Indian J. Pure Appl. Math., **51**(4): 1417-1432, December 2020
 DOI: 10.1007/s13226-020-0474-9

4-TRANSITIVITY AND CROSS-RATIO IN MOUFANG-KLINGENBERG PLANES OVER RINGS OF PLURAL NUMBERS

F.O. Erdoğan

Department of Mathematics, Faculty of Science and Art,
University of Uludağ, 16059, *Bursa, Turkey*
e-mail: fatmaozen@uludag.edu.tr

(*Received* 20 *September* 2017; *accepted* 2 *September* 2019)

In this paper, we carry the obtained results for a certain class of Moufang-Klingenberg planes coordinatized by the local alternative ring of dual numbers, over to a class of Moufang-Klingenberg planes coordinatized by the plural algebra of order m which is a local alternative ring. So, we show that its collineation group is transitive on quadrangles and therefore that coordinatization of the planes is independent of the choice of the coordinatization quadrangle. Moreover, by a conjugacy class we discuss the definition of cross-ratio of four distinct points on the projective lines for the planes.

Key words : local ring; Moufang-Klingenberg plane; 4-transitivity; projective collineation; cross-ratio.

2010 Mathematics Subject Classification : 51A35, 51A10.

1. INTRODUCTION

If a geometric structure is transitive on a class of quadrangle (or 4-gons), then this usually has some interesting consequences in projective geometry. For example, the coordinatization of a projective Klingenberg (PK) plane is independent of the choice of the coordinatization quadrangle (see [21, Corollary 1, p. 79]). The transitivity of the collineation groups on 4-gons in Pappian and Desarguesian planes is given by Theorem 4.4.11 and Theorem 5.2.6 in [26], respectively. As for Moufang planes, the fact that this property is valid has been proved in [23, Satz 14 in Section 7.3]. Besides, for its alternative proof, it can be seen to [12, Theorem 1].

In the paper of [9] Celik *et al.* studied with the certain class (which is denoted by $\mathbb{M}(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by the local alternative ring $\mathcal{A} := \mathbf{A}\,(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$

(an alternative ring $\mathbf{A}$, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [8]. Moreover, they introduced some collineation of $\mathbb{M}(\mathcal{A})$ and so obtained the important result that the collineation group of $\mathbb{M}(\mathcal{A})$ acts transitively on 4-gons.

The first aim of the present paper is to carry the result of 4-transitivity obtained in [9] for a certain class of MK planes coordinatized by a local alternative ring of dual numbers, over the certain class (which is denoted by $\mathbb{M}(\mathcal{A})$) of MK planes coordinatized by the plural algebra (of order m) $\mathcal{A}{:=}\mathbb{O}\left(\eta\right) = \mathbb{O} + \mathbb{O}\eta + \mathbb{O}\eta^{\mathbf{2}} + \cdots + \mathbb{O}\eta^{\mathbf{m-1}}$ defined over a Cayley division algebra $\mathbb{O}$ where $\eta^m = 0$ and $\eta \notin \mathbb{O}$, that is, by the local alternative ring $\mathbb{O}\left[\eta\right]/\langle\eta^m\rangle$.

The cross-ratio of four distinct collinear points A, B, C, D in Euclidean geometry [15, p. 107] is the number defined to be

$$(A, B; C, D) = \frac{AC}{BC} : \frac{AD}{BD}.$$

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria 300 B.C) is invariant under projection [4, p. 133]. The concept of cross-ratio (expressed as a ratio of ratios in [23, p. 122] or anharmonic ratio in [19, p. 150]horadam) has a great importance since it is the unique numerical concept which is a projective invariant. But, since the coordinatizing ring of a projective plane has different algebraic properties a general definition of cross-ratio which is valid in all of projective planes could not be given. The closest one to the definition of the cross-ratio given in the Euclidean plane is that given for Pappian plane whose coordinatizing ring is a field, the cross-ratio of four distinct collinear points $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$ is then given by

$$(A, B; C, D) = \frac{(a_1c_2 - a_2c_1)(b_1d_2 - b_2d_1)}{(a_1d_2 - a_2d_1)(b_1c_2 - b_2c_1)}.$$

If we take non-homogenous coordinates $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, $c = \frac{c_1}{c_2}$, $d = \frac{d_1}{d_2}$ instead of $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$, respectively, then the above definition of cross-ratio can be stated as follows [19, p. 151]:

$$(A, B; C, D) = \frac{(a - c)(b - d)}{(a - d)(b - c)} = (a, b; c, d).$$

(Note that if any one of the points A, B, C, D is ∞, the factors involving ∞ are cancelled [18, p. 81]).

As for Moufang (or Desarguesian) planes which are a more general class than Pappian plane, Ferrar [16] gives the following algebraic definition of the cross-ratio for the points on the line $[0, 0]$

$$(A, B; C, D) = (a, b; c, d) :=< \Big((a - d)^{-1}(b - d)\Big)\Big((b - c)^{-1}(a - c)\Big) >$$

where $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $D = (d, 0)$. Here, $< x >= \left\{ y^{-1}xy \mid y \in R \right\}$ where R is coordinatizing ring. According to the definition, if any one of the points A, B, C, D is ∞, the factors involving ∞ are cancelled. The definition of the cross-ratio given for the points on the line $[0, 0]$ is extended to the whole plane, by considering perspectivities preserve cross-ratio [13, p. 25]. For more information about some well-known properties of cross-ratio in the case of Moufang planes, it can be seen the papers of [5, 12-14, 16].

In the certain class of Moufang-Klingenberg (MK) planes $\mathbb{M}(\mathcal{A})$ coordinatized by the local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$, Blunck, in [8], introduced the definition of the cross-ratio for the points on the line $[1, 0, 0]$ and later Akpinar *et al.*, in [1], extended this definition to the whole plane $\mathbb{M}(\mathcal{A})$, by considering perspectivities preserve cross-ratio. Moreover, in [2], some properties of cross-ratios obtained in [1] for points of $\mathbb{M}(\mathcal{A})$ are carried over lines of $\mathbb{M}(\mathcal{A})$ and so a relation between the harmonicity and the harmonic position is established. Besides, by collineations preserving cross-ratio Basri *et al.*, in [10], generalized the relation between harmonicity and harmonic position which is given for the points of only the line $[1, 0, 0]$ in [1], to any line of $\mathbb{M}(\mathcal{A})$. For more information about some well-known properties of cross-ratio in the case of the MK planes, it can be seen the papers of [1, 2, 6-8, 10] and also [11].

The second aim of the paper is to introduce, by the help of conjugacy classes as given by Blunck [7] in the class of the MK planes, the definition of cross-ratio for four points on the projective lines in the class of MK planes $\mathbb{M}(\mathcal{A})$ coordinatized by the local ring $\mathcal{A}{:=}\mathbb{O}(\eta) = \mathbb{O} + \mathbb{O}\eta + \mathbb{O}\eta^{\mathbf{2}} + \cdots + \mathbb{O}\eta^{\mathbf{m-1}}$.

The paper is organized as follows. In Section 2, there are some basic definitions and results from the literature. In Section 3, some mappings used to show that the collineation group of $\mathbb{M}(\mathcal{A})$, denoted by $\mathcal{G}$, acts transitively on 4-gons are given and it is stated that these mappings are collineations of $\mathbb{M}(\mathcal{A})$. Finally, in this section, it is proved that $\mathcal{G}$ acts transitively on the set of 4-gons. In Section 4, it is discussed the definition of cross-ratio for the MK planes coordinatized by the local ring $\mathcal{A}{:} = \mathbb{O} + \mathbb{O}\eta + \mathbb{O}\eta^{\mathbf{2}} + \cdots + \mathbb{O}\eta^{\mathbf{m-1}}$ with the maximal ideal $\mathcal{A}\eta$.

2. Preliminieries

In this section we will give some definitions and results, which will be the basis of this paper.

Let $\mathbb{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$(points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$. Then $\mathbb{M}$ is called a *projective Klingenberg plane* (PK plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \wedge h$ on both g and h.

(PK3) There is a projective plane $\mathbb{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : \mathbb{M} \to \mathbb{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(g) = \Psi(h) \iff g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line h such that $P \in h$ for some line $h \sim g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\mathbb{M}$. The notion of a *centre* and an *axis of a* collineation is as in ordinary projective planes. A (C, a)-*collineation* ϕ of $\mathbb{M}$ is a collineation with centre C and axis a. If $C \in a$, then ϕ is a (C, a)-*elation* and also if $C \notin a$, then ϕ is a (C, a)-*homology*.

An *Moufang-Klingenberg Plane* (MK plane) is a PK plane $\mathbb{M}$ that generalizes a Moufang plane, and for which $\mathbb{M}^*$ is a Moufang plane, i.e. a projective plane admitting all possible elations.

An *alternative ring (field)* R is a not necessarily associative ring (field) that satisfies the *alternative laws* $a\,(ab) = a^2 b, (ba)\,a = ba^2,\ \forall a, b \in R$. An alternative ring R with identity element 1 is called *local* if the set I of its non-unit elements is an ideal. Then $R\,/I$ is a (skew) field and also either x or $1 - x$ is a unit.

So we can give consecutively two lemmas about alternative rings.

Lemma 1 — The subring generated by any two elements of an alternative ring is associative [26].

Lemma 2 — The identities

$$\begin{aligned} x\,(y\,(xz)) &= (xyx)\,z \\ ((yx)\,z)\,x &= y\,(xzx) \\ (xy)\,(zx) &= x\,(yz)\,x \end{aligned}$$

which are known as Moufang identities are satisfied in every alternative ring [24].

Now we give consecutively four lemmas related to local alternative rings, from [3]. We will need these in showing that transformations $\mathrm{T}_{a,0}$ and $\mathrm{T}_{0,b}$ in Section 3 are collineations.

Lemma 3 — Let $a \in R$, $y \in I$. Then $y\,(ay + 1)^{-1} = (ya + 1)^{-1}\,y$

Lemma 4 — Let $a, y \in R$, $u \in I$ and $n = (ua+1)^{-1}u$ then $y[(ua)n] = [(yu)a]n$.

Lemma 5 — Let $a, u \in R$, $y \in I$ and $t = y(ay+1)^{-1}$ then $[t(ay)]u = t[a(yu)]$

Lemma 6 — Let $a, x, v \in R$; $u\, y \in I$, with $y - u = xv$, $n = (ua+1)^{-1}u$ and $t = y(ay+1)^{-1}$. Then $t - n = [x - t(ax)][v - (va)n]$.

An *n-tuple* ($n \geq 3$) of pairwise non-neighbour points is called an (ordered) *n-gon* if no three of its elements are on neighbour lines [9].

We summarize some basic concepts about the coordinatization of MK-planes from [6].

Let R be a local alternative ring. Then $\mathbb{M}(R) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned} P &= \{(x,y,1) |\, x,y \in R\} \cup \{(1,y,z) |\, y \in R,\ z \in I\} \cup \{(w,1,z) |\, w,z \in I\} \\ L &= \{[m,1,p] \mid m,p \in R\} \cup \{[1,n,p] \mid p \in R,\ n \in I\} \cup \{[q,n,1] \mid q,n \in I\}, \end{aligned}$$

$$\begin{aligned} [m,1,p] &= \{(x, xm+p, 1) |\, x \in R\} \cup \{(1, zp+m, z) |\, z \in I\}, \\ [1,n,p] &= \{(yn+p, y, 1) |\, y \in R\} \cup \{(zp+n, 1, z) |\, z \in I\}, \\ [q,n,1] &= \{(1, y, yn+q) |\, y \in R\} \cup \{(w, 1, wq+n) |\, w \in I\}. \end{aligned}$$

$$\begin{aligned} P &= (x_1,x_2,x_3) \sim (y_1,y_2,y_3) = Q \iff x_i - y_i \in I \ \ (i=1,2,3)), \forall P,Q \in \mathbf{P}; \\ g &= [x_1,x_2,x_3] \sim [y_1,y_2,y_3] = h \Leftrightarrow x_i - y_i \in I \ \ (i=1,2,3)), \forall g,h \in \mathbf{L}. \end{aligned}$$

Baker *et al.* [3] use $(O = (0,0,1), U = (1,0,0), V = (0,1,0), E = (1,1,1))$ as a *coordinatization 4-gon*. We stick to this notation throughout this paper. For more detailed information about the coordinatization see [3, 6].

Now it is time to give the following theorem from [3].

***Theorem* 7** — *$\mathbb{M}(R)$ is a MK plane, and each MK plane is isomorphic to some $\mathbb{M}(R)$.*

Let $\mathbb{F}$ be a field. Let $\eta^m = 0$ for $\eta \notin \mathbb{F}$. Consider $\mathcal{A} := \mathbb{F}(\eta) = \mathbb{F} + \mathbb{F}\eta + \mathbb{F}\eta^2 + \cdots + \mathbb{F}\eta^{m-1}$ with componentwise addition and multiplication modulo η^m. Then $\mathcal{A}$ is a (unital, commutative and associative) local ring with the maximal ideal $\mathbf{I} = \eta\mathcal{A}$ of non-units. Also, the local ring $\mathcal{A}$ can be considered as *plural* $\mathbb{F}-$*algebra of order* m with a basis $\{1, \eta, \eta^2, \cdots, , \eta^{m-1}\}$. Note that the algebra can be seen as quotient ring of the polynomial ring $\mathbb{F}[\eta]$ by the principal ideal $\langle \eta^m \rangle$. For more detailed information about quotient rings, it can be seen to [25]. If we choose $\mathbb{R}$ instead of $\mathbb{F}$, then we have the *real plural algebra of order* m (see [21, Def. 1.1]) and also we reach the result, stated in [21, Prop. 1.7], that $\mathcal{A}$ is isomorphic to the linear algebra of matrix $M_{mm}(\mathbb{R})$ of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-2} & b_{m-1} \\ 0 & b_0 & b_1 & \ddots & \ddots & b_{m-2} \\ \vdots & 0 & b_0 & \ddots & b_2 & \ddots \\ \vdots & \vdots & 0 & \ddots & b_1 & b_2 \\ \vdots & \vdots & \vdots & \ddots & b_0 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & b_0 \end{pmatrix}$$

where $b_i \in \mathbb{R}$ for $0 \leq i \leq m-1$ (for the detailed proof of this, see to [17]).

It is clear that an element x of $\mathcal{A}$ is of the form $x = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{m-1}\eta^{m-1}$ where $a_i \in \mathbb{F}$ for $0 \leq i \leq m-1$.

Now we can consecutively state the following two results, analogues of Proposition 1.3 and 1.5 given in [21], without proof.

Proposition 8 — An element $x = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{m-1}\eta^{m-1} \in \mathcal{A}$ is a unit if and only if $a_0 \neq 0$.

Proposition 9 — $\mathcal{A}$ is a local ring with maximal ideal $\eta\mathcal{A}$. The subsets $\eta^j\mathcal{A}$, $1 \leq j \leq m$, are all ideals in $\mathcal{A}$.

Let $\mathbb{O}$ be a Cayley division algebra over its centre $Z(\mathbb{O}) = \mathbb{F}$, where $\mathbb{F}$ is a field. In this case, $\mathbb{O}$ admits the *proper anti-automorphism* $k : x \longmapsto \bar{x}$ and so on it can be defined the *trace (linear) form* t by the rule $x + \bar{x}$ ($t(x) \in \mathbb{F}$) and the *norm (quadratic) form* n by the rule $x\bar{x}$ ($n(x) \in \mathbb{F}$). So, for any $x, y \in \mathbb{O}$ we can obtain that $x^2 = -n(x) + xt(x)$ and define the *associated form* $n(x,y) := [n(x+y) - n(x) - n(y)]$ where $n(x,y)$ denotes the *symmetric bilinear form* w.r.t. n. Note that $n(x,y) = n(\overline{x}, \overline{y})$ and $n(x,y) = t(x\overline{y})$. Moreover, $\mathbb{O}$ has a basis $(e_0 = 1, e_1, e_2, ..., e_7)$ with multiplication table as in [20, p. 448].

We complete this section by giving definition of an associator. The *associator*

$$\langle x, y, z\rangle = (xy)z - x(yz)$$

of any three elements measures associativity (and lack of it) in an algebra. Note that the associator is the trilinear map.

3. 4-TRANSITIVITY

We can consider as a Cayley plural algebra of order m (that is, the local alternative ring $\mathbb{O}[\eta]/\langle\eta^m\rangle$ or the modular polynomial rings over $\mathbb{O}$) to $\mathcal{A}{:=}\mathbb{O}(\eta)=\mathbb{O}+\mathbb{O}\eta+\mathbb{O}\eta^{\mathbf{2}}+\cdots+\mathbb{O}\eta^{\mathbf{m-1}}$. Then, we have

Proposition 10 — $\mathcal{A}$ is an (unital) alternative local ring with

$$\mathbf{I}=\left\{x_1\eta+x_2\eta^2+\cdots+x_{m-1}\eta^{m-1}\middle|\ x_i\in\mathbb{O}\text{ for }1\leq i\leq m-1\right\}$$

as ideal of non-units.

PROOF : Let us give only the proof of (left) alternativity in $\mathcal{A}$. Then, we must show that $\langle a,a,b\rangle=0$ for every $a=a_0+a_1\eta+a_2\eta^2+\cdots+a_{m-1}\eta^{m-1}$ and $b=b_0+b_1\eta+b_2\eta^2+\cdots+b_{m-1}\eta^{m-1}\in\mathcal{A}$. Since $\mathbb{O}$ is alternative, then the following equalities are satisfied:

1) $\langle a_0,a_0,b_0\rangle=0$,
2) $\langle a_0,a_0,b_1\rangle=0$ and $\langle a_0,a_1,b_0\rangle+\langle a_1,a_0,b_0\rangle=\langle a_0+a_1,a_1+a_0,b_0\rangle=0$,
3) $\langle a_0,a_0,b_2\rangle=0$, $\langle a_1,a_1,b_0\rangle=0$, $\langle a_0,a_1,b_1\rangle+\langle a_1,a_0,b_1\rangle=\langle a_0+a_1,a_1+a_0,b_1\rangle=0$ and $\langle a_0,a_2,b_0\rangle+\langle a_2,a_0,b_0\rangle=\langle a_0+a_2,a_2+a_0,b_0\rangle=0$,
$\vdots$

m-1) $\langle a_0,a_0,b_{m-1}\rangle=0$, $\langle a_1,a_1,b_{m-3}\rangle=0$, $\langle a_2,a_2,b_{m-5}\rangle=0$,..., $\left\langle a_{\frac{m-2}{2}},a_{\frac{m-2}{2}},b_1\right\rangle=0$ (this exists if $m-1$ is odd.), $\left\langle a_{\frac{m-1}{2}},a_{\frac{m-1}{2}},b_0\right\rangle=0$ (this exists if $m-1$ is even.), $\langle a_0+a_1,a_1+a_0,b_{m-2}\rangle=0$, $\langle a_0+a_2,a_2+a_0,b_{m-3}\rangle=0$, $\begin{pmatrix}\langle a_0+a_3,a_3+a_0,b_{m-4}\rangle+\\ \langle a_1+a_2,a_2+a_1,b_{m-4}\rangle=0\end{pmatrix}$,

..., $\begin{pmatrix}\langle a_0+a_{m-1},a_{m-1}+a_0,b_0\rangle+\\ \langle a_1+a_{m-2},a_{m-2}+a_1,b_0\rangle+\cdots+\\ \left\langle a_{\frac{m-2}{2}}+a_{\frac{m}{2}},a_{\frac{m}{2}}+a_{\frac{m-2}{2}},b_0\right\rangle=0\end{pmatrix}$ (if $m-1$ is odd.),

$\begin{pmatrix}\langle a_0+a_{m-1},a_{m-1}+a_0,b_0\rangle+\\ \langle a_1+a_{m-2},a_{m-2}+a_1,b_0\rangle+\cdots+\\ \left\langle a_{\frac{m-1}{2}-1}+a_{\frac{m-1}{2}+1},a_{\frac{m-1}{2}+1}+a_{\frac{m-1}{2}-1},b_0\right\rangle=0\end{pmatrix}$ (if $m-1$ is even.)

From the above equalities, we can reach the result $\langle a,a,b\rangle=0$. This completes the proof.

The centre $\mathbf{Z}$ of $\mathcal{A}$ is defined to be the (commutative and associative) subring of $\mathcal{A}$. Then, it is

$$\begin{aligned}\mathbf{Z}(\mathcal{A})&=Z(\mathbb{O})+Z(\mathbb{O})\eta+Z(\mathbb{O})\eta^2+\cdots+Z(\mathbb{O})\eta^{m-1}\\&=\mathbb{F}+\mathbb{F}\eta+\mathbb{F}\eta^2+\cdots+\mathbb{F}\eta^{m-1}.\end{aligned}$$

So $\mathcal{A}$ admits the *proper anti-automorphism*

$$K : a \longmapsto \overline{a} := \overline{a_0} + \overline{a_1}\eta + \overline{a_2}\eta^2 + \cdots + \overline{a_{m-1}}\eta^{m-1}.$$

for any $a = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{m-1}\eta^{m-1} \in \mathcal{A}$. We denote by T the *trace (linear) form* over the $\mathcal{A}$, which is defined by the rule $a + \overline{a}$, and also denote by N the *norm (quadratic) form* over the $\mathcal{A}$, which is defined by the rule $a\overline{a}$. Obviously t and n are the restrictions of T and N to $\mathbb{O}$. In this case, we have

$$T(a) = t(a_0) + t(a_1)\eta + t(a_2)\eta^2 + \cdots + t(a_{m-1})\eta^{m-1} \in \mathbf{Z}(\mathcal{A})$$

and

$$N(a) = \begin{cases} \begin{array}{l} n(a_0) + [n(a_0,a_1)]\eta + [n(a_0,a_2) + n(a_1)]\eta^2 + \cdots \\ + \left[\begin{array}{c} n(a_0,a_{m-1}) + n(a_1,a_{m-2}) + \cdots \\ +n\left(a_{\frac{m}{2}-1}, a_{\frac{m}{2}}\right) \end{array}\right]\eta^{m-1} \in \mathbf{Z}(\mathcal{A}) \end{array} & \text{if } m-1 \text{ is odd} \\ \begin{array}{l} n(a_0) + [n(a_0,a_1)]\eta + [n(a_0,a_2) + n(a_1)]\eta^2 + \cdots \\ + \left[\begin{array}{c} n(a_0,a_{m-1}) + n(a_1,a_{m-2}) + \cdots \\ +n\left(a_{\frac{m-1}{2}-1}, a_{\frac{m-1}{2}+1}\right) + n\left(a_{\frac{m-1}{2}}\right) \end{array}\right]\eta^{m-1} \in \mathbf{Z}(\mathcal{A}) \end{array} & \text{if } m-1 \text{ is even} \end{cases}$$

for any element $a = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{m-1}\eta^{m-1} \in \mathcal{A}$. Moreover, by Theorem 7 $\mathbb{M}(\mathcal{A})$ is a MK plane. Throughout this paper we restrict ourselves to MK plane $\mathbb{M}(\mathcal{A})$.

Now, we will introduce some collineations on $\mathbb{M}(\mathcal{A})$ where $w, z, q, n \in \mathbf{I}$. The collineations L_a, F_a, $\mathrm{S}_{\alpha,\beta}$, I_1, F and I_2 are similar to the collineations of [9]. Also, the collineation F is same to the collineation given in [3, p. 19-20]. Moreover, the collineations $\mathrm{T}_{a,0}$ and G_s are same to the elations α and σ given in [3, p. 12] , respectively. We will use the collineations $\mathrm{T}_{a,0}$ and $\mathrm{T}_{0,b}$ in $\mathbb{M}(\mathcal{A})$ instead of the collineation $\mathrm{T}_{a,b}$ in [9]. Now we can give the following collineations on $\mathbb{M}(\mathcal{A})$.

For any $a \in \mathcal{A}$, the collineation $\mathrm{T}_{a,0}$ (which is $((1,0,0),[0,0,1])$-elation) transforms points and lines as follows:

$$(x,y,1) \to (x+a,y,1), \ \ (1,y,z) \to \left(1, y - \left(z(1+az)^{-1}\right)(ay), z(1+az)^{-1}\right),$$
$$(w,1,z) \to (w+za,1,z)$$

$$[m,1,k] \to [m,1,k-am], \ \ [1,n,p] \to [1,n,p+a],$$
$$[q,n,1] \to \left[q(1+aq)^{-1}, n-(na)\left(q(1+aq)^{-1}\right), 1\right]$$

For any $b \in \mathcal{A}$, the collineation $\mathrm{T}_{0,b}$ (which is $((0,1,0),[0,0,1])$-elation) transforms points and lines as follows:

$$(x,y,1) \to (x,y+b,1)\,,\ \ (1,y,z) \to (1,y+zb,z)\,,$$
$$(w,1,z) \to \left(w - \left(z\,(1+bz)^{-1}\right)(bw)\,,1,z\,(1+bz)^{-1}\right)$$

$$[m,1,k] \to [m,1,k+b]\,,\ \ [1,n,p] \to [1,n,p-bn]\,,$$
$$[q,n,1] \to \left[q-(qb)\left(n\,(1+bn)^{-1}\right),n\,(1+bn)^{-1}\,,1\right]$$

For any $a \notin \mathbf{I}$, the collineation $\mathrm{L}_a$transforms points and lines as follows:

$$(x,y,1) \to (ax,aya,1)\,,\ \ (1,y,z) \to \left(1,ya,za^{-1}\right),$$
$$(w,1,z) \to \left(a^{-1}w,1,a^{-1}za^{-1}\right)$$

$$[m,1,k] \to [ma,1,aka]\,,\ \ [1,n,p] \to \left[1,a^{-1}n,ap\right],$$
$$[q,n,1] \to \left[qa^{-1},a^{-1}na^{-1},1\right]$$

For any $a \notin \mathbf{I}$, the collineation $\mathrm{F}_a$transforms points and lines as follows:

$$(x,y,1) \to (axa,ay,1)\,,\ \ (1,y,z) \to \left(1,a^{-1}y,a^{-1}za^{-1}\right),$$
$$(w,1,z) \to \left(wa,1,za^{-1}\right)$$

$$[m,1,k] \to \left[a^{-1}m,1,ak\right],\ \ [1,n,p] \to [1,na,apa]\,,$$
$$[q,n,1] \to \left[a^{-1}qa^{-1},na^{-1},1\right]$$

For any $\alpha,\beta \in \mathbf{Z}(\mathcal{A})$, $\alpha,\beta \notin \mathbf{I}$, the collineation $\mathrm{S}_{\alpha,\beta}$ (where $\mathrm{S}_{\alpha,1}$ is $((0,1,0),[0,1,0])$-homology and $\mathrm{S}_{1,\beta}$ is $((1,0,0),[1,0,0])$-homology) transforms points and lines as follows:

$$(x,y,1) \to \left(x^{-1},x^{-1}y,1\right) \quad \text{if} \quad x \notin \mathbf{I}, (x,y,1) \to (1,y,x) \quad \text{if} \quad x \in \mathbf{I},$$

$$[m,1,k] \to \left[\beta^{-1}m\alpha,1,k\alpha\right],\ \ [1,n,p] \to \left[1,\alpha^{-1}n\beta,p\beta\right],$$
$$[q,n,1] \to \left[\beta^{-1}q,\alpha^{-1}n,1\right]$$

The collineation I_1 (which is $((-1,0,1),[1,0,1])$-homology and also involution) transforms points and lines as follows:

$$(x,y,1) \to \left(x^{-1},x^{-1}y,1\right) \quad \text{if} \quad x \notin \mathbf{I}, (x,y,1) \to (1,y,x) \quad \text{if} \quad x \in \mathbf{I},$$
$$(1,y,z) \to (z,y,1)\,,\ \ (w,1,z) \to (z,1,w)$$

$$[m,1,k] \to [k,1,m]\,,\ \ [1,n,p] \to [p,n,1] \quad \text{if} \quad p \in \mathbf{I},$$
$$[1,n,p] \to \left[1,-np^{-1},p^{-1}\right] \quad \text{if} \quad p \notin \mathbf{I}, [q,n,1] \to [1,n,q]$$

The collineation F (which is $((1,-1,0),[1,1,0])$-homology and also involution) transforms points and lines as follows:

$$(x,y,1) \to (y,x,1)\,,\ \ (1,y,z) \to (y,1,z) \quad \text{if} \quad y \in \mathbf{I},$$
$$(1,y,z) \to \left(1,y^{-1},y^{-1}z\right) \quad \text{if} \quad y \notin \mathbf{I}, (w,1,z) \to (1,w,z)$$

$$[m,1,k] \to [1,m,k] \quad \text{if} \quad m \in \mathbf{I}, [m,1,k] \to \left[m^{-1},1,-km^{-1}\right] \quad \text{if} \quad m \notin \mathbf{I},$$
$$[1,n,p] \to [n,1,p]\,,\ \ [q,n,1] \to [n,q,1]$$

For any $s \in \mathcal{A}$, the collineation G_s (which is $((0,1,0),[1,0,0])$-elation) transforms points and lines as follows:

$$\begin{aligned}(x,y,1) \quad &\to \quad (x,y+xs,1)\,,(1,y,z) \to (1,y+s,z)\,, \\ (w,1,z) \quad &\to \quad \left((1+ws)^{-1}w,1,z-\left((1+ws)^{-1}w\right)(sz)\right)\end{aligned}$$

$$[m,1,k] \to [m+s,1,k]\,,\ \ [1,n,p] \to \left[1,(1+ns)^{-1}n,p-(ps)\left((1+ns)^{-1}n\right)\right],$$
$$[q,n,1] \to [q-sn,n,1]$$

The collineation I_2 (which is $((0,-1,1),[0,1,1])$-homology and also involution) transforms points and lines as follows:

$$(x,y,1) \to \left(y^{-1}x,y^{-1},1\right) \quad \text{if} \quad y \notin \mathbf{I},$$
$$(x,y,1) \to \left(1,x^{-1},x^{-1}y\right) \quad \text{if} \quad y \in \mathbf{I} \ \wedge x \notin \mathbf{I},$$
$$(x,y,1) \to (x,1,y) \quad \text{if} \quad y \in \mathbf{I} \ \wedge x \in \mathbf{I},$$
$$(1,y,z) \to \left(y^{-1},y^{-1}z,1\right) \quad \text{if} \quad y \notin \mathbf{I},$$
$$(1,y,z) \to (1,z,y) \quad \text{if} \quad y \in \mathbf{I},\ \ (w,1,z) \longrightarrow (w,z,1)$$

$$[m,1,k] \to \left[-mk^{-1},1,k^{-1}\right] \quad \text{if} \quad k \notin \mathbf{I},$$
$$[m,1,k] \to \left[1,-km^{-1},m^{-1}\right] \quad \text{if} \quad k \in \mathbf{I} \wedge m \notin \mathbf{I},$$
$$[m,1,k] \to [m,k,1] \quad \text{if} \quad k \in \mathbf{I} \wedge m \in \mathbf{I},$$
$$[1,n,p] \to \left[p^{-1},1,-np^{-1}\right] \quad \text{if} \quad p \notin \mathbf{I},$$
$$[1,n,p] \to [1,p,n] \quad \text{if} \quad p \in \mathbf{I}, \ [q,n,1] \to [q,1,n]$$

To show that the transformations $\mathrm{T}_{a,0}$, $\mathrm{T}_{0,b}$, L_a, F_a, $\mathrm{S}_{\alpha,\beta}$, I_1 , F , G_s and I_2 are collineations of $\mathbb{M}(\mathcal{A})$, it is generally enough to see to the proof of Lemma 3 given in [9]. Besides, it can be seen the cases in [3, p. 13] for showing that $\mathrm{T}_{a,0}$ and $\mathrm{T}_{0,b}$ preserve incidence relation. To prove the cases, it has been used the equalities in Lemma 3, Lemma 4, Lemma 5 and Lemma 6.

So, we can give the following theorem, the analogue of Theorem 2 given in [9] for a certain class of MK planes. For its proof, it is enough to use the collineations $\mathrm{T}_{a,0}$ and $\mathrm{T}_{0,b}$ instead of the collineation $\mathrm{T}_{a,b}$ in [9] and also the collineation G_{-s} instead of the collineation G_s in [9].

***Theorem* 11** — *The group $\mathcal{G}$ of collineations of $\mathbb{M}(\mathcal{A})$ acts transitively on 3-gons.*

PROOF : Let (P,Q,R) be a 3-gon in $\mathbb{M}(\mathcal{A})$ and $O=(0,0,1)$, $U=(1,0,0)$, $V=(0,1,0)$. This theorem will be proved by showing the existence of an element of $\mathcal{G}$ which transforms P,Q,R to O,U,V respectively. First, we construct a map J_1 which transforms QR to UV. Such a map is constructed as follows:

If $QR=[a,1,b]$, then $J_1 := \mathrm{I}_1 \circ \mathrm{T}_{-b,0} \circ \mathrm{F} \circ \mathrm{G}_{-a}$:

$$[a,1,b] \xrightarrow{\mathrm{G}_{-a}} [0,1,b] \xrightarrow{\mathrm{F}} [1,0,b] \xrightarrow{\mathrm{T}_{-b,0}} [1,0,0] \xrightarrow{\mathrm{I}_1} [0,0,1] = UV \ ;$$

Similarly, it can be shown that if $QR=[1,n,p]$ then $J_1 := \mathrm{I}_1 \circ \mathrm{T}_{-p,0} \circ \mathrm{F} \circ \mathrm{G}_{-n} \circ \mathrm{F}$ and if $QR=[m,n,1]$ then $J_1 := \mathrm{I}_1 \circ \mathrm{T}_{-m,0} \circ \mathrm{F} \circ \mathrm{G}_{-n} \circ \mathrm{F} \circ \mathrm{I}_1$ are the desired mappings.

Then since $J_1(R) \in UV = [0,0,1]$, $J_1(R)$ is of the form $(1,x,0)$ or $(x,1,0)$. If $J_1(R) = (1,x,0)$, then $J_2 := \mathrm{F} \circ \mathrm{G}_{-x}$ transforms $J_1(R)$ to V as follows:

$$(1,x,0) \xrightarrow{\mathrm{G}_{-x}} (1,0,0) \xrightarrow{\mathrm{F}} (0,1,0) = V$$

If $J_1(R)=(x,1,0)$, then taking $J_2 := \mathrm{F} \circ \mathrm{G}_{-x} \circ \mathrm{F}$, $J_1(R)$ is transformed to V by J_2. Thus $J_2 \circ J_1$ transforms R to V.

Since collineations preserve neighborhoods, $(J_2 \circ J_1)(R)$ and $(J_2 \circ J_1)(Q)$ are non-neighbour points and thus $(J_2 \circ J_1)(Q)$ is of the form $(1,y,0)$. Then under $J_3 := \mathrm{G}_{-y}$, $(1,y,0)$ is transformed to U. Since $(J_3 \circ J_2 \circ J_1)(R) = V$ and $(J_3 \circ J_2 \circ J_1)(Q) = U$, the point $(J_3 \circ J_2 \circ J_1)(P)$ is not near to $UV = [0, 0, 1]$ and so $(J_3 \circ J_2 \circ J_1)(P)$ must have the form $(x, y, 1)$. Then the map J_4 does the following: $J_4 := T_{0,-y} \circ \mathrm{T}_{-x,0}$,

$$(J_4 \circ J_3 \circ J_2 \circ J_1)(P) = J_4(x, y, 1) = (0, 0, 1) = O.$$

Consequently the map $\tau = J_4 \circ J_3 \circ J_2 \circ J_1$ transforms (P, Q, R) to (O, U, V).

Using the definitions of T and N the following lemma is obtained.

Lemma 12 — For $x \in \mathcal{A}$, $x^2 = -N(x) + T(x)x$. In specifically, if $T(x) = 0$ then $x^2 = -N(x)$.

Now we give the following lemma, without the proof from [9], which is very important for the proof of Theorem 15.

Lemma 13 — Let $d \in \mathbb{O}$, $d \notin Z(\mathbb{O})$, $c \in Z(\mathbb{O})$. Then there exists a $s \in \mathbb{O}$, $s \neq 0$, where $t(ds) = t(sd) = c$ and $t(s) = 0$.

Lemma 13 can be generalized to $\mathcal{A}$ by the following lemma.

Lemma 14 — Let $d \in \mathcal{A}$, $d \notin \mathbf{Z}(\mathcal{A})$, $c \in \mathbf{Z}(\mathcal{A})$. Then there exists a $s \in \mathcal{A} \setminus \mathbf{I}$, where $T(sd) = T(ds) = c$ and $T(s) = 0$.

PROOF : If $d = d_0 + d_1\eta + d_2\eta^2 + \cdots + d_{m-1}\eta^{m-1} \notin \mathbf{Z}(\mathcal{A})$ where $d_0, d_1, d_2, ..., d_{m-1} \in \mathbb{O}$, then $\exists d_k \notin Z(\mathbb{O})$ for $0 \leq k \leq m-1$. So, we can write

$$\begin{aligned} T(sd) &= t(s_0d_0) + [t(s_1d_0) + t(s_0d_1)]\eta + [t(s_2d_0) + t(s_1d_1) + t(s_0d_2)]\eta^2 + \\ &\quad \cdots + [t(s_{m-1}d_0) + t(s_{m-2}d_1) + \cdots + t(s_0d_{m-1})]\eta^{m-1} \end{aligned}$$

where $s = s_0 + s_1\eta + s_2\eta^2 + \cdots + s_{m-1}\eta^{m-1}$. From Lemma 13, we know that there exist the elements $0 \neq s_j \in \mathbb{O}$ for $0 \leq j \leq m-1$ such that $t(s_0d_0) = c$, $t(s_1d_0) = -t(s_0d_1)$, $t(s_2d_0) = -t(s_0d_2) - t(s_1d_1)$, ... , $t(s_{m-2}d_0) = -t(s_0d_{m-2}) - t(s_1d_{m-3}) - \cdots - t(s_{m-3}d_1)$, $t(s_{m-1}d_0) = -t(s_0d_{m-1}) - t(s_1d_{m-2}) - \cdots - t(s_{m-2}d_1)$. Moreover, in this case, $T(s) = 0$ because of $t(s_j) = 0$. So, we reach the result: $T(sd) = T(ds) = c$ and $T(s) = 0$ for a $s \in \mathcal{A} \setminus \mathbf{I}$ (for, $s_0 \neq 0$).

Now, we can state the analogue of Theorem 3 given in [9] for a certain class of MK planes.

***Theorem* 15** — *$\mathcal{G}$ acts transitively on 4-gons of $\mathbb{M}(\mathcal{A})$.*

PROOF : Let (P, Q, R, S) be a 4-gon in $\mathbb{M}(\mathcal{A})$. It suffices to show that the points P, Q, R, S can be transformed by an element of $\mathcal{G}$ to $U, V, (1, 1, 1), O$, respectively. From Theorem 11, there exists a collineation σ which transforms P, Q, R to U, V, $(0, 1, 1)$, respectively. Let E denote the intersection point of the lines QR and PS. Then, since $\sigma(E)$ is non-neighbour to the points $\sigma(P)$, $\sigma(Q)$, $\sigma(R)$, it has the form $(0, b, 1)$, where $b - 1 \notin \mathbf{I}$, and so $\sigma(S)$ has the form $(a, b, 1)$, where $a \notin \mathbf{I}$. Therefore σ transforms P, Q, R, S to

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (0, 1, 1)\,,\ (a, b, 1),$$

respectively. Then the mapping $\mathrm{T}_{0,-b} \circ \mathrm{T}_{-a,0}$ transforms these points to

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (-a, 1 - b, 1)\,,\ (0, 0, 1),$$

respectively and $\mathrm{L}_{\overline{a}}$ transforms these points to

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (-N(a), c, 1)\,,\ (0, 0, 1),$$

respectively where $c = \bar{a}(1 - b)\bar{a}$. Then $c \notin \mathbf{I}$, since $a \notin \mathbf{I}$, $1 - b \notin \mathbf{I}$. Then $\mathrm{S}_{1,-N(a)^{-1}}$ transforms

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (-N(a), c, 1)\,,\ (0, 0, 1)$$

to

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (1, c, 1)\,,\ (0, 0, 1).$$

There are two cases for $c = c_0 + c_1\eta + c_2\eta^2 + \cdots + c_{m-1}\eta^{m-1}$: $T(c) \in \mathbf{I}$ or $T(c) \notin \mathbf{I}$.

Case i : If $T(c) = t(c_0) + t(c_1)\eta + t(c_2)\eta^2 + \cdots + t(c_{m-1})\eta^{m-1} \in \mathbf{I}$, then $t(c_0) = 0$ and

Case i.1 : If $t(c_1) = t(c_2) = \cdots = t(c_{m-1}) = 0$, then $T(c) = 0$, and from Lemma 12, $c^2 = -N(c)$. Then F_c transforms

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (1, c, 1)\,,\ (0, 0, 1)$$

to

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (-N(c), -N(c), 1)\,,\ (0, 0, 1),$$

respectively. Finally, $\mathrm{S}_{-N(c)^{-1},-N(c)^{-1}}$ transforms these points to

$$(1, 0, 0)\,,\ (0, 1, 0)\,,\ (1, 1, 1)\,,\ (0, 0, 1),$$

respectively, which are the desired points.

Case i.2 : If at least one of $t(c_1), t(c_2), \ldots, t(c_{m-1})$ is non-zero, then from Lemma 13, there exist the elements $0 \neq s_j \in \mathbb{O}$ and $t(s_j) = 0$ (this means $T(s) = 0$) for $0 \leq j \leq m - 1$ such that $t(s_0c_0) = 0$, $t(s_1c_0) = -t(s_0c_1)$, $t(s_2c_0) = -t(s_0c_2) - t(s_1c_1)$, ... , $t(s_{m-2}c_0) = -t(s_0c_{m-2}) - t(s_1c_{m-3}) - \cdots - t(s_{m-3}c_1)$, $t(s_{m-1}c_0) = -t(s_0c_{m-1}) - t(s_1c_{m-2}) - \cdots - t(s_{m-2}c_1)$. So, we have $T\left(sc\right)=T\left(cs\right)=0$ for a $s\in\mathcal{A}\setminus\mathbf{I}$ (note that $s_0\neq 0$). Then taking the $s=s_0+s_1\eta+s_2\eta^2+\cdots+s_{m-1}\eta^{m-1}$, the points

$$(1,0,0)\,,\;\;(0,1,0)\,,\;\;(1,c,1)\,,\;\;(0,0,1)$$

are transformed to

$$(1,0,0)\,,\;\;(0,1,0)\,,\;\;\left(s^2,sc,1\right)\,,\;\;(0,0,1)$$

under the mapping F_s, and by Lemma 12, $s^2=-N(s)$. Finally the mapping $\mathrm{S}_{1,-N(s)^{-1}}$ transforms these points to

$$(1,0,0)\,,\;\;(0,1,0)\,,\;\;(1,sc,1)\,,\;\;(0,0,1)$$

respectively where $T(sc)=0$. Then the proof follows from Case i.1.

Case ii : If $T(c)=t(c_0)+t(c_1)\eta+t\left(c_2\right)\eta^2+\cdots+t\left(c_{m-1}\right)\eta^{m-1}\notin\mathbf{I}$, then $t(c_0)\neq 0$. By Lemma 13, we can find a $d=d_0+d_1\eta+d_2\eta^2+\cdots+d_{m-1}\eta^{m-1}\in\mathcal{A}\setminus\mathbf{I}$ such that $d_0\neq 0$, $t(c_0d_0)=0$, $t(d_0)=0$, and $t(d_1)=t(d_2)=\cdots=t(d_{m-1})=0$. Then by Lemma 12, $d^2=-N(d)$ because of $T(d)=0$. Thus the mapping F_d transforms

$$(1,0,0)\,,\;\;(0,1,0)\,,\;\;(1,c,1)\,,\;\;(0,0,1)$$

to

$$(1,0,0)\,,\;\;(0,1,0)\,,\;\;(-N(d),dc,1)\,,\;\;(0,0,1),$$

respectively. Finally, under $\mathrm{S}_{1,-N(d)^{-1}}$, these points are transformed to

$$(1,0,0)\,,\;\;(0,1,0)\,,\;\;(1,dc,1)\,,\;\;(0,0,1),$$

where $T(dc)=T(cd)\in\mathbf{I}$ and the proof follows from Case i. □

The following corollary is an obvious result of the last theorem:

Corollary 16 — The coordinatization of $\mathbb{M}(\mathcal{A})$ is independent of the choice of the coordinatization base.

4. CROSS-RATIO

The definition of cross-ratio in the class of MK plane $\mathbb{M}(\mathcal{A})$ where $\mathcal{A}:=\mathbb{O}+\mathbb{O}\eta+\mathbb{O}\eta^2+\cdots+\mathbb{O}\eta^{m-1}$ will be discussed by a conjugacy class as in the class of MK planes, which is an equivalence class w.r.t. the conjugacy relation "$\equiv$" ($a\equiv b:\Leftrightarrow\exists$ unit c: $a=c^{-1}bc$), see [7, at the top of p. 247]. Note that the algebra is not to isomorphic to the algebras $R\left(2\right)$ and so $A\left(\varepsilon,\delta\right)$, used in [7, p. 246-247] by Blunck.

Now, we can give the following result, given as Lemma 1 in [7].

Lemma 17 — Let $\mathbb{O}$ be an Cayley division algebra (or non-associative alternative field). Then for all $a, b \in \mathbb{O}: \ a \equiv b \Leftrightarrow T(a) = T(b)$ and $N(a) = N(b)$.

Note that the relation "$\equiv$" is an equivalence relation on $\mathbb{O}$.

As stated by Blunck in [7, p. 247-248], in a non-associative case, there may be some problems on the condition that the equivalence relation is transitive. Moreover, we have to give a characterization of the conjugacy on $\mathcal{A}$ as in Lemma 17. Because, any elements may have same norm and trace while they are not conjugate (for instance: $1 + 1\eta$, $1 + 1\eta + e_2\eta^2$).

Now, we can give and discuss the following statement which is a generalization of Theorem 1 in [7]:

Let $\mathbb{O}$ be an Cayley division algebra, $a = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{m-1}\eta^{m-1}$, $b = b_0 + b_1\eta + b_2\eta^2 + \cdots + b_{m-1}\eta^{m-1} \in \mathcal{A}$.

i. If $a_0 \in Z(\mathbb{O})$, then: $a \equiv b \Leftrightarrow \left[\begin{array}{c} a_0 = b_0 \wedge \left(a_1 + a_2\eta + \cdots + a_{m-1}\eta^{m-2}\right) \equiv \\ \left(b_1 + b_2\eta + \cdots + b_{m-1}\eta^{m-2}\right) \end{array}\right]$.

ii. If $a_0 \notin Z(\mathbb{O})$, then: $a \equiv b \Leftrightarrow T(a) = T(b) \wedge N(a) = N(b)$."

It can be seen from [7] in the case $m = 2$ of the proof of this statement. If we would like to give a similar proof in the case $m > 2$, i.e. for if part of (ii), let $a_0 \notin Z(\mathbb{O})$. "$\Rightarrow$" is clear from the properties of trace and norm. As for "$\Leftarrow$", from $T(a) = T(b) \wedge N(a) = N(b)$ we have $a(\overline{a} - b) = (\overline{a} - b)b$ which means $a \equiv b$ if $(\overline{a} - b)$ is a unit. Otherwise, we obtain $\overline{a} - b \in \mathbf{I}$ which means $a_0 = \overline{b_0}$ ($\notin Z(\mathbb{O})$). For this reason, in this case we must show the existence of a unit $c = c_0 + c_1\eta + c_2\eta^2 + \cdots + c_{m-1}\eta^{m-1}$ such that $ca = bc$. From the equality $ca = bc$, we find $m - 1$ equalities. But, it is not possible to find a $c_0 \neq 0$ which satisfy the equalities. This means that a is not conjugate to b.

Hence, some new results related to 6-figures in the MK plane $\mathbb{M}(\mathcal{A})$ where $\mathcal{A} = \mathbb{O} + \mathbb{O}\eta + \mathbb{O}\eta^2 + \cdots + \mathbb{O}\eta^{m-1}$ can be obtained in view of the papers [1, 9, 11] except for the notion of cross-ratio.

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