easily seen that $U^{n+1} = 2V^* - U^n$ satisfies the difference scheme (2.1)-(2.3). $\square$

## 6. CONVERGENCE, STABILITY AND UNIQUENESS

### 6.1 *Convergence*

Below the following Lemma will be needed in order to show convergence and uniqueness of approximate solutions (2.1)-(2.3).

*Lemma* 5 — (Discrete Gronwall inequality [27]). Assume $\{G^n/n \geq 0\}$ is non-negative sequences and satisfies

$$G^0 \leq A, \quad G^n \leq A + Bk\sum_{i=0}^{n-1} G^i \quad n = 1, 2, \cdots,$$

where $A$ and $B$ are non negative constants. Then $G$ satisfies

$$G^n \leq Ae^{Bnk}, \quad n = 0, 1, 2, \cdots.$$

We are going to show that the approximate solution of $(2.1)-(2.3)$ converges to the exact solution $u^n$ of $(1.3)-(1.5)$.

***Theorem* 6** — *Suppose that the solution of continuous problem $u(x,t) \in C^{4,3}_{x,t}([0,L]\times[0,T])$, then the solution of the difference scheme (2.1)-(2.3) converges to the solution if the problem (1.3)-(1.5) with the convergence order of $O(h^2+k^2)$ in the discrete $L^\infty$-norm.*

PROOF : Let $r^n \in \mathbb{R}^J_{per}$ be consistency error of the difference scheme $(2.1)-(2.3)$

$$r^n_j = (u^n_j)_t - \mu(u^n_j)_{x\bar{x}t} + (u_j^{n+\frac{1}{2}})_{\hat{x}} + \varphi(u_j^{n+\frac{1}{2}}, u_j^{n+\frac{1}{2}}), \quad j=1,\cdots,J, \quad n=0,\cdots,N-1. \tag{6.1}$$

From Taylor expansion we have

$$\max_{j,n}|r^n_j| \le C(h^2+k^2). \tag{6.2}$$

Let $e^n_j = u^n_j - U^n_j$, then it follows from $(6.1)$ and $(2.1)$ that

$$\begin{aligned}&(e^n_j)_t - \mu(e^n_j)_{x\bar{x}t} + (e_j^{n+\frac{1}{2}})_{\hat{x}} + \varphi(u_j^{n+\frac{1}{2}}, u_j^{n+\frac{1}{2}}) - \varphi(U_j^{n+\frac{1}{2}}, U_j^{n+\frac{1}{2}}) = r^n_j,\\ &\qquad\qquad j=1,\cdots,J, \quad n=0,\cdots,N-1,\end{aligned} \tag{6.3}$$

and

$$e^n_j = e^n_{j+J}, \quad j=1,\cdots,J, \quad n=0,1,\cdots,N, \tag{6.4}$$

$$e^0_j = 0, \quad j=1,\cdots,J. \tag{6.5}$$

Taking the inner product of $(6.3)$ with $2e^{n+\frac{1}{2}}$, we obtain from Lemma 1

$$||e^n||^2_t + \mu||e^n_x||^2_t + 2\big\langle \varphi(u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}) - \varphi(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}), e^{n+\frac{1}{2}}\big\rangle = 2\big\langle r^n, e^{n+\frac{1}{2}}\big\rangle. \tag{6.6}$$

In view of difference properties, the boundary conditions $(2.2)$ and Lemma 1, we have

$$\begin{aligned}&\langle \varphi(u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}) - \varphi(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}), 2e^{n+\frac{1}{2}}\rangle\\ &= \frac{2\delta h}{m+2}\sum_{j=1}^{J}\Big\{(u_j^{n+\frac{1}{2}})^m(u_j^{n+\frac{1}{2}})_{\hat{x}} - (U_j^{n+\frac{1}{2}})^m(U_j^{n+\frac{1}{2}})_{\hat{x}}\Big\}e_j^{n+\frac{1}{2}}\\ &\quad+\frac{2\delta h}{m+2}\sum_{j=1}^{J}\Big\{[(u_j^{n+\frac{1}{2}})^{m+1}]_{\hat{x}} - [(U_j^{n+\frac{1}{2}})^{m+1}]_{\hat{x}}\Big\}e_j^{n+\frac{1}{2}}\\ &= \frac{2\delta h}{m+2}\sum_{j=1}^{J}(U_j^{n+\frac{1}{2}})^m(e_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}} + \frac{2\delta h}{m+2}\sum_{j=1}^{J}[(u_j^{n+\frac{1}{2}})^m - (U_j^{n+\frac{1}{2}})^m](u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}\end{aligned}$$

$$-\frac{2\delta h}{m+2}\sum_{j=1}^{J}[(u_j^{n+\frac{1}{2}})^{m+1}-(U_j^{n+\frac{1}{2}})^{m+1}](e_j^{n+\frac{1}{2}})_{\hat{x}}$$
$$=\frac{2\delta h}{m+2}\sum_{j=1}^{J}(U_j^{n+\frac{1}{2}})^m(e_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}+\frac{2\delta h}{m+2}\sum_{j=1}^{J}[e_j^{n+\frac{1}{2}}\sum_{i=0}^{m-1}(u_j^{n+\frac{1}{2}})^{m-1-i}(U_j^{n+\frac{1}{2}})^i](u_j^{n+\frac{1}{2}})_{\hat{x}}e_j^{n+\frac{1}{2}}$$
$$-\frac{2\delta h}{m+2}\sum_{j=1}^{J}e_j^{n+\frac{1}{2}}[\sum_{i=0}^{m}(u_j^{n+\frac{1}{2}})^{m-i}(U_j^{n+\frac{1}{2}})^i](e_j^{n+\frac{1}{2}})_{\hat{x}}.$$

It follows from $(4.1)$, $(4.3)$ and Cauchy-Schwarz inequality that

$$|\langle\varphi(u^{n+\frac{1}{2}},u^{n+\frac{1}{2}})-\varphi(U^{n+\frac{1}{2}},U^{n+\frac{1}{2}}),2e^{n+\frac{1}{2}}\rangle|\le C(\|e^{n+\frac{1}{2}}\|^2+\|(e^{n+\frac{1}{2}})_{\hat{x}}\|^2). \tag{6.7}$$

Applying Lemma 2 with $(6.5)-(6.6)$, we obtain

$$||e^n||_t^2+\mu||e_x^n||_t^2\le C\Big[\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2\Big]+\|r^n\|^2. \tag{6.8}$$

Let $A^n=\|e^n\|^2+\mu\|e_x^n\|^2$ and summing up $(6.8)$ from 0 to $n-1$, we have

$$A^n\le A^0+Ck\sum_{i=0}^{n}(\|e^i\|^2+\|e_x^i\|^2)+k\sum_{i=0}^{n-1}\|r^i\|^2. \tag{6.9}$$

By assumption $\mu$ is positive constant, thus

$$c_0(\|e^n\|^2+\|e_x^n\|^2)\le A^n, \tag{6.10}$$

where $c_0=\min(1,\mu)$ is positive constant. By $(6.5)$ and $(6.9)-(6.10)$, we obtain

$$\|e^n\|^2+\|e_x^n\|^2\le Ck\sum_{i=0}^{n}(\|e^i\|^2+\|e_x^i\|^2)+Ck\sum_{i=0}^{n-1}\|r^i\|^2, \tag{6.11}$$

where $C$ is constant dependent on $c_0$. Letting now $G^n=\|e^n\|^2+\|e_x^n\|^2$, then (6.11) can be rewritten as follows

$$G^n\le Ck\sum_{i=0}^{n}G^i+Ck\sum_{i=0}^{n-1}\|r^i\|^2.$$

Using $(6.2)$, we have

$$G^n\le Ck\sum_{i=0}^{n}G^i+Cnk\max_{0\le i\le n-1}\|r^i\|^2\le Ck\sum_{i=0}^{n}G^i+CT(h^2+k^2)^2.$$

Hence,

$$(1-Ck)G^n\le Ck\sum_{i=0}^{n-1}G^i+C(h^2+k^2)^2,$$

from which, for $k$ sufficiently small such that $(1 - Ck) > 0$, then

$$G^n \leq Ck \sum_{i=0}^{n-1} G^i + C(h^2 + k^2)^2. \tag{6.12}$$

It follows from Lemma 5 that

$$G^n \leq C(h^2 + k^2)^2 e^{CT} \leq C(h^2 + k^2)^2.$$

Consequently,

$$\|e^n\| \leq C(h^2 + k^2), \quad \|e_x^n\| \leq C(h^2 + k^2). \tag{6.13}$$

According to Lemma 3, we have

$$\|e^n\|_\infty \leq C(h^2 + k^2).$$

This completes the proof. □

6.2 *Stability*

We now show that the difference scheme $(2.1) - (2.3)$ is stable.

***Theorem 7*** — *Assume that $u_0(x) \in H^1_{per}(\Omega)$ and $k$ is small enough, then the difference scheme $(2.1) - (2.3)$ is stable for the initial value by $L^\infty$ discrete norm.*

PROOF : Suppose there are solutions $U_j^n \in \mathbb{R}^J_{per}$ and $\tilde{U}_j^n \in \mathbb{R}^J_{per}$, which satisfy both the nonlinear finite difference scheme $(2.1)$ such that $U_j^0 = u_0(x_j)$, $\tilde{U}_j^0 = \tilde{u}_0(x_j)$. Set

$$\epsilon_j^n = U_j^n - \tilde{U}_j^n\,, \quad j = 1, \cdots, J\,, \quad n = 0, \cdots, N. \tag{6.14}$$

Similarly to the proof of Theorem 6, for small $k$, we have

$$\|\epsilon^n\|_\infty \leq C\|\epsilon^0\|_\infty. \tag{6.15}$$

This completes the proof. □

6.3 *Uniqueness*

***Theorem 8*** — *For $k$ sufficiently small, the approximate solution $U^n$ of $(2.1) - (2.3)$ is unique.*

PROOF : Let $W^n = U^n - V^n$. Assume that $U^n$ and $V^n$ both satisfy the scheme $(2.1) - (2.3)$, then we obtain

$$\begin{aligned}&(W_j^n)_t - \mu(W_j^n)_{x\bar{x}t} + (W_j^{n+\frac{1}{2}})_{\hat{x}} + \left[\varphi(U_j^{n+\frac{1}{2}}, U_j^{n+\frac{1}{2}}) - \varphi(V_j^{n+\frac{1}{2}}, V_j^{n+\frac{1}{2}})\right] = 0,\\ &\qquad j = 1, \dots, J, \quad n = 0, \dots, N-1,\end{aligned} \tag{6.16}$$

$$W_j^n = W_{j+J}^n, \quad j = 1, \dots, J, \quad n = 0, \dots, N, \tag{6.17}$$

$$W_j^0 = 0, \quad j = 1, \dots, J. \tag{6.18}$$

Taking in $(6.16)$ the inner product with $2W^{n+\frac{1}{2}}$ and using $(2.7) - (2.8)$. In the same way as Theorem 6, we have

$$(||W^n||^2)_t + \mu(||W_x^n||^2)_t \le C\big(||W^{n+\frac{1}{2}}||^2 + ||W_x^{n+\frac{1}{2}}||^2\big)$$

Applying Lemma $5$ and (6.18), we obtain for $k$ small enough

$$||W^n||^2 + \mu||W_x^n||^2 = 0.$$

This completes the proof of uniqueness. □

## 7. Numerical Experiments

In this section, we give some numerical experiments to verify our theoretical results which are given in the previous sections. For that purpose, we consider the particular case known as the MRLW equation to show error estimates, order of convergence and discrete conservation laws of the difference scheme $(2.1)-(2.3)$. The computer application program Matlab was used to execute the algorithms that were used with the numerical examples.

### 7.1 *Order of convergence*

7.1.1 *Example* $1$ : We consider the following initial value MRLW equation

$$u_t - u_{xxt} + u_x + u^2 u_x = 0, \quad (x,t) \in [a,b] \times [0,T], \tag{7.1}$$

$$u(x,0) = u_0(x), \ \ x \in [a,b], \tag{7.2}$$

The problem $(7.1) - (7.2)$ has the exact solution

$$u(x,t) = \sqrt{6c}\, sech\Big(\sqrt{\frac{c}{(c+1)}}(x - (c+1)t - x_0)\Big),$$

where $c, x_0$ are arbitrary constants. We choose $x_0 = 40$ and $c = 0.1$. We discretize the previous problem by the following nonlinear finite difference scheme

$$(U_j^n)_t - (U_j^n)_{x\bar{x}t} + (U_j^{n+\frac{1}{2}})_{\hat{x}} + \frac{1}{4}\Big[(U_j^{n+\frac{1}{2}})^2(U_j^{n+\frac{1}{2}})_{\hat{x}} + \big[(U_j^{n+\frac{1}{2}})^3\big]_{\hat{x}}\Big] = 0, \tag{7.3}$$

$$U_j^0 = u_0(x_j). \tag{7.4}$$

We assume that we are interested in solutions of (7.1) that, for the range of time under consideration, are negligible outside an interval $a \leq x \leq b$. (Note that the MRLW equation has solutions $u(x,t)$ that decrease exponentially as $|x| \to \infty$). Then

$$U_0^n = U_1^n = U_{J-1}^n = U_J^n = 0 \tag{7.4}$$

are suitable boundary conditions for $(7.3)$. The situation is not greatly altered if periodic boundary conditions $(2.2)$ rather than $(7.4)$ are considered. In order to compare the numerical scheme more qualitatively, we define the maximum norm errors of the numerical solution

$$e_\infty(h,k) = \max_{0\leq n\leq N} \|u^n - U^n\|_\infty.$$

Denote

$$Order = \log_2\Big(\frac{e_\infty(2h,2k)}{e_\infty(h,k)}\Big),$$

as the rate of convergence when both $h$ and $k$ are sufficiently small.

From Table 1, we see that finite difference scheme (7.3)-(7.4) is convergent with the convergence of two both in space and in time. This is in accordance with theoretical analysis.

| $h=k$ | $e_\infty(h,k)$ | $Order$ |
|---|---|---|
| $0.4$ | $106853 \times 10^{-3}$ | - |
| $0.2$ | $2.6572 \times 10^{-4}$ | $2.0076$ |
| $0.1$ | $6.6174 \times 10^{-5}$ | $2.0056$ |
| $0.05$ | $1.6527 \times 10^{-5}$ | $2.0014$ |

Table 1. Errors estimates and order of convergence for the scheme $(7.3)-(7.4)$ to the MRLW equation $(7.1)-(7.2)$ when $h=k$, $T=2,\ c=0.1$, $x_0=40$ and $x\in[0,100]$.

7.1.2 *Example* 2 : Consider the following periodic initial value problem for the inhomogeneous GRLW equation

$$u_t - u_{xxt} + u_x + u^2u_x = g(x,t),\ (x,t) \in [0,1]\times[0,1], \tag{7.5}$$

with the initial data

$$u(x,0) = \sin\Big(2\pi x\Big). \tag{7.6}$$

where

$$g(x,t) = -\exp(-t)\sin(2\pi x)[1+4\pi^2] + 2\pi\exp(-t)\cos(2\pi x)[1+\exp(-2t)\sin^2(2\pi x)]. \tag{7.7}$$

The exact solution of the problem $(7.5)-(7.7)$ is

$$u(x,t)=e^{-t}\sin\Big(2\pi x\Big).$$

Some numerical results are presented in Table 2.

| $h=k$ | $e_\infty(h,k)$ | $Order$ |
|---|---|---|
| $1/20$ | $5.4329\times 10^{-3}$ | - |
| $1/40$ | $1.3739\times 10^{-3}$ | 1.9834 |
| $1/80$ | $3.4314\times 10^{-4}$ | 2.0013 |
| $1/160$ | $8.5765\times 10^{-5}$ | 2.0003 |
| $1/320$ | $2.1440\times 10^{-5}$ | 2.0000 |

Table 2. Error estimates and order of convergence for $h=k$ and $T=1$.

as the rate of convergence when both $h$ and $k$ are sufficiently small.
From Table 2, we see that finite difference scheme is convergent with the convergence of two both in space and in time. This is in accordance with theoretical analysis.

### 7.2 *Results of conservation*

As it is shown in [29], the MRLW equation has three conservation laws which can be expressed in the form:

$$\begin{cases} I_1 &= \displaystyle\int_a^b u dx, \\ I_2 &= \displaystyle\int_a^b (u^2+u_x^2)dx, \\ I_3 &= \displaystyle\int_a^b (u^4-6u_x^2)dx. \end{cases}$$

We are going to test our numerical method via these invariants. For that we present some numerical experiments to assign the numerical solution of single solitary wave, in addition we determine the conservations laws for the solution of two and three interactions at different time levels.

#### 7.2.1 *Single solitary waves*

To illustrate the validity of our scheme in case of a singleton soliton, we measure the quantities $I_1$, $I_2$ and $I_3$. For the single solitary waves we consider two cases in each one we choose different values of $c$, $h$ and $k$.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0.0 | 8.070838220015190 | 4.100503142518572 | 0.868659548237246 |
| 1.0 | 8.070838220015190 | 4.100503142518572 | 0.868659501151140 |
| 2.0 | 8.070838217458110 | 4.100503149556048 | 0.868659366552122 |
| 3.0 | 8.070838213444239 | 4.100503159959637 | 0.868659161339300 |
| 4.0 | 8.070838208261398 | 4.100503172377464 | 0.868658906010821 |
| 5.0 | 8.070838202199381 | 4.100503185679988 | 0.868658619088283 |
| 6.0 | 8.070838195509559 | 4.100503199099017 | 0.868658314705490 |
| 7.0 | 8.070838188392989 | 4.100503212186401 | 0.868658002620573 |
| 8.0 | 8.070838181002820 | 4.100503224714228 | 0.868657689193868 |
| 9.0 | 8.070838173452927 | 4.100503236587453 | 0.868657378432991 |
| 10 | 8.070838165827132 | 4.100503247784894 | 0.868657072799598 |

Table 3. Invariants for single solitary wave : $x_0 = 40,\ h = k = 0.1, c = 0.1, x \in [0, 80]$.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0.0 | 8.07084 | 4.1005 | 0.868660 |
| 1.0 | 8.07083 | 4.1005 | 0.868661 |
| 2.0 | 8.07082 | 4.1005 | 0.868659 |
| 3.0 | 8.07081 | 4.1005 | 0.868659 |
| 4.0 | 8.07078 | 4.1005 | 0.868659 |
| 5.0 | 8.07074 | 4.1005 | 0.868659 |
| 6.0 | 8.07068 | 4.1005 | 0.868660 |
| 7.0 | 8.07059 | 4.1005 | 0.868660 |
| 8.0 | 8.07047 | 4.1005 | 0.868660 |
| 9.0 | 8.07029 | 4.1005 | 0.868660 |
| 10 | 8.07004 | 4.1005 | 0.868659 |

Table 4. Invariants for single solitary wave with scheme in [29]: $x_0 = 40$, $h = k = 0.1, c = 0.1, x \in [0, 80]$.

For the first case, we choose $x_0 = 40,\ h = k = 0.1, c = 0.1$ and $x \in [0, 80]$. The simulations are done up to $t = 10$. The invariants $I_1$, $I_2$ and $I_3$ are changed by less than $10^{-6}$, $10^{-6}$ and $10^{-5}$ percent, respectively, Our results are recorded in Table 3 and the motion of the solitary wave is plotted at different levels in Figure 1. In the second case, we choose $c = 0.03$ , $h = 0.2$, $k = 0.025$, $x_0 = 0$ and $x \in [-40, 60]$. The simulations are also done up to $t = 10$. The variation of the invariants $I_1$, $I_2$ and $I_3$ are all changed less than $10^{-6}$ percent. The results for the second case are shown numerically in Table $5$ and graphically in Figure $2$ to illustrate the motion of the solitary wave at different time levels.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0.0 | 7.804215913258073 | 2.129875605533693 | 0.130304848275567 |
| 1.0 | 7.804215913270137 | 2.129875605524225 | 0.130304846482890 |
| 2.0 | 7.804215912955137 | 2.129875605755783 | 0.130304841134546 |
| 3.0 | 7.804215912956964 | 2.129875605754446 | 0.130304841188457 |
| 4.0 | 7.804215912461224 | 2.129875606133308 | 0.130304832408943 |
| 5.0 | 7.804215911788766 | 2.129875606651847 | 0.130304820290645 |
| 6.0 | 7.804215910944756 | 2.129875607299324 | 0.130304805002403 |
| 7.0 | 7.804215908754416 | 2.129875608937994 | 0.130304765455233 |
| 8.0 | 7.804215907425825 | 2.129875609897376 | 0.130304741856029 |
| 9.0 | 7.804215905947061 | 2.129875610934696 | 0.130304715939643 |
| 10 | 7.804215904324714 | 2.129875612038237 | 0.130304687919353 |

Table 5. Invariants for single solitary wave: $x_0 = 0,\ c = 0.03, h = 0.2,\ k = 0.025,\ x \in [-40, 60]$.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0.0 | 7.80421 | 2.12988 | 0.130304 |
| 1.0 | 7.80513 | 2.12988 | 0.130307 |
| 2.0 | 7.80592 | 2.12988 | 0.130309 |
| 3.0 | 7.80660 | 2.12988 | 0.130311 |
| 4.0 | 7.80720 | 2.12988 | 0.130312 |
| 5.0 | 7.80771 | 2.12988 | 0.130313 |
| 6.0 | 7.80816 | 2.12988 | 0.130314 |
| 7.0 | 7.80854 | 2.12988 | 0.130314 |
| 8.0 | 7.80886 | 2.12988 | 0.130315 |
| 9.0 | 7.80912 | 2.12988 | 0.130315 |
| 10 | 7.80923 | 2.12988 | 0.130315 |

Table 6. Invariants for single solitary wave with scheme in [29]: $x_0 = 0, c = 0.03, h = 0.2,\ k = 0.025,\ x \in [-40, 60]$.

As we can see in comparing our numerical results with those found in [29], for the numerical values of the laws $I_1, I_2$ and $I_3$ we can see that our scheme provides more stable results in both cases.

### 7.2.2 *Interaction of two solitary waves*

We consider in this case, the MRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$u(x, 0) = \sum_{i=1}^{2} \sqrt{6c_i}\, sech\Big(k_i(x - x_i)\Big),$$

where $k_i = \sqrt{\dfrac{c_i}{c_i + 1}}$, $c_i$ and $x_i$ are constants, $i = 1, 2$.

For this section we have chosen $c_1 = 2,\ c_2 = 0.5,\ x_1 = 15,\ x_2 = 35$ and $h = k = 0.1$ with interval $[0, 250]$. The results of the conservation of the three laws for two solitary waves are given in Table 7 and in Figure 3, the interactions of these solitary waves are plotted at different time levels.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0.0 | 22.753384373649666 | 47.452130065054625 | 209.9640040124943 |
| 5 | 22.753330587923273 | 47.452725591446253 | 209.9208206560400 |
| 10 | 22.752643148339001 | 47.458820462234485 | 209.0070165394541 |
| 15 | 22.753319720974226 | 47.452622909513117 | 209.8413114880513 |
| 20 | 22.753390559088615 | 47.452421689237156 | 209.9449720868477 |
| 25 | 22.753360285925911 | 47.452555536570266 | 209.9422333882808 |
| 30 | 22.753345805004560 | 47.452603420904225 | 209.9405644127809 |

Table 7. Invariants for interactions of two solitary waves with $x_1 = 15,\ x_2 = 35, h = k = 0.1,\ c_1 = 2,\ c_2 = 0.5\,, x \in [0, 100]$.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0.0 | 22.7534 | 47.4521 | 209.964 |
| 5 | 22.7539 | 47.5119 | 210.453 |
| 10 | 22.7505 | 47.1418 | 210.706 |
| 15 | 22.7536 | 47.4993 | 210.340 |
| 20 | 22.7540 | 47.5217 | 210.511 |
| 25 | 22.7546 | 47.5206 | 210.509 |
| 30 | 22.7544 | 47.5200 | 210.508 |

Table 8. Invariants for interactions of two solitary waves with scheme in [29] with $x_1 = 15,\ x_2 = 35,$

$h = k = 0.1,\ c_1 = 2,\ c_2 = 0.5\,, x \in [0, 100]$.

It is seen that the obtained values of the invariants, by our scheme are more stable than the values obtained in [29] (Table 8).

### 7.2.3 *Interaction of three solitary waves*

We consider the MRLW equation with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes

$$u(x, 0) = \sum_{i=1}^{3} \sqrt{6c_i}\, sech\Big(k_i(x - x_i)\Big),$$

where $k_i = \sqrt{\dfrac{c_i}{c_i + 1}}$, $c_i$ and $x_i$ are constants, $i = 1, 2, 3$.

We choose $c_1 = 2,\ c_2 = 0.75,\ c_3 = 0.2,\ x_1 = 10,\ x_2 = 45,\ x_3 = 65,\ h = k = 0.1$ with interval $[0, 250]$. Figure 4 shows the plot of these solitary interaction waves at different time levels. In Table 9 we report the three invariants for MRLW equation. Numerical checks on the conservation mass $I_1$, momentum $I_2$, and energy $I_3$ show that the three quantities remain reasonably constant as the solitary waves evolve in time. By comparing these results with those found in [29] (Table 10) calculated under the same initial conditions we can also say that numerical results by our scheme are relatively more stable.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0 | 31.935836692358894 | 57.837356622408443 | 228.3938414699506 |
| 10 | 31.935769256372033 | 57.837975975080575 | 228.3601778665868 |
| 20 | 31.935511489058182 | 57.839256082438681 | 228.2469894220832 |
| 30 | 31.934874094559728 | 57.843827916852000 | 227.7021891619049 |
| 40 | 31.935783227158971 | 57.837918116222347 | 228.3459762915063 |
| 50 | 31.935835447591472 | 57.837781612721017 | 228.3607291085764 |
| 60 | 31.935823498634271 | 57.837802653375022 | 228.3609912390971 |
| 70 | 31.935812261995626 | 57.837818609793530 | 228.3605239193500 |

Table 9. Invariants for interactions of three solitary waves with $x_1 = 10,\ x_2 = 45,\ x_3 = 65,$ $h = k = 0.1,\ c_1 = 2,\ c_2 = 0.75,\ c_3 = 0.2,\ x \in [0, 250]$.

| Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|
| 0 | 31.9358 | 57.8374 | 228.394 |
| 10 | 31.9510 | 57.9028 | 228.924 |
| 20 | 31.9592 | 57.8440 | 228.592 |
| 30 | 31.9621 | 57.7630 | 228.119 |
| 40 | 31.9675 | 57.8983 | 228.879 |
| 50 | 31.9716 | 57.9040 | 228.898 |
| 60 | 31.9764 | 57.9040 | 228.899 |
| 70 | 31.9659 | 57.9038 | 228.899 |

Table 10. Invariants for interactions of three solitary waves for scheme in [29] with $x_1 = 10,\ x_2 = 45,\ x_3 = 65,$ $h = k = 0.1,\ c_1 = 2,\ c_2 = 0.75,\ c_3 = 0.2,\ x \in [0, 250]$.

#### 7.2.4 *The maxwellian initial condition*

Consider the following MRLW equation

$$u_t - \mu u_{xxt} + u_x + u^2 u_x = 0, \quad a \le x \le b. \tag{7.8}$$

where $\mu$ is a real. As a last problem, we consider the equation $(7.8)$ with the following Maxwallian condition.

$$u(x, 0) = exp[-(x - 40)^2].$$

In this case, the behavior of the solution depends on the values of the parameters $\mu$. Therefore the values $\mu = 0.05$, $\mu = 0.01$ and $\mu = 0.004$ are picked. The numerical computations are done up to $T = 12$. The values of the three invariants of motion for different $\mu$ are presented in Table $10$. These three invariants remain relatively more stable by our scheme than the scheme in [29], as it shown in Table $11$. Also Figures $5(a) - (c)$

illustrate the development of the maxwellian condition into solitary waves. In Figure $5(a)$, for $\mu = 0.05$ a single stable solution has appeared, when $\mu = 0.01$ two stable solitary waves occurred in Figure $5(b)$ and finally for $\mu = 0.004$ three solitary waves are shown in Figure $5(c)$. So we can figure out from these Figures that, as the value of $\mu$ decreases, the number of the stable solitaty waves increases.

| $\mu$ | Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|---|
| 0.05 | 4 | 1.772541188604176 | 1.315064412578247 | 0.520350044979594 |
| | 8 | 1.772473727139606 | 1.315172532526903 | 0.516980951601557 |
| | 12 | 1.772437630575408 | 1.315211183569450 | 0.515388464305743 |
| 0.01 | 4 | 1.774194084124291 | 1.263014369222264 | 0.960016191143626 |
| | 8 | 1.774544007896905 | 1.262651090496481 | 0.985831050587402 |
| | 12 | 1.774542673254300 | 1.262638717985662 | 0.986993332215430 |
| 0.004 | 4 | 1.775738947125951 | 1.254847110855574 | 1.153766090018281 |
| | 8 | 1.776433331177086 | 1.254463916243288 | 1.193976394010478 |
| | 12 | 1.776568530702599 | 1.254424052827068 | 1.198848010749434 |

Table 11. Invariants of MRLW equation using the Maxwellian initial condition .

| $\mu$ | Time | $I_1$ | $I_2$ | $I_3$ |
|---|---|---|---|---|
| 0.05 | 4 | 1.77290 | 1.31677 | 0.521201 |
| | 8 | 1.77266 | 1.31621 | 0.518514 |
| | 12 | 1.77253 | 1.31593 | 0.517189 |
| 0.01 | 4 | 1.78150 | 1.29806 | 1.011870 |
| | 8 | 1.78317 | 1.30336 | 1.044710 |
| | 12 | 1.78323 | 1.30355 | 1.046000 |
| 0.004 | 4 | 1.79303 | 1.33791 | 1.39875 |
| | 8 | 1.79710 | 1.34971 | 1.46035 |
| | 12 | 1.79803 | 1.35219 | 1.480062 |

Table 12. Invariants of MRLW equation using the Maxwellian initial condition provided by the scheme in [29].

## 8. Conclusion

In this paper, a nonlinear Crank-Nicolson finite difference scheme has been presented to solving a model of nonlinear dispersive MRLW equation. It is proved that the scheme is conservative, unconditionally stable and convergent with second order both in time and in space in maximum norm. Numerical results have illustrated the order of accuracy and efficiency of the scheme. Also the efficiency of the difference scheme is tested on the problems of propagation of the single soliton and the interaction of two and three solitons for MRLW equations. We also investigate the conservation quantities $I_1,\ I_2$ and $I_3$ which were satisfactorily maintained. Furthermore, a Maxwellian initial condition has been used and a relation between $\mu$ and the number of waves was explored. Results show the interaction of two and three solitons, with conservation laws adhered to satisfaction. Let's not forget to mention that our numerical results were compared to those obtained by another difference scheme and the numerical results have proved that our scheme gives more stable values in calculating the quantities $I_1,\ I_2$ and $I_3$.

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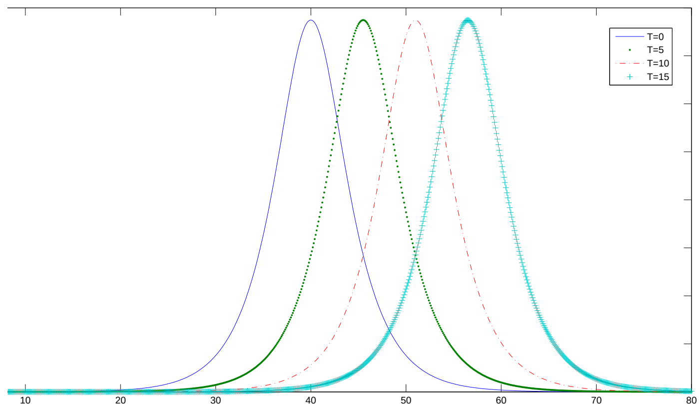



Figure 1: Single solitary wave with $x_0 = 40$, $c = 0.1$, $h = k = 0.1$ at times T=0, T=5, T=10 and T=15.

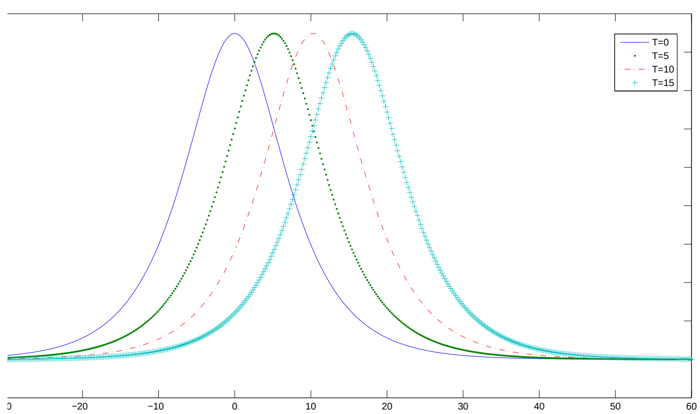
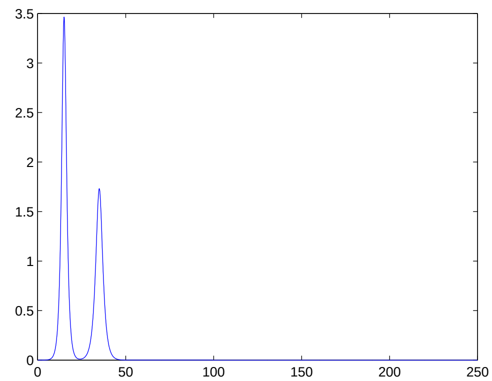
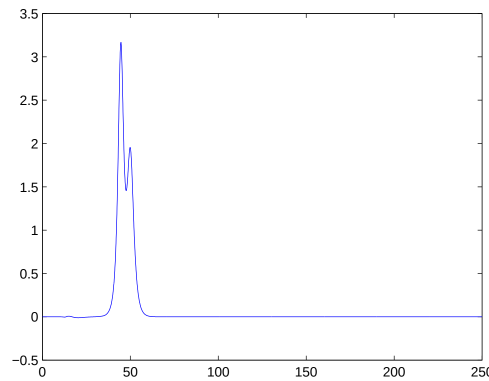


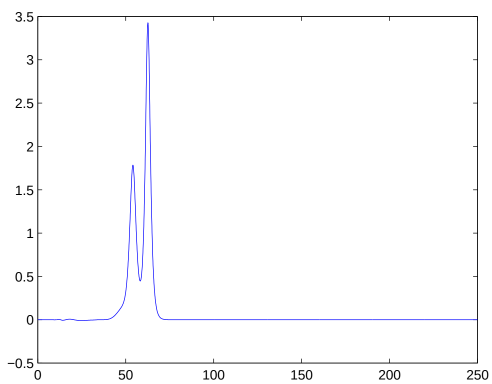

Figure 2: Single solitary wave with $x_0 = 0, c = 0.03$, $h = 0.2,\ k = 0.025$ at times T=0, T=5, T=10 and T=15



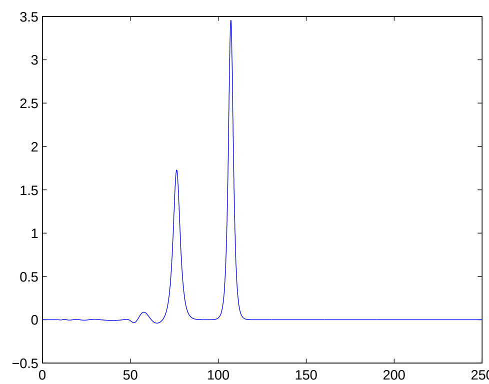

(a)




(b)




(c)




(d)

Figure 3: Interaction with two solitary waves with $c_1 = 2$, $c_2 = 0.5$, $x_1 = 15$, $x_2 = 35$, $h = k = 0.1$, $x \in [0, 250]$ at times (a) T=0, (b) T=10 (c) T=15 (d) T=30.

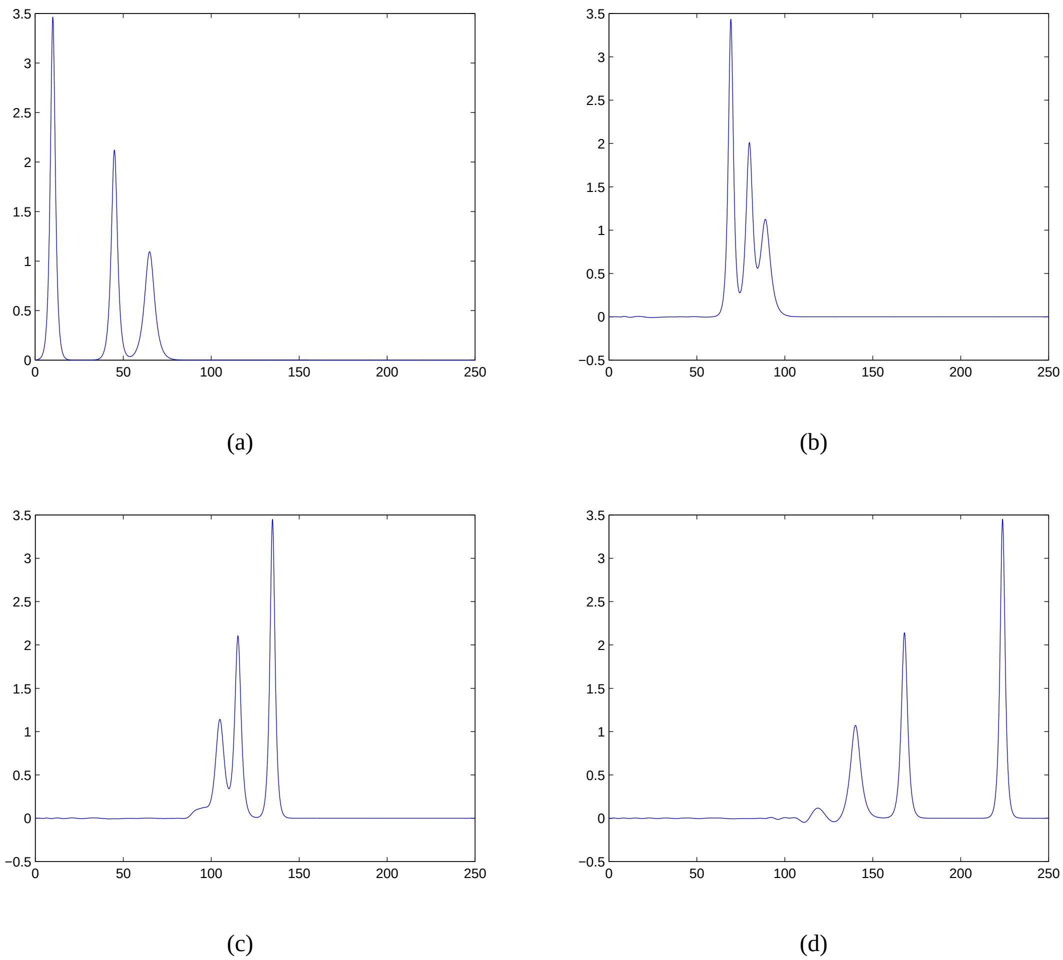



Figure 4: Interaction with three solitary waves with $c_1 = 2$, $c_2 = 0.5$, $x_1 = 10$, $x_2 = 45$, $x_3 = 65$, $h = k = 0.1$, $x \in [0, 250]$ at times (a) T=0, (b) T=20 (c) T=40 (d) T=70.

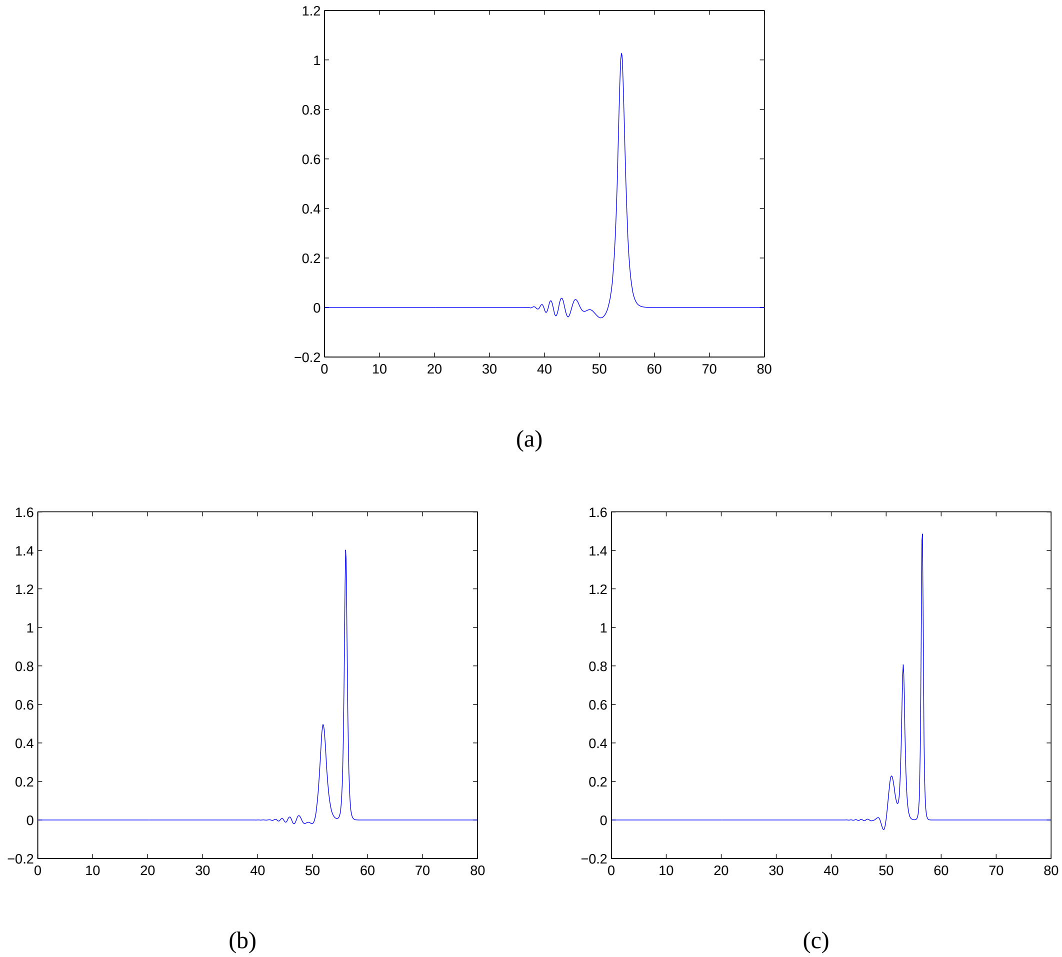



Figure 5: (a) T=12, $\mu = 0.05$ (b) T=12, $\mu = 0.01$ (c) T=12, $\mu = 0.004$.