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NEW INTEGRAL NORM ESTIMATES OF BERNSTEIN-TYPE INEQUALITIES FOR A CERTAIN CLASS OF POLYNOMIALS

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In this paper, we establish some results on algebraic polynomials in the complex plane that are inspired by some classical Bernstein-type inequalities that relate the sup-norm of a polynomial to that of its derivative on the unit circle. The obtained results relate the L^{γ}-norm of the polar derivative and the polynomial.

Key words : Polar derivative; Bernstein-inequality; L^{γ}-inequalities.

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1. INTRODUCTION

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n in the complex plane and $P'(z)$ its derivative. The study of Bernstein-type inequalities that relate the norm of a polynomial to that of its derivative and their various versions are a classical topic in analysis. One basic result is that: if $P(z)$ is a polynomial of degree n, then on $|z| = 1$,

$$|P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) is a famous result due to Bernstein [4], who proved it in 1912. Later, in 1930 (see [5]), Bernstein revisited his inequality (1.1) and proved that, for two polynomials $P(z)$ and $Q(z)$ with degree of $P(z)$ not exceeding that of $Q(z)$ and $Q(z) \neq 0$ for $|z| > 1$, the majorization $|P(z)| \leq |Q(z)|$ on the unit circle $|z| = 1$ implies the majorization of their derivatives $|P'(z)| \leq |Q'(z)|$ on $|z| = 1$. It is worth mentioning that equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin, so it is natural to seek improvements under appropriate assumption on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials $P(z)$ having no zero in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

whereas if $P(z)$ has no zeros in $|z| > 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.2) was conjectured by Erdös and later proved by Lax [13], whereas inequality (1.3) is due to Turán [21].

As an extension of (1.2) and (1.3), Malik [14] proved that if $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.4}$$

whereas if $P(z)$ has all its zeros in $|z| \leq k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

In the literature, there already exists some refinements and generalizations of all the above inequalities, for example see Chan and Malik [7], Qazi [19], Gardner, Govil and Weems [10], Dewan, Singh and Mir [8], Mir and Hussain [17] etc. By considering a more general class of polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, not vanishing in $|z| < k,\ k \geq 1$, Gardner, Govil and Weems [10] generalized and sharpened inequality (1.4) and proved that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+S_1} \left\{ \max_{|z|=1} |P(z)| - m \right\}, \tag{1.6}$$

where here and throughout $m = \min_{|z|=k} |P(z)|$ and

$$S_1 = k^{\mu+1} \left\{ \frac{\left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0|-m} k^{\mu-1} + 1}{\left(\frac{\mu}{n}\right) \frac{|a_\mu|}{|a_0|-m} k^{\mu+1} + 1} \right\}. \tag{1.7}$$

On the other hand, for the class of polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$ of degree n having all its zeros in $|z| \leq k,\ k \leq 1$, Aziz and Shah [2] strengthened (1.5) and proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |P(z)| + \frac{m}{k^{n-\mu}} \right\}. \tag{1.8}$$

Different versions of these Bernstein and Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomials are replaced by more general classes of functions. The one such generalization is moving from the ordinary derivative to their polar derivative. Before mentioning few such generalizations of the said inequalities, let us first introduce the concept of the polar derivative involved. For a polynomial $P(z)$ of degree n, we define

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point α. The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\} = P'(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

Aziz [1] was among the first to extend some of the above inequalities by replacing the derivative with the polar derivative of polynomial. In fact, in 1988, Aziz [1] proved that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for any complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1}|D_\alpha P(z)| \leq n\left(\frac{|\alpha|+k}{1+k}\right)\max_{|z|=1}|P(z)|. \tag{1.9}$$

If we divide both sides of (1.9) by $|\alpha|$ and make $|\alpha| \to \infty$, we get (1.4).

The latest research and development on this topic can be found in the papers [11, 15-18, 22, 23]. In 2009, Dewan, Singh and Mir [8] extended (1.6) to the polar derivative of a polynomial and obtained that if $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for any complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1}|D_\alpha P(z)| \leq \frac{n}{1+S_1}\left\{(|\alpha|+S_1)\max_{|z|=1}|P(z)| - (|\alpha|-1)\min_{|z|=k}|P(z)|\right\}, \tag{1.10}$$

where S_1 is defined by formula (1.7).

As a generalization of (1.8), Dewan, Singh and Mir in the same paper proved that if $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}, 1 \leq \mu \leq n,\ a_0 \neq 0$, is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \leq 1$, and β is any complex number with $|\beta| \leq 1$, then

$$\max_{|z|=1}|D_\beta P(z)| \leq n\left(\frac{k^\mu + |\beta|}{1+k^\mu}\right)\max_{|z|=1}|P(z)| - n\left(\frac{1-|\beta|}{k^{n-\mu}(1+k^\mu)}\right)\min_{|z|=k}|P(z)|. \tag{1.11}$$

As mentioned earlier, Bernstein and Turán-type inequalities have been extended and generalized in different domains, different norms and for different classes of functions. Zygmund [24] extended the Bernstein-inequality (1.1) to L^γ-norms of $P(z)$ as

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leq n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}},\ \ \gamma\geq 1.$$

As an extension of (1.2) to L^γ-norms, de-Bruijn [6] proved an analogue of Zygmund's result for the class of polynomials not vanishing in $|z|<1$. Govil and Rahman [12] generalized and sharpened the inequality due to de-Bruijn for polynomials of degree n not vanishing in $|z|<k$, $k\geq 1$ and for any $\gamma\geq 1$. Gardner and Weems [9] not only generalized the above result of Govil and Rahman to Lucanary-tpye of polynomials but also showed that it holds for $0<\gamma<1$ as well. The author is curious to know how the above mentioned inequalities in the sup-norm especially (1.10) and (1.11), as well as some other inequalities, can be obtained from general L^γ-norm inequalities. Indeed, this paper is mainly motivated by the desire to establish some more general Zygmund-type inequalities for polynomials with restricted zeros. Our method of proof may be quite different from the previous methods and the results obtained here derive polar derivative analogues of some classical Bernstein-type inequalities for the sup-norm on the unit circle as well.

2. Main Results

Here, we first prove the following Zygmund-type integral inequality for polynomials not vanishing in a disk. Our result in particular gives the L^γ-analogue of (1.10) and other related inequalities. As special cases, some known inequalities that relate the sup-norm of the derivative of a polynomial on the unit circle to that of the polynomial itself, follows as consequences from this result.

Theorem 1 — *If $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j, 1\leq\mu\leq n$, is a polynomial of degree n having no zeros in $|z|<k,\ k\geq 1$, then for any complex numbers $\alpha,\ \beta,\ \delta$ with $|\alpha|\geq 1,\ |\delta|\leq 1$ and $\gamma\geq 1$,*

$$\left\{\int_0^{2\pi}\left|e^{i\theta}D_\alpha P(e^{i\theta})+mn\delta\left(\frac{|\alpha|-1}{1+S_1}\right)+n\beta\left(\frac{|\alpha|-1}{1+S_1}\right)P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq n\Big\{(|\alpha|+S_1)+|\beta|(|\alpha|-1)\Big\}C_\gamma(S_1,\mu)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{2.1}$$

where

$$C_\gamma(S_1,\mu)=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|S_1+e^{it}\right|^\gamma dt\right\}^{\frac{-1}{\gamma}},$$

and S_1 is as defined by formula (1.7).

Taking $\beta = 0$ in Theorem 1, we get the following generalization of (1.10).

Corollary 1 — If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \ge 1$, then for any complex numbers $\alpha,\ \delta$ with $|\alpha| \ge 1$, $|\delta| \le 1$ and $\gamma \ge 1$,

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + \frac{mn(|\alpha|-1)\delta}{1+S_1} \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$
$$\le n(|\alpha| + S_1) C_\gamma(S_1,\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.2}$$

where $C_\gamma(S_1,\mu)$ is as defined in Theorem 1 and S_1 is defined by formula (1.7).

Remark 2 : If we divide both sides of (2.2) by $|\alpha|$ and let $|\alpha| \to \infty$, we get a result of Mir, Dewan and Singh ([16], Theorem 1).

If we let $\gamma \to \infty$ in (2.2), noting that $C_\gamma(S_1,\mu) \to \frac{1}{1+S_1}$ and choose the argument of δ suitably with $|\delta| = 1$, we get (1.10). If we do not have the knowledge of $\min_{|z|=k} |P(z)|$, we get as special cases from Corollary 1 the following polar derivative analogues of some integral inequalities due to Gardner and Weems [9].

Corollary 2 — If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \le \mu \le n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge 1$ and $\gamma \ge 1$,

$$\left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \le n(|\alpha| + S_0) C_\gamma(S_0,\mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.3}$$

where

$$S_0 = k^{\mu+1} \left\{ \frac{\left(\frac{\mu}{n}\right) \left|\frac{a_\mu}{a_0}\right| k^{\mu-1} + 1}{\left(\frac{\mu}{n}\right) \left|\frac{a_\mu}{a_0}\right| k^{\mu+1} + 1} \right\}$$

and $C_\gamma(S_0,\mu) = \left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} \left| S_0 + e^{it} \right|^\gamma dt \right\}^{-\frac{1}{\gamma}}$.

Remark 3 : If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \ne 0$ in $|z| < k,\ k \ge 1$, then (see [19]) $\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^\mu \le 1$, which can also be taken as equivalent to $S_0 \ge k^\mu$. Taking $a = |\alpha| \ge 1, b = S_0, c = k^\mu$ in Lemma 2, we get for each $\gamma \ge 1$,

$$\frac{|\alpha| + S_0}{\left\{ \int_0^{2\pi} |S_0 + e^{it}|^\gamma dt \right\}^{\frac{1}{\gamma}}} \le \frac{|\alpha| + k^\mu}{\left\{ \int_0^{2\pi} |k^\mu + e^{it}|^\gamma dt \right\}^{\frac{1}{\gamma}}}.$$

Thus, from Corollary 2, we get the following result.

Corollary 3 — If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ \mu \geq 1$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for any complex number α with $|\alpha| \geq 1$ and $\gamma \geq 1$,

$$\left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq n(|\alpha| + k^\mu) C_\gamma(k, \mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.4}$$

where

$$C_\gamma(k, \mu) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| k^\mu + e^{it} \right|^\gamma dt \right\}^{\frac{-1}{\gamma}}.$$

Remark 4 : Dividing both sides of (2.3) and (2.4) by $|\alpha|$ and let $|\alpha| \to \infty$, we get respectively Theorem 2.1 and Corollary 2.2 of Gardner and Weems [9].

If we take $\delta = 0$ in Theorem 1, we get the following generalization of a result due to Mir and Wani [18].

Corollary 4 — If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$, is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for any complex numbers $\alpha,\ \beta$ with $|\alpha| \geq 1$ and $\gamma \geq 1$,

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta \left(\frac{|\alpha| - 1}{1 + S_1} \right) P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n \Big\{ (|\alpha| + S_1) + |\beta|(|\alpha| - 1) \Big\} C_\gamma(S_1, \mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.5}$$

where $C_\gamma(S_1, \mu)$ is as defined in Theorem 1 and S_1 is given by formula (1.7).

Remark 5 : For $k = \mu = 1$, the above corollary reduces to a result of Mir and Wani [18]. Dividing both sides of (2.5) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 5 — If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1$, then for any complex number β and $\gamma \geq 1$,

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} P'(e^{i\theta}) + \frac{n\beta}{1 + S_1} P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n(|\beta| + S_1) C_\gamma(S_1, \mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.6}$$

where $C_\gamma(S_1, \mu)$ is as defined in Theorem 1 and S_1 is given by formula (1.7).

Remark 6 : The above inequality (2.6) generalizes a result of Mir and Wani [18] and some inequalities obtained by de-Brujin [6] as well as Rahman and Schmeisser [20].

As an application of Corollary 1, we get the following Zygmund-type integral inequality for polynomials having all zeros in a closed disk involving the polar derivative.

***Theorem* 2** — *If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j},\ 1 \leq \mu \leq n,\ a_0 \neq 0$, is a polynomial of degree n having all zeros in $|z| \leq k,\ k \leq 1$, then for any complex numbers $\beta,\ \delta$ with $|\beta| \leq 1$, $|\delta| \leq 1$ and $\gamma \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\beta P(e^{i\theta}) + \frac{\delta(1-|\beta|)mnA_1}{k^n(1+A_1)} \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}$$

$$\leq n(|\beta| + A_1) C_\gamma(A_1, \mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \tag{2.7}$$

where

$$A_1 = \frac{n\left(|a_n| - \frac{m}{k^n}\right) k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n\left(|a_n| - \frac{m}{k^n}\right) k^{\mu-1} + \mu |a_{n-\mu}|}$$

and

$$C_\gamma(A_1, \mu) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| A_1 + e^{i\phi} \right|^\gamma d\phi \right\}^{\frac{-1}{\gamma}}.$$

Making $\gamma \to \infty$ in (2.7) and choosing the argument of δ with $|\delta| = 1$ and noting that $C_\gamma(A_1, \mu) \to \frac{1}{1+A_1}$, we get the following refinement of (1.11).

Corollary 7 — If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j},\ 1 \leq \mu \leq n,\ a_0 \neq 0$, is a polynomial of degree n having all zeros in $|z| \leq k,\ k \leq 1$, and β is any complex number with $|\beta| \leq 1$, then,

$$\max_{|z|=1} |D_\beta P(z)| \leq n \frac{(|\beta| + A_1)}{1 + A_1} \max_{|z|=1} |P(z)| - \frac{nA_1(1-|\beta|)}{k^n(1+A_1)} m, \tag{2.8}$$

where A_1 is as defined in Theorem 2.

Remark 7 : It is easy to verify, for example by the derivative test, that for every β with $|\beta| \leq 1$, the function $\left(\frac{|\beta|+x}{1+x}\right) \max_{|z|=1} |P(z)| - \frac{x(1-|\beta|)}{k^n(1+x)} \min_{|z|=k} |P(z)|$, is a non-decreasing function of x. If we combine this with the fact $A_1 \leq k^\mu$, that follows by Lemma 5, we get (1.11) from (2.8). If we take $\beta = 0$ in (2.8), we get for $|z| = 1$,

$$|nP(z) - zP'(z)| \leq \frac{nA_1}{1 + A_1} \max_{|z|=1} |P(z)| - \frac{mnA_1}{k^n(1+A_1)}. \tag{2.9}$$

Let z_0 be on $|z|=1$ be such that $\max_{|z|=1}|P(z)|=|P(z_0)|$, then from (2.9), we get

$$|z_0P'(z_0)|\geq\frac{n}{1+A_1}\left\{\max_{|z|=1}|P(z)|+\frac{mnA_1}{k^n}\right\}. \tag{2.10}$$

Since $\max_{|z|=1}|P'(z)|\geq|P'(z_0)|$ and it is well known (see inequality (1.17) in [8]) that

$$\max_{|z|=1}|P(z)|\geq\frac{m}{k^n},$$

we get from (2.10) that

$$\begin{aligned}\max_{|z|=1}|P'(z)|\geq{}&\frac{n}{1+k^\mu}\left\{\max_{|z|=1}|P(z)|+\frac{m}{k^{n-\mu}}\right\}\\&+\frac{n(k^\mu-A_1)}{(1+k^\mu)(1+A_1)}\left\{\max_{|z|=1}|P(z)|-\frac{m}{k^n}\right\}.\end{aligned} \tag{2.11}$$

Clearly (2.11) is a refinement of (1.8) in view of Lemma 5.

3. Lemmas

We need the following Lemmas to prove our theorems.

Lemma 1 — If $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j, 1\leq\mu\leq n$, is a a polynomial of degree n not vanishing in $|z|<k,\ k\geq1$ and $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$, then for $|z|=1$,

$$S_1|P'(z)|\leq|Q'(z)|-nm,$$

where S_1 is defined by formula (1.7).

The above lemma is due to Gardner, Govil and Weems [10].

The following lemma is due to Govil and Kumar [11].

Lemma 2 — If $a\geq1$, $b\geq c\geq1$ and $\gamma>0$, then

$$\frac{a+b}{\left\{\int\limits_0^{2\pi}|e^{i\theta}+b|^\gamma d\theta\right\}^{\frac{1}{\gamma}}}\leq\frac{a+c}{\left\{\int\limits_0^{2\pi}|e^{i\theta}+c|^\gamma d\theta\right\}^{\frac{1}{\gamma}}}.$$

Lemma 3 — Let p,q be any two positive real numbers such that $(q-h)\geq(p+h)x$, where $h\geq0$ and $x\geq1$. If β is any real number such that $0\leq\beta<2\pi$, then

$$[(q-h)+(p+h)y]|x+e^{i\beta}|\leq(x+y)|q+pe^{i\beta}|, \tag{3.1}$$

for any $y \geq 1$.

PROOF OF LEMMA 3 : For $h = 0$, the above lemma was recently proved by Govil and Kumar [11]. Hence, assume that $h > 0$. We have for real number β,

$$\left|\frac{x+e^{i\beta}}{q+pe^{i\beta}}\right|^2 = \frac{x^2+1+2x\cos\beta}{q^2+p^2+2pq\cos\beta}.$$

It is easy to see that the hypothesis $(q-h) \geq (p+h)x$ with $h > 0$ and $x \geq 1$ imply $q \geq px$ and $qx \geq p$. Therefore

$$\frac{x^2+1+2x\cos\beta}{q^2+p^2+2pq\cos\beta},$$

is a non-decreasing function of cosβ and hence

$$\left|\frac{x+e^{i\beta}}{q+pe^{i\beta}}\right|^2 \leq \frac{x^2+1+2x}{q^2+p^2+2pq} = \left(\frac{x+1}{q+p}\right)^2,$$

which implies

$$\left|\frac{x+e^{i\beta}}{q+pe^{i\beta}}\right| \leq \frac{x+1}{q+p}.$$

Again, by using $(q-h) \geq (p+h)x$ and $y \geq 1$, one can easily check that

$$\frac{x+1}{q+p} \leq \frac{x+y}{q+py+(y-1)h}.$$

Thus we have

$$\left|\frac{x+e^{i\beta}}{q+pe^{i\beta}}\right| \leq \frac{x+y}{(q-h)+(p+h)y},$$

which is (3.1) and this completes the proof of Lemma 3.

Lemma 4 — If $P(z)$ is a polynomial of degree n and $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then for every $\gamma > 0$ and β real,

$$\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta}) + e^{i\beta}P'(e^{i\theta})\right|^\gamma d\theta d\beta \leq 2\pi n^\gamma \int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta.$$

The above lemma is due to Aziz and Shah [3].

Lemma 5 — If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j},\ 1 \leq \mu \leq n$, is a polynomial of degree n having all zeros in $|z| \leq k,\ k \leq 1$, then

$$A_1 \leq k^\mu,$$

where A_1 is as defined in Theorem 2.

The above lemma is due to Dewan, Singh amd Mir [8].

4. PROOFS OF THE THEOREMS

PROOF OF THEOREM 1 : By hypothesis $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \neq 0$ in $|z| < k,\ k \geq 1$. If $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then $P(z) = z^n \overline{Q(\frac{1}{\bar{z}})}$ and it can be easily verified that for $0 \leq \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}. \tag{4.1}$$

For any complex number α and $0 \leq \theta < 2\pi$, we have

$$D_\alpha P(e^{i\theta}) = nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}),$$

which on using (4.1) gives,

$$\begin{aligned} |D_\alpha P(e^{i\theta})| &\leq |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + |\alpha||P'(e^{i\theta})| \\ &= |Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})|. \end{aligned} \tag{4.2}$$

Now for $\gamma \geq 1$ and $\beta,\ \delta \in \mathbb{C}$ with $|\delta| \leq 1$ and real t, we have by Minkowski's inequality,

$$\begin{aligned} &\left\{ \int_0^{2\pi} \left|S_1 + e^{it}\right|^\gamma dt \int_0^{2\pi} \left|e^{i\theta}D_\alpha P(e^{i\theta}) + \delta mn \left(\frac{|\alpha|-1}{1+S_1}\right) + n\beta\left(\frac{|\alpha|-1}{1+S_1}\right)P(e^{i\theta})\right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \\ &\quad = \left\{ \int_0^{2\pi}\int_0^{2\pi} \left|S_1 + e^{it}\right|^\gamma \left|e^{i\theta}D_\alpha P(e^{i\theta}) + \delta mn \left(\frac{|\alpha|-1}{1+S_1}\right) + n\beta\left(\frac{|\alpha|-1}{1+S_1}\right)P(e^{i\theta})\right|^\gamma d\theta dt \right\}^{\frac{1}{\gamma}} \\ &\quad \leq \left\{ \int_0^{2\pi}\int_0^{2\pi} \left|S_1 + e^{it}\right|^\gamma \left|e^{i\theta}D_\alpha P(e^{i\theta}) + \delta mn \left(\frac{|\alpha|-1}{1+S_1}\right) \right|^\gamma d\theta dt \right\}^{\frac{1}{\gamma}} \\ &\qquad + n|\beta|(|\alpha|-1)\left\{ \int_0^{2\pi}\int_0^{2\pi} \left|\frac{S_1 + e^{it}}{1+S_1}\right|^\gamma \left|P(e^{i\theta})\right|^\gamma d\theta dt \right\}^{\frac{1}{\gamma}} \\ &\quad \leq \left\{ \int_0^{2\pi}\int_0^{2\pi} \left|S_1 + e^{it}\right|^\gamma \left|e^{i\theta}D_\alpha P(e^{i\theta}) + \delta mn \left(\frac{|\alpha|-1}{1+S_1}\right) \right|^\gamma d\theta dt \right\}^{\frac{1}{\gamma}} \\ &\qquad + n|\beta|(|\alpha|-1)(2\pi)^{\frac{1}{\gamma}}\left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}. \end{aligned} \tag{4.3}$$

Recall that $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j \neq 0$ in $|z| < k,\ k \geq 1$, by Lemma 1, we have for $0 \leq \theta < 2\pi$,

$$S_1|P'(e^{i\theta})| \leq |Q'(e^{i\theta})| - mn,$$

which is equivalent to

$$S_1\left\{|P'(e^{i\theta})| + \frac{mn}{1+S_1}\right\} \leq |Q'(e^{i\theta})| - \frac{mn}{1+S_1}.$$

This gives by taking $q = |Q'(e^{i\theta})|,\ p = |P'(e^{i\theta})|,\ h = \frac{mn}{1+S_1},\ x = S1 \geq 1$ and $y = |\alpha| \geq 1$ in Lemma 3, we get for t real that

$$\begin{aligned}\Big\{\left(|Q'(e^{i\theta})| - \frac{mn}{1+S_1}\right) + |\alpha|\left(|P'(e^{i\theta})| + \frac{mn}{1+S_1}\right)\Big\}&|S_1 + e^{it}|\\ &\leq (S_1 + |\alpha|)\big||Q'(e^{i\theta})| + e^{it}|P'(e^{i\theta})|\big|. \qquad (4.4)\end{aligned}$$

On applying (4.2) and (4.4), we get for each $\gamma \geq 1$ and t real,

$$\begin{aligned}&\int_0^{2\pi}\left|S_1 + e^{it}\right|^{\gamma} dt \int_0^{2\pi}\left[\left|D_\alpha P(e^{i\theta})\right| + mn\left(\frac{|\alpha|-1}{1+S_1}\right)\right]^{\gamma} d\theta\\ &= \int_0^{2\pi}\int_0^{2\pi}\left|S_1 + e^{it}\right|^{\gamma}\left[\left|D_\alpha P(e^{i\theta})\right| + mn\left(\frac{|\alpha|-1}{1+S_1}\right)\right]^{\gamma} dtd\theta\\ &\leq \int_0^{2\pi}\int_0^{2\pi}\left|S_1 + e^{it}\right|^{\gamma}\left[|Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})| +\ mn\left(\frac{|\alpha|-1}{1+S_1}\right)\right]^{\gamma} dtd\theta\\ &= \int_0^{2\pi}\int_0^{2\pi}\left|S_1 + e^{it}\right|^{\gamma}\left[\left(|Q'(e^{i\theta})| - \frac{mn}{1+S_1}\right) + |\alpha|\left(|P'(e^{i\theta})| + \frac{mn}{1+S_1}\right)\right]^{\gamma} dtd\theta\\ &\leq (S_1 + |\alpha|)^{\gamma}\int_0^{2\pi}\int_0^{2\pi}\left||Q'(e^{i\theta})| + e^{it}|P'(e^{i\theta})|\right|^{\gamma} dtd\theta. \qquad (4.5)\end{aligned}$$

Observe that for every $\gamma \geq 1$ and $a, b \in \mathbb{C}$ with t real, we have

$$\int_0^{2\pi}\left|a + e^{it}b\right|^{\gamma} dt = \int_0^{2\pi}\left||a| + e^{it}|b|\right|^{\gamma} dt. \qquad (4.6)$$

Inequality (4.5) gives with the help of (4.6) and Lemma 4 for each $\gamma \geq 1$, t real and $|\alpha| \geq 1$,

$$\begin{aligned}\int_0^{2\pi}\left|S_1 + e^{it}\right|^{\gamma} dt \int_0^{2\pi}&\left[\left|D_\alpha P(e^{i\theta})\right| + mn\left(\frac{|\alpha|-1}{1+S_1}\right)\right]^{\gamma} d\theta\\ &\leq (S_1 + |\alpha|)^{\gamma}\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta}) + e^{it}P'(e^{i\theta})\right|^{\gamma} dtd\theta\\ &\leq (S_1 + |\alpha|)^{\gamma} 2\pi n^{\gamma}\int_0^{2\pi}\left|P(e^{i\theta})\right|^{\gamma} d\theta, \qquad (4.7)\end{aligned}$$

which on raising the power $\frac{1}{\gamma}$ on both sides after using the obvious inequality for $|\delta| \leq 1$,

$$\left|e^{i\theta}D_\alpha P(e^{i\theta}) + \frac{(|\alpha|-1)\delta mn}{1+S_1}\right| \leq |D_\alpha P(e^{i\theta})| + \frac{(|\alpha|-1)mn}{1+S_1},$$

in (4.3) gives (2.1) and this completes the proof of Theorem 1.

PROOF OF THEOREM 2 : By hypothesis $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j},\ 1 \leq \mu \leq n$, has all its zeros in $|z| \leq k,\ k \leq 1$ and $a_0 \neq 0.$ Therefore, the polynomial $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$ of degree n has no zeros in $|z| < \frac{1}{k},\ \frac{1}{k} \geq 1.$ Applying inequality (4.7) to $Q(z)$, we get for every complex number α with $|\alpha| \geq 1$ and $\gamma \geq 1$,

$$\begin{aligned}&\left\{\int_0^{2\pi}\left[\left|D_\alpha Q(e^{i\theta})\right| + \frac{m'n(|\alpha|-1)}{1+S'(1)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}} \\ &\qquad \leq n\left(|\alpha| + S'(1)\right) C_\gamma\left(S'(1),\mu\right)\left\{\int_0^{2\pi}\left|Q(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \end{aligned} \tag{4.8}$$

where

$$m' = \min_{|z|=\frac{1}{k}} |Q(z)| = \min_{|z|=\frac{1}{k}} \left|z^n \overline{P\left(\frac{1}{\bar{z}}\right)}\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m}{k^n},$$

$$S'(1) = \frac{1}{k^{\mu+1}}\left\{\frac{\frac{\mu}{n}\left(\frac{|a_{n-\mu}|}{|a_n|-m'}\right)\frac{1}{k^{\mu-1}}+1}{\frac{\mu}{n}\left(\frac{|a_{n-\mu}|}{|a_n|-m'}\right)\frac{1}{k^{\mu+1}}+1}\right\}$$

$$= \frac{\mu|a_{n-\mu}| + n\left(|a_n| - \frac{m}{k^n}\right)k^{\mu-1}}{\mu|a_{n-\mu}|k^{\mu-1} + n\left(|a_n| - \frac{m}{k^n}\right)k^{2\mu}}$$

$$= \frac{1}{A_1}$$

and

$$\begin{aligned}
C_\gamma(S'(1),\mu) &= C_\gamma\left(\frac{1}{A_1},\mu\right)\\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{A_1}+e^{i\phi}\right|^\gamma d\phi\right\}^{\frac{-1}{\gamma}}\\
&= A_1\left\{\frac{1}{2\pi}\int_0^{2\pi}|1+A_1e^{i\phi}|^\gamma d\phi\right\}^{\frac{-1}{\gamma}}\\
&= A_1\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|A_1+e^{i\phi}\right|^\gamma d\phi\right\}^{\frac{-1}{\gamma}}\\
&= A_1C_\gamma(A_1,\mu).
\end{aligned}$$

Using these observations and $|Q(e^{i\theta})|=|P(e^{i\theta})|$ for $0\le\theta<2\pi$ in (4.8), we obtain for $|\alpha|\ge 1$ and $\gamma\ge 1$, that

$$\begin{aligned}
&\left\{\int_0^{2\pi}\left[\left|D_\alpha Q(e^{i\theta})\right|+\frac{mn(|\alpha|-1)A_1}{k^n(1+A_1)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}\\
&\qquad\le n(|\alpha|A_1+1)C_\gamma(A_1,\mu)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \qquad (4.9)
\end{aligned}$$

It is easy to verify (for example, see [1]), that for $|\alpha|\neq 0$,

$$|D_\alpha Q(e^{i\theta})|=|\overline{\alpha}|\left|D_{\frac{1}{\overline{\alpha}}}P(e^{i\theta})\right|.$$

Using this in (4.9), we get for $|\alpha|\ge 1$ and $\gamma\ge 1$,

$$\begin{aligned}
&|\alpha|\left\{\int_0^{2\pi}\left[\left|D_{\frac{1}{\overline{\alpha}}}P(e^{i\theta})\right|+\frac{mn\left(1-\frac{1}{|\alpha|}\right)A_1}{k^n(1+A_1)}\right]^\gamma d\theta\right\}^{\frac{1}{\gamma}}\\
&\qquad\le n(|\alpha|A_1+1)C_\gamma(A_1,\mu)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}. \qquad (4.10)
\end{aligned}$$

Replacing $\frac{1}{\alpha}$ by β so that $|\beta| \leq 1$, we obtain from (4.10) that for $\gamma \geq 1$,

$$\left\{ \int_0^{2\pi} \left[\left| D_\beta P(e^{i\theta}) \right| + \frac{mn(1-|\beta|)A_1}{k^n(1+A_1)} \right]^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leq n(|\beta| + A_1) C_\gamma(A_1, \mu) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}},$$

which on using for any $|\delta| \leq 1$,

$$\left| e^{i\theta} D_\beta P(e^{i\theta}) + \frac{mn\delta(1-|\beta|)A_1}{k^n(1+A_1)} \right| \leq \left| D_\beta P(e^{i\theta}) \right| + \frac{mn(1-|\beta|)A_1}{k^n(1+A_1)},$$

gives (2.7) and this completes the proof of Theorem 2.

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