Indian J. Pure Appl. Math., **51**(4): 1383-1397, December 2020
 DOI: 10.1007/s13226-020-0472-y

GLOBAL NONEXISTENCE OF SOLUTIONS FOR A CLASS OF VISCOELASTIC LAMÉ SYSTEM

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(*Received* 20 *September* 2018; *after final revision* 16 *July* 2019;
accepted 21 *August* 2019)

The aim of this work is to study the blow-up result for a Lamé system of viscoelastic equation with variable-exponent nonlinearities and strong damping. Under some suitable conditions on the coefficients, memory term and initial data, we proved a global nonexistence result for the weak solution with positive initial energy.

Key words : Global nonexistence; variable-exponent; viscoelastic; Lamé system.

2010 Mathematics Subject Classification : 35B44, 35L72, 35N10.

1. Introduction and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We consider the following initial and boundary value problem:

$$u_{tt} - \Delta_{e,m(x)}u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(x,t,u,\nabla u) - \beta\Delta u_t$$
$$= \phi_{p(x)}(u), \quad x \in \Omega, t > 0 \tag{1.1}$$

$$u(x,t) = 0 \qquad x \in \partial\Omega, t > 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega \tag{1.3}$$

where $\Delta_{e,m(x)}$ denotes the variable-exponent elasticity operator, which is the differential operator defined by

$$\Delta_{e,m(x)}u = \mu\Delta u + (\alpha+\mu)\nabla(div\ u) + div\big(|\nabla u|^{m(x)-2}\nabla u\big),$$

α and μ are the Lamé constants which satisfy the following conditions

$$\mu > 0, \ \ \alpha + \mu \geq 0.$$

Also $\phi_{p(x)}(z) = |z|^{p(x)-2}z$. Here, β is a positive constant and the exponents $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ such that:

(A1)

$$2 \leq m^- \leq m(x) \leq m^+$$

$$2 \leq p^- \leq p(x) \leq p^+$$

with

$$m^- := essinf_{x\in\overline{\Omega}} m(x), \ \ m^+ := esssup_{x\in\overline{\Omega}} m(x),$$

$$p^- := essinf_{x\in\overline{\Omega}} p(x), \ \ p^+ := esssup_{x\in\overline{\Omega}} p(x).$$

In addition, $g(t)$ and $h(x,t,u,\nabla u)$ are functions satisfying the following conditions:

(A2) The kernel of memory, $g : R^+ \to R^+$, is a differentiable and non-increasing function satisfying

$$g(0) > 0, \quad \mu - \int_0^{+\infty} e^{-\lambda s} g(s) ds = l > 0,$$

for some positive λ.

(A3) $h(x,t,u,\nabla u)$ is a function that satisfies

$$|h(x,t,u,\nabla u)| \leq M_1|u| + M_2|\nabla u|,$$

for some positive M_1, M_2.

Before going any further, it is worth pointing out some classical results. In bounded domains, there is an extensive literature on the existence, asymptotic behaviour and nonexistence of solution for the following equation

$$u_{tt} - \Delta u + h(u_t) = f(u), \ \text{ in } \ \Omega \times (0, +\infty)$$

In the absence of the source term i.e. ($f = 0$), it is well known that the damping term $h(u_t)$ assures global existence and decay of the solution energy for arbitrary initial data (see [15, 16]). In contrast, in the absence of the damping term, the source term causes finite-time blow-up of solutions with a large initial data (negative initial energy) (see [5]). The interaction between the damping term and the source term was first considered by Levine [19] in the linear damping case $h(u_t) = au_t$ and a polynomial source term of the form $f(u) = b|u|^{(p-2)}u$. He showed that solutions with negative initial energy blow up in finite time (see also [20]). Georgiev and Todorova [13] extended Levines result to the nonlinear damping case $h(u_t) = a|u_t|^{(m-2)}u_t$ with Dirichlet boundary conditions. They showed that solutions with negative energy continue to exist globally in time if $m \geq p \geq 2$ and blow up in finite time if $p \geq m \geq 2$ and the initial energy is sufficiently negative.

In another study, the following m-Laplacian fourth-order equation with dissipative boundary condition was examined by Shahrouzi [26]

$$u_{tt} + \Delta[(a_0 + a|\Delta u|^{m-2})\Delta u] - b\Delta u_t = g(x,t,u,\Delta u) + |u|^{p-2}u, \quad x \in \Omega,\ t > 0,$$

$$u(x,t) = 0, \ \ \Delta u(x,t) = -c_0 \partial_\nu u(x,t), \quad x \in \partial\Omega,\ t > 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega.$$

He showed that when $p > m + 1 > 3$, there are solutions which blow up in finite time with positive initial energy. In all of these papers, the authors obtained the global existence and the finite time blow up of solutions by potential well theory combined with Galerkin method and concavity method.

It is known that viscoelastic materials show natural damping properties, which is due to the special property of these substances in keeping memory of their past history. From mathematical point of view, these damping effects are modeled by integro-differential operators. But modeling of some physical phenomena such as flows of electro-rheological fluids, nonlinear viscoelasticity and image processing give rise to equation with nonstandard growth conditions, that is, equations with variable exponents of nonlinearities. These models include hyperbolic, parabolic or elliptic equations that are nonlinear in gradient of the unknown solution and with variable exponents of nonlinearity. These problems have largely been discussed by several authors in the last decades, and have made a lot of progress. Fabrizio and Polidoro [21] studied the following equation with homogenous boundary condition

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = 0 \quad \text{in } \Omega \times (0,\infty)$$

and showed that the exponential decay of the memory term is a necessary condition for the exponential decay of the solution energy. Messaoudi [21] considered the following initial-boundary value problem:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m-2}u_t = |u|^{p-2}u, \ \text{ in } \ \Omega \times (0,+\infty)$$

$$u(x,t) = 0 \ \text{ on } \ \partial\Omega$$

$$u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x),\ \ \text{in}\ \ \Omega$$

where Ω was a bounded domain of $R^N(N\geq 1)$ with smooth boundary $\partial\Omega$, $m\geq 2,\ \ p>2$ and $g:R^+\to R^+$ was a positive nonincreasing function. He proved a blow up result for the solution with the negative initial energy if $p>m$, and a global existence result for $p\leq m$. Berrimi and Messaoudi [8] considered the same problem without damping term $|u_t|^{m-2}u_t$ and established a local existence result for $p>2$ and showed that the local solution was global and decays uniformly if the initial data were small enough.

In the case of variable exponent, Antontsev Stanislav [4] discussed the following problem:

$$u_{tt}=div(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u)+\alpha\Delta u_t+b(x,t)|u|^{\sigma(x,t)-2}u+f(x,t),\ x\in\Omega,t>0$$

$$u(x,t)=0,\ \ x\in\partial\Omega$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega$$

and proved the existence and blow up of weak solutions with negative initial energy under suitable conditions on the functions a,b,f,p and σ.

Shahrouzi [25] has recently considered the nonlinear viscoelastic equation with variable-exponent nonlinearities

$$u_{tt}-\Delta u-div(|\nabla u|^{m(x)}\nabla u)+\int_0^t g(t-\tau)\Delta u(\tau)d\tau+h(x,t,u,\nabla u)+\beta u_t=|u|^{p(x)}u,$$

under appropriate conditions, he proved a general decay result associated to solution energy. It is also shown that regarding arbitrary positive initial energy, solutions blow-up in a finite time. We refer the interested reader to 22, 28].

On the other hand, Lamé system has attracted considerable attention in recent years, where diverse types of dissipative mechanisms have been introduced and several stability and boundedness results have been obtained. Bisognin *et al*. [9] established that the solutions of a system in elasticity theory with a nonlinear localized dissipation decay in an algebraic rate to zero using some energy identities associated with localized multipliers.

In [6] Bchatnia and Daoulatli studied behavior of the energy for solutions to a Lamé system on a bounded domain with localized nonlinear damping and external force, that is

$$u''(x,t)-\Delta_e u(x,t)+a(x)g(u'(x,t))=f(t,x)\qquad \text{in}\ \Omega\times R^+,$$

$$u(x,t)=0\qquad \text{on}\ \partial\Omega\times R^+,$$

$$u(x,0) = u_0(x) \quad u'(x,0) = u_1(x) \qquad \text{on } \Omega.$$

Recently, Beniani *et al.* [7] considered the following Lamé system with viscoelastic term and strong damping in R^3:

$$u_{tt} + \alpha v - \Delta_e u(x,t) + \int_0^t g_1(t-s)\Delta u(x,s)ds - \mu_1 \Delta u_t(x,t) = 0, \qquad \text{in } \Omega \times (0,+\infty),$$

$$v_{tt} + \alpha u - \Delta_e v(x,t) + \int_0^t g_2(t-s)\Delta v(x,s)ds - \mu_2 \Delta v_t(x,t) = 0, \qquad \text{in } \Omega \times (0,+\infty),$$

$$u(x,t) = v(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,+\infty),$$

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)),\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) \qquad \text{in } \Omega.$$

They proved well-posedness of the problem by using Faedo-Galerkin method and established the exponential decay of the energy by introducing a suitable Lyapunov functional. For more results on the energy decay for the Lamé system with linear or nonlinear damping we refer the reader to [1-3, 14, 17] and references therein.

Motivated by the aforementioned works, in this paper, we try to extend the previous results from constant-exponent nonlinearities to variable-exponent nonlinearities. Indeed, this paper provide the sufficient conditions for the global nonexistence result to a variable-exponent Lamé system with strong damping term.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(.)}(\Omega)$, the Sobolev space, $W^{1,p(.)}(\Omega)$, that be use for main result. In Section 3, by using the modified concavity argument, the global nonexistence of solutions for the problem (1.1)-(1.3) under the conditions of positive initial data and appropriate domain for parameters is verified.

2. Preliminaries

In order to study problem (1.1)-(1.3), we need some theories about Lebesgue and Sobolev spaces with variable-exponents (for detailed, see [10, 27]). Let $p(x) \geq 1$ and measurable, we assume that

$$C_+(\overline{\Omega}) = \{h | h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$

$$h^+ = \max_{\overline{\Omega}} h(x), \quad h^- = \min_{\overline{\Omega}} h(x) \quad \text{for any } h \in C(\overline{\Omega}),$$

$$L^{p(x)}(\Omega) = \Big\{ u |\ u \text{ is a measurable real} - \text{valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(x)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)} := \inf\Big\{\lambda > 0 \Big| \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\Big\}.$$

Lemma 2.1 — [10, 27]. Let Ω be a bounded domain in R^n

(i) the space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$|\int_\Omega uvdx| \leq (\frac{1}{p^-} + \frac{1}{q^-})\|u\|_{p(x)}\|v\|_{q(x)}.$$

(ii) If $p, q \in C_+(\overline{\Omega})$, $q(x) \leq p(x)$ for any $x \in \overline{\Omega}$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and the imbedding is continuous.

The variable-exponent Lebesgue Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Furthermore, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The dual of $W_0^{1,p(x)}(\Omega)$ is defined as $W^{-1,p'(x)}(\Omega)$, by the same way as the usual Sobolev spaces, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

If we define

$$p^*(x) = \begin{cases} \frac{Np(x)}{esssup_{x\in\overline{\Omega}}(N-p(x))}, & p^+ < N \\ \infty, & p^+ \geq N, \end{cases}$$

then we have

Lemma 2.2 — [10, 27]. Let Ω be a bounded domain in R^n then for any measurable bounded exponent $p(x)$ we have

(i) $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable Banach spaces;

(ii) if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding

$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous;

(iii) if $p(x)$ is uniformly continuous in Ω then there exists a constant $C > 0$, such that

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (iii) of Lemma 2.2, we know that the space $W_0^{1,p(x)}(\Omega)$ has an equivalent norm given by $\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$.

We sometimes use the Young's inequality

$$ab \leq \beta a^q + C(\theta, q) b^{q'}, \quad a, b \geq 0, \quad \theta > 0, \quad \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.1}$$

where $C(\theta, q) = \frac{1}{q'}(\theta q)^{-\frac{q'}{q}}$ are constants.

The following lemma was introduced in [18]; it will be used in Section 3 in order to prove the global nonexistence result (see also [23, 24]).

Lemma 2.3 — Let $\gamma_0 > 0, c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Assume that $\psi(t)$ is a twice differentiable positive function such that

$$\psi^{''}\psi - (1+\gamma_0)[\psi^{'}]^2 \geq -2c_1\psi\psi' - c_2[\psi]^2,$$

for all $t \geq 0$. If

$$\psi(0) > 0 \qquad \text{and} \qquad \psi'(0) + \gamma_2\gamma_0^{-1}\psi(0) > 0,$$

then there exists a finite time t^*, such that

$$\psi(t) \to +\infty \qquad as \qquad t \to t^* \leq t_1 = \frac{1}{2\sqrt{c_1^2 + \gamma_0 c_2}} \log \frac{\gamma_1\psi(0) + \gamma_0\psi'(0)}{\gamma_2\psi(0) + \gamma_0\psi'(0)}.$$

Here

$$\gamma_1 = -c_1 + \sqrt{c_1^2 + \gamma_0 c_2} \qquad \text{and} \qquad \gamma_2 = -c_1 - \sqrt{c_1^2 + \gamma_0 c_2}.$$

For completeness we state, without proof, the existence result for the problem (1.1)-(1.3):(see [12, 22]).

***Theorem* 2.4** — *(Local existence) Let $u_0 \in H_0^1(\Omega) \cap W^{1,m(x)}(\Omega)$ and $u_1 \in H_0^1(\Omega) \cap L^2(\Omega)$ be given. Assume that (A1)-(A3) are satisfied; then the problem (1.1)-(1.3) has at least one weak solution such that*

$$u \in L^\infty\big((0,T), H_0^1(\Omega) \cap W^{1,m(x)}\big), \qquad u_t \in L^\infty\big((0,T), H_0^1(\Omega) \cap L^2(\Omega)\big),$$

$$u_{tt} \in L^\infty\big((0,T), H^2(\Omega) \cap W_0^{-1,m'(x)}(\Omega)\big),$$

where $\frac{1}{m(x)} + \frac{1}{m'(x)} = 1.$

3. Global Nonexistence

As we mentioned earlier, in this section we shall prove the global nonexistence of solutions for appropriate initial data. To prove the main result for positive initial energy, we assumed that:

(B1)

$$2 \leq m^- \leq m(x) \leq m^+ < p^- \leq p(x) \leq p^+ < m^*(x)$$

such that for some $\varepsilon, \lambda > 0$, $m^+ - 2 < \frac{\varepsilon}{\lambda} < p^- - 2$. Also

$$m^*(x) = \begin{cases} \frac{Nm(x)}{esssup_{x\in\overline{\Omega}}(N-m(x))}, & m^+ < N \\ \infty, & m^+ \geq N. \end{cases}$$

In order to carry the proof of blow-up result, the following problem is obtained from (1.1)-(1.3) when $e^{\lambda t}v(x,t)$ is substituted for $u(x,t)$:

$$v_{tt} + 2\lambda v_t + \lambda^2 v - (\lambda^2 + \beta\lambda + \mu)\Delta v - \Delta v_{tt} - (\alpha + \mu)\nabla(div\ v)$$

$$-div\Big(e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)-2}\nabla v\Big) + \int_0^t g_1(t-\tau)\Delta v(\tau)d\tau + e^{-\lambda t}\widehat{h}(t,v)$$

$$-(2\lambda+\beta)\Delta v_t = e^{\lambda(p(x)-2)t}|v|^{p(x)-2}v, \text{ in } \Omega \times (0,+\infty) \tag{3.1}$$

$$v(x,t) = 0, \text{ in } \partial\Omega \times (0,+\infty) \tag{3.2}$$

$$v(x,0) = u_0(x),\ \ v_t(x,0) = u_1(x) - \lambda u_0(x), \quad \text{in } \Omega. \tag{3.3}$$

where

$$g_1(s) = e^{-\lambda s}g(s), \qquad \hat{h}(t,v) = h(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v), \tag{3.4}$$

and the value of the parameter λ will be prescribed later.

The energy function related with problem (3.1)-(3.3) is given by

$$E_\lambda(t) = \int_\Omega \frac{e^{\lambda(p(x)-2)t}}{p(x)}|v|^{p(x)}dx - \frac{1}{2}I(t), \tag{3.5}$$

where

$$I(t) = \|v_t\|^2 + \lambda^2\|v\|^2 + \|\nabla v_t\|^2 + (\lambda^2 + \beta\lambda + \mu - \int_0^t g_1(s)ds)\|\nabla v\|^2 + (g_1 * \nabla v)(t)$$

$$+(\alpha+\mu)\int_\Omega |div\ v|^2 dx + 2\int_\Omega \frac{e^{\lambda(m(x)-2)t}}{m(x)}|\nabla v|^{m(x)}dx, \tag{3.6}$$

and

$$(g_1 * \nabla v)(t) = \int_0^t g_1(t-s)\|\nabla v(t) - \nabla v(s)\|^2 ds.$$

Now we are in a position to state our blow-up result as follows:

***Theorem* 3.1** — *Let the conditions (A1)-(A3) and (B1), are satisfied. Assume that $E_\lambda(0) > 0$ and $\mu < p^- + l$. Also suppose that for any $\varepsilon > 0$ and sufficiently large λ:*

$$\lambda \geq \max\Big\{\frac{M_1}{\varepsilon},\ \kappa^*\Big\},$$

where κ^* is a number that for all $\kappa > \kappa^*$ the following cubic polynomial is positive

$$s(\kappa) = \kappa^3 + \beta\kappa^2 + l\kappa - \frac{M_2^2}{4\varepsilon}.$$

Then there exists a finite time t^* such that

$$\|u(t)\| + \|\nabla u(t)\| \to +\infty \ \ as \ \ t \to t^*. \tag{3.7}$$

To prove the main result for positive initial energy, the following Lemma is required.

Lemma 3.2 — Under the conditions of Theorem 3.1, the energy functional $E_\lambda(t)$, defined by (3.5), satisfies

$$E_\lambda(t) \geq E_\lambda(0) \qquad\qquad \forall t \in R^+. \tag{3.8}$$

PROOF : A multiplication of equation (3.1) by v_t and integrating over Ω gives

$$\begin{aligned} E'_\lambda(t) = 2\lambda\|v_t\|^2 - \int_\Omega \frac{\lambda(m(x)-2)}{m(x)} e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx + \frac{1}{2}g(t)\|\nabla v\|^2 \\ -\frac{1}{2}(g_1' * \nabla v)(t) + \int_\Omega \frac{\lambda(p(x)-2)}{p(x)} e^{\lambda(p(x)-2)t}|v|^{p(x)+2}dx \\ +e^{-\lambda t}\int_\Omega v_t\hat{h}(t,v)dx + (2\lambda+\beta)\|\nabla v_t\|^2, \end{aligned} \tag{3.9}$$

using the condition (A2), we get

$$\begin{aligned} E'_\lambda(t) \geq 2\lambda\|v_t\|^2 + (2\lambda+\beta)\|\nabla v_t\|^2 - \int_\Omega \frac{\lambda(m(x)-2)}{m(x)} e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx \\ + \int_\Omega \frac{\lambda(p(x)-2)}{p(x)} e^{\lambda(p(x)-2)t}|v|^{p(x)+2}dx \\ +e^{-\lambda t}\int_\Omega v_t\hat{h}(t,v)dx. \end{aligned} \tag{3.10}$$

Employing the last inequality, we obtain from (3.5) the following inequality for some $\varepsilon > 0$

$$E'_\lambda(t) - \varepsilon E_\lambda(t) \geq \int_\Omega \Big(\frac{\lambda(p(x)-2)-\varepsilon}{p(x)}\Big) e^{\lambda(p(x)-2)t}|v|^{p(x)}dx + (2\lambda + \frac{\varepsilon}{2})\|v_t\|^2$$

$$+\frac{\varepsilon\lambda^2}{2}\|v\|^2+\int_\Omega\Big(\frac{\varepsilon-\lambda(m(x)-2)}{m(x)}\Big)e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx$$
$$+\frac{\varepsilon}{2}\big(\lambda^2+\beta\lambda+\mu-\int_0^t g_1(s)ds\big)\|\nabla v\|^2+(2\lambda+\beta+\frac{\varepsilon}{2})\|\nabla v_t\|^2$$
$$+\frac{\varepsilon}{2}(\alpha+\mu)\int_\Omega|div\ v|^2dx+\frac{\varepsilon}{2}(g_1*\nabla v)(t)$$
$$+e^{-\lambda t}\int_\Omega v_t\hat{h}(t,v)dx \tag{3.11}$$

Taking into account (A2) and the assumption (B1) on the exponents $p(x)$ and $m(x)$, we deduce

$$E'_\lambda(t)-\varepsilon E_\lambda(t)\geq[\lambda(p^--2)-\varepsilon]\int_\Omega\frac{1}{p(x)}e^{\lambda(p(x)-2)t}|v|^{p(x)}dx+(2\lambda+\frac{\varepsilon}{2})\|v_t\|^2$$
$$+\frac{\varepsilon\lambda^2}{2}\|v\|^2+[\varepsilon-\lambda(m^+-2)]\int_\Omega\frac{1}{m(x)}e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx$$
$$+\frac{\varepsilon}{2}\big(\lambda^2+\beta\lambda+l\big)\|\nabla v\|^2+e^{-\lambda t}\int_\Omega v_t\hat{h}(t,v)dx. \tag{3.12}$$

It is easy to verify that

$$e^{-\lambda t}|\int_\Omega v_t\hat{h}(t,v)dx|\leq(2\lambda+\frac{\varepsilon}{2})\|v_t\|^2+\frac{M_1^2}{2\varepsilon}\|v\|^2+\frac{M_2^2}{8\lambda}\|\nabla v\|^2, \tag{3.13}$$

where (A3) and Young's inequality (2.1) have been used.

By virtue of (3.13), we obtain from (3.12) the following inequality

$$E'_\lambda(t)-\varepsilon E_\lambda(t)\geq[\lambda(p^--2)-\varepsilon]\int_\Omega\frac{1}{p(x)}e^{\lambda(p(x)-2)t}|v|^{p(x)}dx$$
$$+[\varepsilon-\lambda(m^+-2)]\int_\Omega\frac{1}{m(x)}e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx$$
$$+(\frac{\varepsilon\lambda^2}{2}-\frac{M_1^2}{2\varepsilon})\|v\|^2+\frac{\varepsilon}{2}\big(\lambda^2+\beta\lambda+l-\frac{M_2^2}{4\varepsilon\lambda}\big)\|\nabla v\|^2. \tag{3.14}$$

Thanks to the hypothesises of Theorem 3.1, we get

$$E'_\lambda(t)-\varepsilon E_\lambda(t)\geq 0. \tag{3.15}$$

$E_\lambda(t)\geq e^{\varepsilon t}E_\lambda(0)\geq E_\lambda(0)$ is obtained from (3.15) and the proof of this Lemma is completed. □

PROOF OF THEOREM 3.1 : For obtaining the nonexistence result, we set

$$\psi(t)=\|v\|^2+\|\nabla v\|^2, \tag{3.16}$$

then

$$\psi'(t) = 2\int_\Omega v v_t dx + 2\int_\Omega \nabla v \nabla v_t dx,$$

$$\psi''(t) = 2(\|v_t\|^2 + \|\nabla v_t\|^2) + 2\int_\Omega v(v_{tt} - \Delta v_{tt})dx.$$

A multiplication of equation (3.1) by v and integrating over Ω gives

$$\begin{aligned}\int_\Omega v(v_{tt}-\Delta v_{tt})dx = -2\lambda\int_\Omega v v_t dx - \lambda^2\|v\|^2 - (\lambda^2+\beta\lambda+\mu)\|\nabla v\|^2 \\ -(\alpha+\mu)\int_\Omega |div\ v|^2 dx - \int_\Omega e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx \\ +\int_\Omega \nabla v\int_0^t g_1(t-\tau)\nabla v(\tau)d\tau dx - (2\lambda+\beta)\int_\Omega \nabla v\nabla v_t dx \\ +\int_\Omega e^{\lambda(p(x)-2)t}|v|^{p(x)}dx - e^{-\lambda t}\int_\Omega v\hat{h}(t,v)dx. \qquad (3.17)\end{aligned}$$

For any $\delta > 0$, we have

$$\begin{aligned}\int_\Omega v(v_{tt}-\Delta v_{tt})dx = \delta E_\lambda(t) - \delta\int_\Omega \frac{1}{p(x)}e^{\lambda(p(x)-2)t}|v|^{p(x)}dx \\ +\delta\int_\Omega \frac{1}{m(x)}e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx + \frac{\delta}{2}\|v_t\|^2 + \lambda^2(\frac{\delta}{2}-1)\|v\|^2 \\ +\frac{\delta}{2}\|\nabla v_t\|^2 + [(\frac{\delta}{2}-1)(\lambda^2+\beta\lambda+\mu) - \frac{\delta}{2}\int_0^t g_1(s)ds]\|\nabla v\|^2 \\ +\frac{\delta}{2}(g_1 * \nabla v)(t) + (\frac{\delta}{2}-1)(\alpha+\mu)\int_\Omega |div\ v|^2dx - 2\lambda\int_\Omega v v_t dx \\ -\int_\Omega e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx + \int_\Omega \nabla v\int_0^t g_1(t-\tau)\nabla v(\tau)d\tau dx \\ -(2\lambda+\beta)\int_\Omega \nabla v\nabla v_t dx + \int_\Omega e^{\lambda(p(x)-2)t}|v|^{p(x)}dx \\ -e^{-\lambda t}\int_\Omega v\hat{h}(t,v)dx. \qquad (3.18)\end{aligned}$$

According to (A1), (A2) and definition (3.16), we deduce

$$\begin{aligned}\psi''(t) \geq 2\delta E_\lambda(t) + 2(1-\frac{\delta}{p^-})\int_\Omega e^{\lambda(p(x)-2)t}|v|^{p(x)}dx + \lambda^2(\delta-2)\|v\|^2 \\ +2(\frac{\delta}{m^+}-1)\int_\Omega e^{\lambda(m(x)-2)t}|\nabla v|^{m(x)}dx + (\delta+2)\|v_t\|^2 \\ +[(\delta-2)(\lambda^2+\beta\lambda)+\delta l-2\mu]\|\nabla v\|^2 + (\delta+2)\|\nabla v_t\|^2\end{aligned}$$

$$
\begin{aligned}
&+\delta(g_1 * \nabla v)(t) - 2(2\lambda+\beta)\int_\Omega \nabla v \nabla v_t dx \\
&+(\delta-2)(\alpha+\mu)\int_\Omega |div\ v|^2 dx - 4\lambda \int_\Omega v v_t dx \\
&\quad +2\int_\Omega \nabla v \int_0^t g_1(t-\tau)\nabla v(\tau) d\tau dx - 2e^{-\lambda t}\int_\Omega v\hat{h}(t,v)dx. \qquad (3.19)
\end{aligned}
$$

Now if we choose $\delta = p^-$ and apply (B1) then we arrive at

$$
\begin{aligned}
\psi''(t) \geq\ & 2p^- E_\lambda(t) + (p^-+2)(\|v_t\|^2 + \|\nabla v_t\|^2) + \lambda^2(p^- - 2)\|v\|^2 \\
&+[(p^- - 2)(\lambda^2+\beta\lambda) + p^- l - 2\mu]\|\nabla v\|^2 - 2(2\lambda+\beta)\int_\Omega \nabla v\nabla v_t dx \\
&+p^-(g_1 * \nabla v)(t) + (p^- - 2)(\alpha+\mu)\int_\Omega |div\ v|^2 dx - 4\lambda\int_\Omega v v_t dx \\
&\quad +2\int_\Omega \nabla v\int_0^t g_1(t-\tau)\nabla v(\tau)d\tau dx - 2e^{-\lambda t}\int_\Omega v\hat{h}(t,v)dx. \qquad (3.20)
\end{aligned}
$$

Due to the conditions (A1),(A2) and Cauchy-Schwarz, Young's inequalities, and the fact that $\int_0^t g_1(s)ds \leq \int_0^\infty g_1(s)ds = \mu - l$, we obtain

$$
4\lambda\int_\Omega v v_t dx + 2(2\lambda+\beta)\int_\Omega \nabla v\nabla v_t dx \leq (2\lambda+\beta)\psi'(t), \qquad (3.21)
$$

$$
\int_\Omega \nabla v\int_0^t g_1(t-\tau)\nabla v(\tau)d\tau dx \geq -\frac{\mu - l}{2}(g_1 * \nabla v)(t) - l\|\nabla v\|^2, \qquad (3.22)
$$

$$
e^{-\lambda t}\left|\int_\Omega v\hat{h}(t,v)dx\right| \leq \frac{(p^- - 2)(\beta\lambda + l)}{2}\|\nabla v\|^2 + \Big(M_1 + \frac{M_2^2}{2(p^- - 2)(\beta\lambda + l)}\Big)\|v\|^2. \qquad (3.23)
$$

By applying (3.21)-(3.23) into (3.20), we see that

$$
\begin{aligned}
\psi''(t) \geq\ & 2p^- E_\lambda(t) + (p^- + 2)(\|v_t\|^2 + \|\nabla v_t\|^2) + [(p^- - 2)\lambda^2 - 2\mu]\|\nabla v\|^2 \\
&+\Big((p^- - 2)\lambda^2 - \frac{M_2^2}{(p^- - 2)(\beta\lambda + l)} - M_1\Big)\|v\|^2 \\
&+\big(p^- - (\mu - l)\big)(g_1 * \nabla v)(t) - (2\lambda+\beta)\psi'(t),
\end{aligned}
$$

at this point by choosing λ sufficiently large and $\mu < p^- + l$, we deduce

$$
\psi''(t) \geq 2p^- E_\lambda(t) + (p^- + 2)(\|v_t\|^2 + \|\nabla v_t\|^2) - (2\lambda+\beta)\psi'(t). \qquad (3.24)
$$

By using the Cauchy-Schwarz and Young's inequalities, it can be clearly be observed that

$$
\Big(\int_\Omega v v_t dx\Big)\Big(\int_\Omega \nabla v\nabla v_t dx\Big) \leq \frac{1}{2}\big(\|v_t\|^2\|\nabla v\|^2 + \|v\|^2\|\nabla v_t\|^2\big), \qquad (3.25)
$$

$$(\psi'(t))^2 \leq 4\big(\|v_t\|^2 + \|\nabla v_t\|^2\big)\big(\|v\|^2 + \|\nabla v\|^2\big). \tag{3.26}$$

Taking into account (3.25), (3.26) and Lemma 3.2 in relation with (3.24) we get

$$\begin{aligned}\psi(t)\psi''(t) &\geq 2p^- E_\lambda(t) + \frac{p^-+2}{4}(\psi'(t))^2 - (2\lambda+\beta)\psi(t)\psi'(t)\\ &\geq \frac{p^-+2}{4}(\psi'(t))^2 - (2\lambda+\beta)\psi(t)\psi'(t).\end{aligned} \tag{3.27}$$

By definition of $\psi(t)$, (3.16) and since $E_\lambda(0) > 0$, it is easy to see that

$$0 < \psi(0) < \frac{p^- - 2}{4(2\lambda+\beta)}\psi'(0),$$

for some u_0 and u_1.

Hence, we see that the hypotheses of Lemma 2.3 are fulfilled with

$$\gamma_0 = \frac{p^- - 2}{4}, \qquad c_1 = \frac{2\lambda+\beta}{2}, \qquad c_2 = 0,$$

and the conclusion of Lemma 2.3 demonstrates that

$$\|u(t)\| + \|\nabla u(t)\| \to +\infty \;\; as \;\; t \to t^*.$$

Since this system is equivalent to (1.1)-(1.3), the proof is complete.

ACKNOWLEDGEMENT

The author would like to thank the anonymous referee for the careful reading of this paper and for the valuable comments and suggestions to improve the presentation of the paper.

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