

## JORDAN LEFT $\{g, h\}$ -DERIVATION OVER SOME ALGEBRAS

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In this article, left  $\{g, h\}$ -derivation and Jordan left  $\{g, h\}$ -derivation on algebras are introduced. It is shown that there is no Jordan left  $\{g, h\}$ -derivation over  $\mathcal{M}_n(C)$  and  $\mathbb{H}_{\mathbb{R}}$ , for  $g \neq h$ . Examples are given which show that every Jordan left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$ ,  $\mathcal{M}_n(C)$  and  $\mathbb{H}_{\mathbb{R}}$  are not left  $\{g, h\}$ -derivations. Also, the Jordan left  $\{g, h\}$ -derivations over  $\mathcal{T}_n(C)$ ,  $\mathcal{M}_n(C)$  and  $\mathbb{H}_{\mathbb{R}}$  are right centralizers, where  $C$  is a 2-torsionfree commutative ring. Moreover, we prove the result of Jordan left  $\{g, h\}$ -derivation to be a left  $\{g, h\}$ -derivation over tensor products of algebras as well as for algebra of polynomials.

**Key words** : Derivation; Jordan derivation; left  $\{g, h\}$ -derivation; Jordan left  $\{g, h\}$ -derivation; Tensor product of algebras.

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### 1. INTRODUCTION

Throughout this paper,  $C$  represents a 2-torsion free commutative ring with unity unless otherwise stated. A ring  $R$  is a 2-torsion free if  $2a = 0$  for  $a \in R$  implies  $a = 0$ . Let  $R$  be a ring. An additive map  $D : R \rightarrow R$  is said to be a derivation if  $D(ab) = D(a)b + aD(b)$ , for all  $a, b \in R$  and a Jordan derivation if  $D(a^2) = D(a)a + aD(a)$ , for all  $a \in R$ . Jordan derivation over rings and algebras have been studied by many researchers in [2, 3, 5, 7, 9, 11, 14, 15]. In 1990, Jordan left derivation was introduced by Brešar and Vukman [6]. They proved that the existence of a nonzero left derivation of a prime ring of characteristic not 2 implies the commutativity of the ring. After that many new results have been established on Jordan left derivations over different rings and algebras [1, 4, 8, 10, 12, 13]. Recently, in 2016, Brešar introduced  $\{g, h\}$ -derivation and studied over semiprime algebras and tensor product of algebras [6]. Let  $A$  be an algebra over  $C$  and  $f, g, h : A \rightarrow A$  be linear maps.

Then  $f$  is said to be a left derivation if  $f(ab) = af(b) + bf(a)$  and  $f$  is said to be a  $\{g, h\}$ -derivation if  $f(ab) = g(a)b + ah(b) = h(a)b + ag(b)$ .

Recall that a linear map  $f : A \rightarrow A$  is said to be a left centralizer if  $f(ab) = f(a)b$ , and it is a right centralizer if  $f(ab) = af(b)$ , for all  $a, b \in A$ . It is a two sided centralizer if  $f$  is both left as well as right centralizers. Note that if  $A$  has an identity element, then  $f$  is a left centralizer if and only if there exists an element  $\alpha \in A$  such that  $f(a) = \alpha a$ , for all  $a \in A$ . Also,  $f$  is a right centralizer if and only if there exists an element  $\beta \in A$  such that  $f(a) = a\beta$ , for all  $a \in A$ .

Motivated by left derivation [4] and  $\{g, h\}$ -derivation [6], we introduce left  $\{g, h\}$ -derivation over  $A$  as follows:

*Definition 1* — The map  $f$  is a left  $\{g, h\}$ -derivation if

$$f(ab) = ag(b) + bh(a) = ah(b) + bg(a), \text{ for all } a, b \in A. \quad (1)$$

Clearly, if  $f = g = h$ , then  $f$  is a left derivation. Similarly, we define Jordan left  $\{g, h\}$ -derivation.

*Definition 2* — The map  $f$  is said to be a Jordan left  $\{g, h\}$ -derivation if

$$f(a \circ b) = 2(ag(b) + bh(a)), \text{ for all } a, b \in A \quad (2)$$

(where,  $a \circ b = ab + ba$ ).

Since  $a \circ b = b \circ a$  whenever  $f$  is a Jordan left  $\{g, h\}$ -derivation on  $A$ , then  $f(a \circ b) = 2(bg(a) + ah(b))$  and  $f(a \circ b) = ag(b) + bh(a) + bg(a) + ah(b)$ , for all  $a, b \in A$ . Also,  $f(a^2) = a(g(a) + h(a))$ , for all  $a \in A$ .

Note that if  $A$  is a 2-torsion free commutative algebra over  $C$ , then every Jordan left  $\{g, h\}$ -derivation is a left  $\{g, h\}$ -derivation. This is not true when  $A$  is not 2-torsion free. For example:

*Example 3* : Let  $A = \mathbb{Z}_4$ . Then  $A$  is a commutative  $\mathbb{Z}$ -algebra which is not 2-torsion free. Now, we define  $f : A \rightarrow A$  by  $f(x) = 2x$ , for all  $x \in A$ . Then  $f$  is a Jordan left  $\{f, f\}$ -derivation. But,  $f([1][1]) = [2] \neq [4] = [1]f([1]) + [1]f([1])$ . Therefore,  $f$  is not a left  $\{f, f\}$ -derivation.

*Remark 4* : If  $f$  is a Jordan left  $\{g, h\}$ -derivation over  $A$ , then  $f$  is a left  $\{g + h, g + h\}$ -derivation.

## 2. JORDAN LEFT $\{g, h\}$ -DERIVATION ON MATRIX ALGEBRAS

Here, we start with an example of a left  $\{g, h\}$ -derivation which is not a  $\{g, h\}$ -derivation.

*Example 5 :* Let  $A = \mathcal{T}_2(C)$ , algebra of  $2 \times 2$  upper triangular matrices over  $C$ . Let  $e_{ij}$  be the matrix whose  $(i, j)$ -th entry is 1, otherwise 0 and  $X = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix} \in A$ . We define  $f, g, h : A \rightarrow A$  by  $f(X) = \begin{pmatrix} x_1 & x_1 \\ 0 & 0 \end{pmatrix}$ ,  $g(X) = \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $h(X) = \begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix}$ , respectively. Then  $f$  is a left  $\{g, h\}$ -derivation. Now,  $f(e_{11}(e_{11} + e_{12})) = f(e_{11} + e_{12}) = e_{11} + e_{12} \neq e_{11} + 2e_{12} = g(e_{11})(e_{11} + e_{12}) + e_{11}h(e_{11} + e_{12})$ . Therefore,  $f$  is not a  $\{g, h\}$ -derivation.

Now, we give an example of a  $\{g, h\}$ -derivation which is not a left  $\{g, h\}$ -derivation.

*Example 6 :* Define  $f, g : \mathcal{T}_2(C) \rightarrow \mathcal{T}_2(C)$  by  $f(x) = x + g(x)$  and  $g(x) = ax - xa$  for all  $x \in \mathcal{T}_2(C)$  where  $a = e_{11} + e_{12} + e_{22}$ . Then  $f$  is  $\{f, g\}$ -derivation. But,  $e_{12}f(e_{11}) + e_{11}g(e_{12}) = e_{12} \neq 0 = f(e_{12}e_{11})$ , therefore,  $f$  is not left  $\{f, g\}$ -derivation.

Note that every left  $\{g, h\}$ -derivation over  $A$  is a Jordan left  $\{g, h\}$ -derivation but the converse is not true.

*Example 7 :* Let  $A = \mathcal{T}_2(C)$ . Now, let  $X = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix} \in A$  and  $f, g, h : A \rightarrow A$  are defined by  $f(X) = \begin{bmatrix} 5x_1 & 7x_1 + 6x_2 \\ 0 & 6x_3 \end{bmatrix}$ ,  $g(X) = \begin{bmatrix} x_1 & 2x_1 + 3x_2 \\ 0 & 3x_3 \end{bmatrix}$  and  $h(X) = \begin{bmatrix} 4x_1 & 5x_1 + 3x_2 \\ 0 & 3x_3 \end{bmatrix}$ , respectively. Then  $f$  is a Jordan left  $\{g, h\}$ -derivation. Also,  $f(e_{12}e_{11}) = f(0) = 0$ , but  $e_{12}g(e_{11}) + e_{11}h(e_{12}) = 3e_{12} \neq 0$ . Hence,  $f$  is not a left  $\{g, h\}$ -derivation.

Note that in Example 7,  $f(e_{12} \circ e_{11}) = f(e_{12}) = 6e_{12} \neq 7e_{12} = g(e_{12}) \circ e_{11} + e_{12} \circ h(e_{11})$ . Therefore,  $f$  is not a Jordan  $\{g, h\}$ -derivation. In next results, we characterize Jordan left  $\{g, h\}$ -derivation and left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$ .

**Theorem 8** — Let  $\mathcal{T}_n(C)$ ,  $n \geq 2$ , be the algebra of  $n \times n$  upper triangular matrices over  $C$ . Then  $f$  is a Jordan left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$  if and only if there exist  $\frac{n(n+3)}{2}$  elements in  $C$  such that

$$g(A) = \sum_{1 \leq i \leq j \leq n} \left( \sum_{k=i}^j a_{ik}g_{kj}^{(kk)} \right) e_{ij};$$

$$h(A) = a_{11}h_{11}^{(11)} e_{11} + \sum_{j=2}^n (a_{11}h_{1j}^{(11)} + \sum_{k=2}^j a_{1k}g_{kj}^{(kk)}) e_{1j} + \sum_{1 < i \leq j \leq n} \left( \sum_{k=i}^j a_{ik}g_{kj}^{(kk)} \right) e_{ij};$$

$$f(A) = (g + h)(A), \text{ for all } A \in \mathcal{T}_n(C)$$

where  $g_{lm}^{(np)}, h_{lm}^{(np)} \in C$  and  $A = \sum_{1 \leq i \leq j \leq n} a_{ij}e_{ij}$ .

Moreover, if  $f$  is a Jordan left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$ , then  $f, g$  and  $h$  are right centralizers.

PROOF : Let  $f$  be a Jordan left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$ . Let

$$g(e_{ij}) = \sum_{1 \leq m \leq p \leq n} g_{mp}^{(ij)} e_{mp}, \quad (3)$$

$$h(e_{ij}) = \sum_{1 \leq m \leq p \leq n} h_{mp}^{(ij)} e_{mp}, \quad (4)$$

and

$$f(e_{ij}) = \sum_{1 \leq m \leq p \leq n} f_{mp}^{(ij)} e_{mp}, \text{ where } g_{mp}^{(ij)}, h_{mp}^{(ij)}, f_{mp}^{(ij)} \in C. \quad (5)$$

Since  $e_{ii}^2 = e_{ii}$  for  $i \in \{1, 2, \dots, n\}$  and  $f$  is a Jordan left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$ , so  $f(e_{ii}) = e_{ii}g(e_{ii}) + e_{ii}h(e_{ii})$  and this implies

$$\begin{aligned} f_{ii}^{(ii)} &= g_{ii}^{(ii)} + h_{ii}^{(ii)}, f_{i,i+1}^{(ii)} = g_{i,i+1}^{(ii)} + h_{i,i+1}^{(ii)}, \dots, f_{in}^{(ii)} = g_{in}^{(ii)} + h_{in}^{(ii)}, \\ &\text{and other entries of } f(e_{ii}) \text{ are zero except } (ii)\text{-th to } (in)\text{-th entries} \\ &\text{(by (3), (4) and (5)).} \end{aligned} \quad (6)$$

Let  $i \neq j$ . Since  $e_{ii} \circ e_{jj} = 0$ , by using (3), (4), (5) and (2), we have

$$g_{ii}^{(jj)} = g_{i,i+1}^{(jj)} = \dots = g_{in}^{(jj)} = 0 = h_{jj}^{(ii)} = h_{j,j+1}^{(ii)} = \dots = h_{jn}^{(ii)}. \quad (7)$$

Let  $i < j$ . Since  $e_{ij} \circ e_{ii} = e_{ij} = e_{ii} \circ e_{ij}$ , we have

$$\begin{aligned} 2h_{ik}^{(ij)} &= f_{ik}^{(ij)} = 2g_{ik}^{(ij)}, \text{ for } k = \{i, i+1, \dots, j-1\}. \\ \text{Also, } 2(g_{jl}^{(ii)} + h_{il}^{(ij)}) &= f_{il}^{(ij)} = 2(h_{jl}^{(ii)} + g_{il}^{(ij)}), \text{ for } l = \{j, j+1, \dots, n\} \\ \implies 2h_{il}^{(ij)} &= f_{il}^{(ij)} = 2g_{il}^{(ij)}, \text{ for } l = \{j, j+1, \dots, n\} \text{ (by (7))} \end{aligned} \quad (8)$$

and other entries of  $f(e_{ij})$  are zero except  $(ii)$ -th to  $(in)$ -th entry.

Similarly, from  $e_{jj} \circ e_{ij} = e_{ij} = e_{ij} \circ e_{jj}$ ,

$$\begin{aligned} 2h_{jk}^{(jj)} &= f_{jk}^{(ij)} = 2g_{jk}^{(jj)}, \text{ for } k = \{j, j+1, \dots, n\} \text{ (by (8))}, \\ g_{jl}^{(ij)} &= 0 = h_{jl}^{(ij)}, \text{ for } l = \{j, j+1, \dots, n\} \text{ (by (8))} \end{aligned} \quad (9)$$

and other entries of  $f(e_{ij})$  are zero except  $(ij)$ -th to  $(in)$ -th entry.

Therefore, for any  $i < j$ , by (8) and (9),

$$\begin{aligned} g_{ik}^{(ij)} &= h_{ik}^{(ij)} = h_{jk}^{(jj)} = g_{jk}^{(jj)}, \text{ for } k = \{j, j+1, \dots, n\}, \\ g_{il}^{(ij)} &= 0 = h_{il}^{(ij)}, \text{ for } l = \{i, i+1, \dots, j-1\}. \end{aligned} \quad (10)$$

Let  $k \neq i, j$ . Since  $e_{kk} \circ e_{ij} = 0$ ,

$$g_{kl}^{(ij)} = 0 = h_{kl}^{(ij)}, \text{ for } l = \{k, k + 1, \dots, n\}. \tag{11}$$

Hence, from (6)-(7) and (9)-(11), all entries of  $g(e_{ij})$ ,  $h(e_{ij})$  and  $f(e_{ij})$  are zero except  $(ij)$ -th to  $(in)$ -th entry, for all  $1 \leq i \leq j \leq n$ .

Now, let  $A \in \mathcal{T}_n(C)$ . Then  $A = \sum_{1 \leq i \leq j \leq n} a_{ij}e_{ij}$ , where  $a_{ij} \in C$ . Since  $g, h$  and  $f$  are linear,

$$\begin{aligned} g(A) &= \sum_{1 \leq i \leq j \leq n} \left( \sum_{k=i}^j a_{ik}g_{kj}^{(kk)} \right) e_{ij}, \\ h(A) &= a_{11}h_{11}^{(11)}e_{11} + \sum_{j=2}^n (a_{11}h_{1j}^{(11)} + \sum_{k=2}^j a_{1k}g_{kj}^{(kk)})e_{1j} + \sum_{1 < i \leq j \leq n} \left( \sum_{k=i}^j a_{ik}g_{kj}^{(kk)} \right) e_{ij}, \\ f(A) &= (g + h)(A), \text{ (by (6), (9) and (10)).} \end{aligned} \tag{12}$$

Therefore, the number of elements of  $C$  require to express  $g, h$  and  $f$  is equal to  $n + n + (n - 1) + \dots + 1 = \frac{n(n+3)}{2}$ .

Conversely, let  $g, h$  and  $f$  be of the form (12), for all  $A \in \mathcal{T}_n(C)$ , where  $A = \sum_{1 \leq i \leq j \leq n} a_{ij}e_{ij}$  and  $a_{ij} \in C$ . Then  $g(A) = A\alpha$ ,  $h(A) = A\alpha'$  and  $f(A) = A(\alpha + \alpha')$  where  $\alpha = \sum_{1 \leq i \leq j \leq n} g_{ij}^{(ii)}e_{ij}$  and  $\alpha' = \sum_{j=1}^n h_{1j}^{(11)}e_{1j} + \sum_{1 < i \leq j \leq n} g_{ij}^{(ii)}e_{ij}$ . Now, let  $B \in \mathcal{T}_n(C)$ . Then  $B = \sum_{1 \leq i \leq j \leq n} b_{ij}e_{ij}$ , where  $b_{ij} \in C$  and by direct computation,

$$f(AB + BA) = 2(Ag(B) + Bh(A)).$$

The last conclusion can be easily seen from the converse part. □

**Theorem 9** — Let  $\mathcal{T}_n(C)$ ,  $n \geq 2$ , be the algebra of  $n \times n$  upper triangular matrices over  $C$ . Then  $f$  is a left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$  if and only if there exists  $2n$  elements in  $C$  such that

$$\begin{aligned} g(A) &= \sum_{i=1}^n a_{11}g_{1i}^{(11)}e_{1i}; \quad h(A) = \sum_{i=1}^n a_{11}h_{1i}^{(11)}e_{1i}; \\ f(A) &= (g + h)(A), \text{ for all } A \in \mathcal{T}_n(C), \\ \text{where } g_{lm}^{(np)}, h_{lm}^{(np)} &\in C \text{ and } A = \sum_{1 \leq i \leq j \leq n} a_{ij}e_{ij}. \end{aligned}$$

PROOF : Let  $f$  be a left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$  where  $g, h$  and  $f$  be of the form (3), (4) and (5), respectively. Since every left  $\{g, h\}$ -derivation over  $\mathcal{T}_n(C)$  is a Jordan left  $\{g, h\}$ -derivation, so (6) and (7) hold.

Let  $i < j$ . Since  $e_{ij}e_{ii} = 0$ , from  $e_{ij}g(e_{ii}) + e_{ii}h(e_{ij}) = 0 = e_{ij}h(e_{ii}) + e_{ii}g(e_{ij})$  (by (1)), we have

$$\begin{aligned} h_{ik}^{(ij)} &= 0 = g_{ik}^{(ij)}, \text{ for } k = \{i, i+1, \dots, j-1\} \\ \text{and } g_{jl}^{(ii)} + h_{il}^{(ij)} &= 0 = h_{jl}^{(ii)} + g_{il}^{(ij)}, \text{ for } l = \{j, j+1, \dots, n\} \\ \implies h_{il}^{(ij)} &= 0 = g_{il}^{(ij)}, \text{ for } l = \{j, j+1, \dots, n\} \text{ (by (7)).} \end{aligned} \quad (13)$$

Similarly, from the identity  $e_{jj}e_{ij} = 0$ ,

$$\begin{aligned} h_{jk}^{(jj)} &= 0 = g_{jk}^{(jj)}, \text{ for } k = \{j, j+1, \dots, n\} \\ g_{jl}^{(ij)} &= 0 = h_{jl}^{(ij)}, \text{ for } l = \{j, j+1, \dots, n\}. \end{aligned} \quad (14)$$

By (6) and (14),

$$f(e_{jj}) = 0, \text{ for all } j > 1. \quad (15)$$

By (7) and (14),

$$g(e_{jj}) = h(e_{jj}) = 0, \text{ for all } j > 1. \quad (16)$$

Let  $k \neq i, j$ . Since  $e_{kk}e_{ij} = 0$ ,

$$g_{kk}^{(ij)} = g_{k,k+1}^{(ij)} = \dots = g_{kn}^{(ij)} = 0 = h_{kk}^{(ij)} = h_{k,k+1}^{(ij)} = \dots = h_{kn}^{(ij)}. \quad (17)$$

By (13), (14) and (17),

$$g(e_{ij}) = h(e_{ij}) = 0, \text{ for all } i < j. \quad (18)$$

Now,

$$f(e_{ij}) = e_{ii}g(e_{ij}) + e_{ij}h(e_{ii}) = 0, \text{ for all } i < j \text{ (by (18), (4), (7)).} \quad (19)$$

Let  $A \in \mathcal{T}_n(C)$ . Then  $A = \sum_{1 \leq i \leq j \leq n} a_{ij}e_{ij}$  where  $a_{ij} \in C$ . Since  $g, h$  and  $f$  are linear, by (3)-(7), (15), (16), (18) and (19), we have

$$\begin{aligned} g(A) &= \sum_{i=1}^n a_{11}g_{1i}^{(11)}e_{1i}, \quad h(A) = \sum_{i=1}^n a_{11}h_{1i}^{(11)}e_{1i}, \\ f(A) &= (g+h)(A), \text{ for all } A \in \mathcal{T}_n(C). \end{aligned}$$

Thus, the number of elements from  $C$  require to express  $g, h$  and  $f$  is  $n + n = 2n$ .

Converse can be proved easily by direct computation.

Now, we give an example of a Jordan left  $\{g, h\}$ -derivation  $f$  on  $\mathcal{T}_n(C)$  where  $g, h$  and  $f$  are not left centralizers.

*Example 10* : Consider  $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \in \mathcal{T}_2(C)$  and  $g, h, f : \mathcal{T}_2(C) \rightarrow \mathcal{T}_2(C)$  are defined by  $g(A) = \begin{bmatrix} a_{11} & -a_{12} \\ 0 & -a_{22} \end{bmatrix}$ ,  $h(A) = \begin{bmatrix} -a_{11} & a_{11} - a_{12} \\ 0 & -a_{22} \end{bmatrix}$  and  $f(A) = \begin{bmatrix} 0 & a_{11} - 2a_{12} \\ 0 & -2a_{22} \end{bmatrix}$ , respectively.

Then  $f$  is a Jordan left  $\{g, h\}$ -derivation on  $\mathcal{T}_2(C)$ . Here, if  $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$ , then  $g(AB) \neq g(A)B$ ,  $h(AB) \neq h(A)B$  and  $f(AB) \neq f(A)B$ . Therefore,  $g, h$  and  $f$  are not left centralizers.

Our next result characterize Jordan left  $\{g, h\}$ -derivation on full matrix algebras  $\mathcal{M}_n(C)$ .

**Theorem 11** — *Let  $f$  be a Jordan left  $\{g, h\}$ -derivation over  $\mathcal{M}_n(C)$ ,  $n \geq 2$ , the algebra of  $n \times n$  matrices over  $C$ . Then  $g = h$ .*

PROOF : Suppose  $f$  is a Jordan left  $\{g, h\}$ -derivation over  $\mathcal{M}_n(C)$ . Now, let

$$g(e_{ij}) = \sum_{k=1}^n \sum_{l=1}^n g_{kl}^{(ij)} e_{kl} \tag{20}$$

and

$$h(e_{ij}) = \sum_{k=1}^n \sum_{l=1}^n h_{kl}^{(ij)} e_{kl}, \text{ where } g_{kl}^{(ij)}, h_{kl}^{(ij)} \in C. \tag{21}$$

Let  $i \neq j$ . Since  $C$  is 2-torsion free,

$$2(g(e_{ii})e_{jj} + e_{jj}h(e_{ii})) = f(e_{ii} \circ e_{jj}) = 0 \implies g(e_{ii})e_{jj} + e_{jj}h(e_{ii}) = 0. \tag{22}$$

Comparing the coefficients of  $e_{i1}, \dots, e_{in}, e_{j1}, \dots, e_{jn}$  from (22),

$$g_{i1}^{(jj)} = \dots = g_{in}^{(jj)} = 0 = h_{j1}^{(ii)} = \dots = h_{jn}^{(ii)}. \tag{23}$$

Similarly, from  $h(e_{ii})e_{jj} + e_{jj}g(e_{ii}) = 0$ ,

$$h_{i1}^{(jj)} = \dots = h_{in}^{(jj)} = 0 = g_{j1}^{(ii)} = \dots = g_{jn}^{(ii)}. \tag{24}$$

Now,  $e_{ij} = e_{ii} \circ e_{ij}$ . Using (23) and (24),

$$f(e_{ij}) = 2(g_{i1}^{(ij)} e_{i1} + \cdots + g_{in}^{(ij)} e_{in}) = 2(h_{i1}^{(ij)} e_{i1} + \cdots + h_{in}^{(ij)} e_{in}). \quad (25)$$

In a similar way, we have

$$\begin{aligned} f(e_{ij}) &= 2(g_{j1}^{(jj)} e_{i1} + \cdots + g_{jn}^{(jj)} e_{in} + h_{j1}^{(ij)} e_{j1} + \cdots + h_{jn}^{(ij)} e_{jn}) \\ &= 2(h_{j1}^{(jj)} e_{i1} + \cdots + g_{jn}^{(jj)} e_{in} + h_{j1}^{(ij)} e_{j1} + \cdots + h_{jn}^{(ij)} e_{jn}). \end{aligned} \quad (26)$$

From (26),

$$g_{j1}^{(jj)} = h_{j1}^{(jj)}, \dots, g_{jn}^{(jj)} = h_{jn}^{(jj)}, \text{ for all } j = 1, \dots, n. \quad (27)$$

From (23), (24) and (27),

$$g(e_{jj}) = h(e_{jj}) \text{ for all } j = 1, \dots, n. \quad (A1)$$

From (25),

$$g_{i1}^{(ij)} = h_{i1}^{(ij)}, \dots, g_{in}^{(ij)} = h_{in}^{(ij)} \text{ for all } i \neq j. \quad (28)$$

From (26),

$$g_{j1}^{(ij)} = h_{j1}^{(ij)}, \dots, g_{jn}^{(ij)} = h_{jn}^{(ij)} \text{ for all } i \neq j. \quad (29)$$

Now, let  $k \neq i$  and  $k \neq j$ . From  $e_{kk} \circ e_{ij} = 0$ ,

$$g_{k1}^{(ij)} = \cdots = g_{kn}^{(ij)} = 0 = h_{k1}^{(ij)} = \cdots = h_{kn}^{(ij)}. \quad (30)$$

From (28), (29) and (30),

$$g(e_{ij}) = h(e_{ij}) \text{ for all } i \neq j. \quad (A2)$$

Since  $g$  and  $h$  are linear maps, using (A1) and (A2),  $g(X) = h(X)$ , for all  $X \in \mathcal{M}_n(C)$ .  $\square$

Now, we give an example of a Jordan left  $\{g, g\}$ -derivation over  $\mathcal{M}_2(C)$  which is not a left  $\{g, g\}$ -derivation.

*Example 12* : Let  $A = \mathcal{M}_2(C)$  and  $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in A$ . We consider  $f, g : A \rightarrow A$  defined as

$$f(X) = \begin{bmatrix} 2x_1 + 6x_2 & 4x_1 + 8x_2 \\ 2x_3 + 6x_4 & 4x_3 + 8x_4 \end{bmatrix} \text{ and } g(X) = \begin{bmatrix} x_1 + 3x_2 & 2x_1 + 4x_2 \\ x_3 + 3x_4 & 2x_3 + 4x_4 \end{bmatrix}, \text{ respectively. Then } f \text{ is a}$$

Jordan left  $\{g, g\}$ -derivation. Now,  $f(e_{12}e_{11}) = f(0) = 0$ , but  $e_{12}g(e_{11}) + e_{11}g(e_{12}) = 3e_{11} + 4e_{12} \neq 0$ . Therefore,  $f$  is not a left  $\{g, g\}$ -derivation.

*Remark 13 :* It is known that if  $C$  is a prime (semiprime) ring, then  $\mathcal{M}_n(C)$  is a prime (semiprime) ring, for  $n \geq 2$ . In Example 12, if  $C$  is a commutative prime (semiprime) ring, then  $\mathcal{M}_2(C)$  becomes a prime (semiprime) algebra over  $C$ . Therefore, Example 12 shows that every Jordan left  $\{g, h\}$ -derivation over a prime (semiprime) algebra need not be a left  $\{g, h\}$ -derivation.

Also, we characterize Jordan left  $\{g, g\}$ -derivation and left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$ .

**Theorem 14** — *Let  $\mathcal{M}_n(C)$ ,  $n \geq 2$ , be the algebra of  $n \times n$  full matrices over  $C$ . Then  $f$  is a Jordan left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$  if and only if there exist  $n^2$  elements in  $C$  such that*

$$g(A) = \sum_{j=1}^n \sum_{i=1}^n \left( \sum_{k=1}^n a_{ik} g_{kj}^{(kk)} \right) e_{ij},$$

$$f(A) = 2g(A), \text{ for all } A \in \mathcal{T}_n(C) \text{ where, } g_{lm}^{(np)} \in C, A = \sum_{j=1}^n \sum_{i=1}^n a_{ij} e_{ij}.$$

Moreover, if  $f$  is a Jordan left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$ , then  $f$  and  $g$  are right centralizers.

PROOF : Let  $f$  be a Jordan left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$ . Now, let  $g$  be of the form (20) and

$$f(e_{ij}) = \sum_{k=1}^n \sum_{l=1}^n f_{kl}^{(ij)} e_{kl}, \text{ where } f_{kl}^{(ij)} \in C. \tag{31}$$

Let  $i \in \{1, 2, \dots, n\}$ . Since  $e_{ii}^2 = e_{ii}$  and  $f$  is a Jordan left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$ ,

$$f(e_{ii}) = 2 \sum_{k=1}^n g_{ik}^{(ii)} e_{ik}. \tag{32}$$

Let  $i \neq j$ . Then from  $e_{ij} = e_{ii} \circ e_{ij}$ ,

$$f(e_{ij}) = 2 \sum_{k=1}^n (g_{ik}^{(ij)} + g_{jk}^{(ii)}) e_{ik}. \tag{33}$$

Similarly, as  $e_{ij} = e_{ij} \circ e_{jj}$ ,

$$f(e_{ij}) = 2 \left( \sum_{k=1}^n g_{jk}^{(jj)} e_{ik} + \sum_{k=1}^n g_{jk}^{(ij)} e_{jk} \right). \tag{34}$$

Now, by (33) and (34),

$$g_{jk}^{(ij)} = 0, \text{ for } k = 1, 2, \dots, n. \tag{35}$$

Let  $k \neq i, j$ . Since  $0 = e_{ij} \circ e_{kk}$ ,

$$g_{kl}^{(ij)} = 0 = g_{jl}^{(kk)}, \text{ for } l = 1, 2, \dots, n. \quad (36)$$

Now, by (33), (34) and (36),

$$g_{ik}^{(ij)} = g_{jk}^{(jj)}, \text{ for } k = 1, 2, \dots, n. \quad (37)$$

Therefore, for  $i \neq j$ , by using (34)-(37),

$$f(e_{ij}) = 2 \sum_{k=1}^n g_{jk}^{(jj)} e_{ik}, \quad g(e_{ij}) = \sum_{k=1}^n g_{jk}^{(jj)} e_{ik}, \quad g(e_{ii}) = \sum_{k=1}^n g_{ik}^{(ii)} e_{ik}. \quad (38)$$

Let  $A \in \mathcal{M}_n(C)$ . Then  $A = \sum_{i=1}^n \sum_{j=1}^n a_{ij} e_{ij}$ , where  $a_{ij} \in C$ . Hence, by (32) and (38),

$$g(A) = \sum_{j=1}^n \sum_{i=1}^n \left( \sum_{k=1}^n a_{ik} g_{kj}^{(kk)} \right) e_{ij}, \quad (39)$$

$$f(A) = 2g(A).$$

Also, the required number of elements from  $C$  to express  $g$  and  $f$  are  $n + n + \dots + n$  ( $n$  times) =  $n^2$ .

Conversely, let  $g$  and  $f$  be of the form (39), for all  $A \in \mathcal{M}_n(C)$ , where  $A = \sum_{i=1}^n \sum_{j=1}^n a_{ij} e_{ij}$  and  $a_{ij} \in C$ . Then  $g(A) = A\alpha$  and  $f(A) = 2A\alpha$ , where  $\alpha = \sum_{i=1}^n \sum_{j=1}^n g_{ij}^{(ii)} e_{ij}$ . Now, let  $B \in \mathcal{M}_n(C)$ . Then  $B = \sum_{1 \leq i \leq j \leq n} b_{ij} e_{ij}$ , where  $b_{ij} \in C$ . Thus, by direct computation

$$f(AB + BA) = 2(Ag(B) + Bg(A)).$$

The last part can easily be derived from the converse part. □

**Theorem 15** — Let  $\mathcal{M}_n(C)$ ,  $n \geq 2$ , be the algebra of  $n \times n$  full matrices over  $C$ . Then  $f$  is a left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$  if and only if  $f = g = 0$ .

PROOF : Let  $f$  be a left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$  where  $g$  and  $f$  be of the form (20) and (31), respectively. Since every left  $\{g, g\}$ -derivation over  $\mathcal{M}_n(C)$  is a Jordan left  $\{g, g\}$ -derivation, therefore, (32) and (39) hold.

Let  $i \neq j$ . Since  $e_{ij} = e_{ii}e_{ij}$ ,

$$f(e_{ij}) = e_{ii}g(e_{ij}) + e_{ij}g(e_{ii}) = f(e_{ij}e_{ii}) = 0. \quad (40)$$

Now, since  $e_{ii}e_{ji} = 0$ ,

$$g_{ik}^{(ii)} = 0, \text{ for } k = 1, 2, \dots, n. \tag{41}$$

By (32) and (39)-(41),  $f = g = 0$ .

The next example is of a Jordan left  $\{g, g\}$ -derivation  $f$  on  $\mathcal{M}_n(C)$  where  $g$  and  $f$  are not left centralizers.

*Example 16 :* Let  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}_2(C)$  and  $g, f : \mathcal{M}_2(C) \rightarrow \mathcal{M}_2(C)$  are defined by  $g(A) = \begin{bmatrix} a_{11} - a_{12} & -a_{11} \\ a_{21} - a_{22} & -a_{21} \end{bmatrix}$  and  $f(A) = 2g(A)$  respectively. Then  $f$  is a Jordan left  $\{g, g\}$ -derivation on  $\mathcal{M}_2(C)$ . In this case, for  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ ,  $g(AB) \neq g(A)B$  and  $f(AB) \neq f(A)B$ . Therefore,  $g$  and  $f$  are not left centralizers.

### 3. JORDAN LEFT $\{g, h\}$ -DERIVATION ON TENSOR PRODUCTS OF ALGEBRAS

Now, we discuss a result on Jordan left  $\{g, h\}$ -derivation over tensor products of algebras. Let  $R$  be a commutative ring with unity and  $M, N, K$  be  $R$ -modules. A map  $\theta : M \times N \rightarrow K$  is  $R$ -bilinear if  $\theta(x, y)$  is  $R$ -linear in  $x$  for each fixed  $y \in N$  and  $R$ -linear in  $y$  for each fixed  $x \in M$ . The tensor product of  $M$  and  $N$  is a pair  $(T, \theta)$ , where  $T$  is an  $R$ -module and  $\theta : M \times N \rightarrow T$  an  $R$ -bilinear map with the property that for any  $R$ -module  $K$  and  $R$ -bilinear map  $f : M \times N \rightarrow K$ , there exists a unique  $R$ -linear map  $\tilde{f} : T \rightarrow K$  with  $\tilde{f} \circ \theta = f$ . For  $x, y \in M$  the image  $\theta(x, y)$  is denoted by  $x \otimes y$ . If  $A$  and  $B$  are two  $R$ -algebras, the tensor product  $A \otimes_R B$  has a well defined multiplication  $(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$ ,  $a, a' \in A, b, b' \in B$ . This makes  $A \otimes_R B$  into an  $R$ -algebra, called the tensor product of algebras  $A$  and  $B$ .

**Theorem 17** — *If an algebra  $A$  over a field  $\mathbb{F}$  with  $\text{char}(\mathbb{F}) \neq 2$  has the property that every Jordan left  $\{g, h\}$ -derivation of  $A$  is a left  $\{g, h\}$ -derivation, then the algebra  $A \otimes S$  has the same property where  $S$  is a commutative algebra over  $\mathbb{F}$ .*

PROOF : Let  $f$  be a Jordan left  $\{g, h\}$ -derivation on  $A \otimes S$ . Let  $\{b_t \mid t \in T, T \text{ an index set}\}$  be a basis of  $S$  and  $u \in A \otimes S$ . Then

$$f(u) = \sum_{t \in T} f_t(u) \otimes b_t, \quad g(u) = \sum_{t \in T} g_t(u) \otimes b_t \text{ and } h(u) = \sum_{t \in T} h_t(u) \otimes b_t \tag{42}$$

where  $f_t(u) = g_t(u) = h_t(u) = 0$  for all but finitely many  $t \in T$ .

Now, let  $x, y \in A$  and  $r, s \in S$ . Since  $f$  is a Jordan left  $\{g, h\}$ -derivation on  $A \otimes S$ ,

$$\begin{aligned} & f((xy + yx) \otimes rs) \\ &= f((x \otimes r)(y \otimes s) + (y \otimes s)(x \otimes r)) = 2[(x \otimes r)g(y \otimes s) + (y \otimes s)h(x \otimes r)], \\ & \text{which implies, } \sum_{t \in T} f_t((xy + yx) \otimes rs) \otimes b_t \\ &= 2[(x \otimes r)(\sum_{w \in T} g_w(y \otimes s) \otimes b_w) + (y \otimes s)(\sum_{w \in T} h_w(x \otimes r) \otimes b_w)] \\ &= 2[(\sum_{w \in T} xg_w(y \otimes s)) \otimes b_w r + (\sum_{w \in T} yh_w(x \otimes r)) \otimes b_w s] \end{aligned} \tag{43}$$

where,  $b_w r = \sum_{t \in T} \alpha_{tw} b_t$ ,  $b_w s = \sum_{t \in T} \beta_{tw} b_t$  and  $\alpha_{tw}, \beta_{tw} \in \mathbb{F}$ .

$$\begin{aligned} \text{R.H.S. of (43)} &= 2[\sum_{t \in T} x(\sum_{w \in T} \alpha_{tw} g_w(y \otimes s)) \otimes b_t] \\ &+ [\sum_{t \in T} y(\sum_{w \in T} \beta_{tw} h_w(x \otimes r)) \otimes b_t] \\ \implies f_t((xy + yx) \otimes rs) &= 2[x(\sum_{w \in T} \alpha_{tw} g_w(y \otimes s)) + y(\sum_{w \in T} \beta_{tw} h_w(x \otimes r))]. \end{aligned} \tag{44}$$

Let  $\tilde{f}(x) = f_t(x \otimes rs)$ ,  $\tilde{g}(y) = \sum_{w \in T} \alpha_{tw} g_w(y \otimes s)$  and  $\tilde{h}(x) = \sum_{w \in T} \beta_{tw} h_w(x \otimes r)$ , for all  $x, y \in A$ . Then, by (44),  $\tilde{f}$  is a Jordan left  $\{\tilde{g}, \tilde{h}\}$ -derivation on  $A$ . So,  $\tilde{f}$  is a left  $\{\tilde{g}, \tilde{h}\}$ -derivation, by assumption. Hence,

$$f_t(xy \otimes rs) = x(\sum_{w \in T} \alpha_{tw} g_w(y \otimes s)) + y(\sum_{w \in T} \beta_{tw} h_w(x \otimes r)). \tag{45}$$

Also,

$$\begin{aligned} f((x \otimes r)(y \otimes s)) &= f(xy \otimes rs) = \sum_{t \in T} f_t(xy \otimes rs) \otimes b_t \\ &= \sum_{t \in T} [x(\sum_{w \in T} \alpha_{tw} g_w(y \otimes s)) + y(\sum_{w \in T} \beta_{tw} h_w(x \otimes r))] \text{ (by (45))} \\ &= x(\sum_{w \in T} g_w(y \otimes s)) \otimes (\sum_{t \in T} \alpha_{tw} b_t) + y(\sum_{w \in T} h_w(x \otimes r)) \otimes (\sum_{t \in T} \beta_{tw} b_t) \\ &= \sum_{w \in T} xg_w(y \otimes s) \otimes b_w r + \sum_{w \in T} yh_w(x \otimes r) \otimes b_w s \\ &= (x \otimes r)(\sum_{w \in T} g_w(y \otimes s) \otimes b_w) + (y \otimes s)(\sum_{w \in T} h_w(x \otimes r) \otimes b_w) \end{aligned} \tag{46}$$

(since  $S$  is commutative)

$$= (x \otimes r)g(y \otimes s) + (y \otimes s)h(x \otimes r).$$

Therefore,  $f(uv) = ug(v) + vh(u)$  for all  $u, v \in A \otimes S$ . Similarly,  $f(uv) = uh(v) + vg(u)$  for all  $u, v \in A \otimes S$ . Thus,  $f$  is a left  $\{g, h\}$ -derivation on  $A \otimes S$ .  $\square$

4. JORDAN LEFT  $\{g, h\}$ -DERIVATION ON ALGEBRA OF POLYNOMIALS

Let  $A$  be an algebra over  $C$  and  $A[x]$ , the ring of polynomials over  $A$ . Then  $A[x]$  becomes an algebra over  $C$  where scalar multiplication is defined as  $\alpha(a_0 + a_1x + a_2x^2 + \dots + a_rx^r) = \alpha a_0 + (\alpha a_1)x + (\alpha a_2)x^2 + \dots + (\alpha a_r)x^r$ ,  $\alpha \in C$  and  $a_0, a_1, a_2, \dots, a_r \in A$ . First, we derive a result on left  $\{g, h\}$ -derivation on  $A[x]$ .

**Theorem 18** — *If  $f$  is left  $\{g, h\}$ -derivation on  $A$ , then  $\tilde{f}$  is a left  $\{\tilde{g}, \tilde{h}\}$ -derivation on  $A[x]$  where  $\tilde{F}(\sum_{j=0}^r a_j x^j) = \sum_{j=0}^r F(a_j)x^j$ , for  $F = f, g, h$  and  $a_j \in A$ .*

PROOF : Let  $P, Q \in A[x]$ . Then  $P = \sum_{j=0}^r a_j x^j$  and  $Q = \sum_{k=0}^s b_k x^k$ , where  $a_j, b_k \in A$ . Therefore,  $PQ = \sum_{n=0}^{r+s} c_n x^n$  where  $c_n = \sum_{k=0}^s \sum_{j=0}^r a_j b_k$ .

By definition of  $\tilde{f}, \tilde{g}, \tilde{h}$  and the assumption that  $f$  is a left  $\{g, h\}$ -derivation on  $A$ ,

$$\begin{aligned} & P\tilde{g}(Q) + Q\tilde{h}(P) \\ &= \left(\sum_{j=0}^r a_j x^j\right)\left[\sum_{k=0}^s g(b_k)x^k\right] + \left(\sum_{k=0}^s b_k x^k\right)\left[\sum_{j=0}^r h(a_j)x^j\right] \\ &= \sum_{n=0}^{r+s} \left[\sum_{k=0}^s \sum_{j=0}^r a_j g(b_k)\right]x^n + \sum_{n=0}^{r+s} \left[\sum_{k=0}^s \sum_{j=0}^r b_k h(a_j)\right]x^n \\ &= \sum_{n=0}^{r+s} \left[\sum_{k=0}^s \sum_{j=0}^r (a_j g(b_k) + b_k h(a_j))\right]x^n \\ &= \sum_{n=0}^{r+s} \left[\sum_{k=0}^s \sum_{j=0}^r f(a_j b_k)\right]x^n \\ &= \tilde{f}\left[\sum_{n=0}^{r+s} \left(\sum_{k=0}^s \sum_{j=0}^r a_j b_k\right)x^n\right] \\ &= \tilde{f}(PQ). \end{aligned}$$

Similarly,  $\tilde{f}(PQ) = P\tilde{h}(Q) + Q\tilde{g}(P)$ . Thus,  $\tilde{f}$  is a left  $\{\tilde{g}, \tilde{h}\}$ -derivation on  $A[x]$ .  $\square$

Our next result characterizes Jordan left  $\{g, h\}$ -derivation on  $A[x]$ .

**Theorem 19** — If  $f$  is Jordan left  $\{g, h\}$ -derivation on  $A$ , then  $\tilde{f}$  is a Jordan left  $\{\tilde{g}, \tilde{h}\}$ -derivation on  $A[x]$  where  $\tilde{F}(\sum_{j=0}^r a_j x^j) = \sum_{j=0}^r F(a_j) x^j$ , for  $F = f, g, h$  and  $a_j \in A$ .

PROOF : Let  $P, Q \in A[x]$ . Then  $P = \sum_{j=0}^r a_j x^j$  and  $Q = \sum_{k=0}^s b_k x^k$ , where  $a_j, b_k \in A$ . Therefore,  $PQ + QP = \sum_{n=0}^{r+s} d_n x^n$ , where  $d_n = \sum_{\substack{k=0 \\ j+k=n}}^s \sum_{j=0}^r (a_j b_k + b_k a_j)$ .

By definition of  $\tilde{f}, \tilde{g}, \tilde{h}$  and the assumption that  $f$  is a Jordan left  $\{g, h\}$ -derivation on  $A$ ,

$$\begin{aligned} & 2(P\tilde{g}(Q) + Q\tilde{h}(P)) \\ &= \sum_{n=0}^{r+s} \left[ \sum_{k=0}^s \sum_{\substack{j=0 \\ j+k=n}}^r 2(a_j g(b_k) + b_k h(a_j)) \right] x^n \\ &= \sum_{n=0}^{r+s} \left[ \sum_{k=0}^s \sum_{\substack{j=0 \\ j+k=n}}^r f(a_j b_k + b_k a_j) \right] x^n \\ &= \tilde{f}(PQ + QP). \end{aligned}$$

Hence,  $\tilde{f}$  is a Jordan left  $\{\tilde{g}, \tilde{h}\}$ -derivation on  $A[x]$ . □

### 5. JORDAN LEFT $\{g, h\}$ -DERIVATION ON QUATERNION ALGEBRA

Let  $\mathbb{H}_{\mathbb{R}} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$  be the quaternion algebra over the field of real numbers.

**Proposition 20** — Let  $f$  be a left  $\{g, h\}$ -derivation over  $\mathbb{H}_{\mathbb{R}}$ . Then the image of  $f$  is the subalgebra of  $\mathbb{H}_{\mathbb{R}}$  generated by  $f(1)$ .

PROOF : Since  $ij + ji = 0$  and  $f$  is a left  $\{g, h\}$ -derivation,

$$0 = f(ij + ji) = 2(ig(j) + jh(i)) \implies f(k) = f(ij) = ig(j) + jh(i) = 0. \quad (47)$$

Similarly, we get  $f(i) = f(j) = 0$ . Now, let  $q = a + bi + cj + dk \in \mathbb{H}_{\mathbb{R}}$ , where  $a, b, c, d \in \mathbb{R}$ . Since  $f$  is linear,  $f(q) = af(1)$ . Hence,  $f(\mathbb{H}_{\mathbb{R}}) = \langle f(1) \rangle$ .

Now, we present the necessary and sufficient condition for Jordan left  $\{g, h\}$ -derivation to be a left  $\{g, h\}$ -derivation on  $\mathbb{H}_{\mathbb{R}}$ .

**Theorem 21** — Let  $f$  be a Jordan left  $\{g, h\}$ -derivation over  $\mathbb{H}_{\mathbb{R}}$ . Then  $f$  is a left  $\{g, h\}$ -derivation on  $\mathbb{H}_{\mathbb{R}}$  if and only if  $f(i) = f(j) = f(k) = 0$ .

PROOF : Let  $f(i) = f(j) = f(k) = 0$  and  $q = a + bi + cj + dk \in \mathbb{H}_{\mathbb{R}}$ , where  $a, b, c, d \in \mathbb{R}$ . So,  $f(q) = af(1)$ .

Let  $r = x + yi + zj + tk \in \mathbb{H}_{\mathbb{R}}$ . We have to prove  $f(qr) = qg(r) + rh(q) = qh(r) + rg(q)$ . Since

$$\begin{aligned} 2(qg(r) + rh(q)) &= f(qr + rq) = 2(ax - by - cz - dt)f(1) \\ \implies qg(r) + rh(q) &= (ax - by - cz - dt)f(1) = f(qr). \end{aligned}$$

Similarly,  $qh(r) + rg(q) = f(qr)$ . Therefore,  $f$  is a left  $\{g, h\}$ -derivation on  $\mathbb{H}_{\mathbb{R}}$ .

The converse is true by the proof of Proposition 20. □

Now, we characterize Jordan left  $\{g, h\}$ -derivation on  $\mathbb{H}_{\mathbb{R}}$ .

**Theorem 22** — *Let  $\mathbb{H}_{\mathbb{R}}$  be the quaternion algebra over the field of real numbers. Then  $f$  is a Jordan left  $\{g, h\}$ -derivation over  $\mathbb{H}_{\mathbb{R}}$  if and only if  $g = h$  and there exist 4 elements in  $\mathbb{R}$  such that*

$$\begin{aligned} g(q) &= (aa_g^{(1)} - bb_g^{(1)} - cc_g^{(1)} - dd_g^{(1)}) + (ab_g^{(1)} + ba_g^{(1)} + cd_g^{(1)} - dc_g^{(1)})i \\ &\quad + (ac_g^{(1)} - bd_g^{(1)} + ca_g^{(1)} + db_g^{(1)})j + (ad_g^{(1)} + bc_g^{(1)} - cb_g^{(1)} + da_g^{(1)})k, \\ f(q) &= 2g(q), \text{ for all } q \in \mathbb{H}_{\mathbb{R}} \\ \text{where } q &= a + bi + cj + dk \text{ and } a_g^{(1)}, b_g^{(1)}, c_g^{(1)}, d_g^{(1)} \in \mathbb{R}. \end{aligned}$$

Moreover, if  $f$  is a Jordan left  $\{g, g\}$ -derivation over  $\mathbb{H}_{\mathbb{R}}$ , then  $f$  and  $g$  are right centralizers.

PROOF : Let

$$F(l) = a_F^{(l)} + b_F^{(l)}i + c_F^{(l)}j + d_F^{(l)}k, \text{ for } F = f, g, h \text{ and } l = 1, i, j, k. \tag{48}$$

Since  $f(1) = g(1) + h(1)$ ,

$$a_f^{(1)} = a_g^{(1)} + a_h^{(1)}, b_f^{(1)} = b_g^{(1)} + b_h^{(1)}, c_f^{(1)} = c_g^{(1)} + c_h^{(1)}, d_f^{(1)} = d_g^{(1)} + d_h^{(1)}. \tag{49}$$

Since,  $i^2 = j^2 = k^2 = -1$ ,

$$\begin{aligned} a_f^{(1)} &= b_g^{(i)} + b_h^{(i)}, b_f^{(1)} = -a_g^{(i)} - a_h^{(i)}, c_f^{(1)} = d_g^{(i)} + d_h^{(i)}, d_f^{(1)} = -c_g^{(i)} - c_h^{(i)}, \\ a_f^{(1)} &= c_g^{(j)} + c_h^{(j)}, b_f^{(1)} = -d_g^{(j)} - d_h^{(j)}, c_f^{(1)} = -a_g^{(j)} - a_h^{(j)}, d_f^{(1)} = b_g^{(j)} + b_h^{(j)}, \\ a_f^{(1)} &= d_g^{(k)} + d_h^{(k)}, b_f^{(1)} = c_g^{(k)} + c_h^{(k)}, c_f^{(1)} = -b_g^{(k)} - b_h^{(k)}, d_f^{(1)} = -a_g^{(k)} - a_h^{(k)}. \end{aligned} \tag{50}$$

Since  $2i = 1 \circ i = i \circ 1$ , using (48),

$$\begin{aligned} a_f^{(i)} &= a_h^{(i)} - b_g^{(1)} = a_g^{(i)} - b_h^{(1)}, & b_f^{(i)} &= b_h^{(i)} + a_g^{(1)} = b_g^{(i)} + a_h^{(1)} \\ c_f^{(i)} &= c_h^{(i)} - d_g^{(1)} = c_g^{(i)} - d_h^{(1)}, & d_f^{(i)} &= d_h^{(i)} + c_g^{(1)} = d_g^{(i)} + c_h^{(1)}. \end{aligned} \quad (51)$$

By using (49)-(51),

$$\begin{aligned} 2a_f^{(i)} &= a_h^{(i)} - b_g^{(1)} + a_g^{(i)} - b_h^{(1)} = (a_g^{(i)} + a_h^{(i)}) - (b_g^{(1)} + b_h^{(1)}) = -2b_f^{(1)} \\ \implies a_f^{(i)} &= -b_f^{(1)}. \text{ Similarly, } b_f^{(i)} = a_f^{(1)}, c_f^{(i)} = -d_f^{(1)}, d_f^{(i)} = c_f^{(1)}. \end{aligned} \quad (52)$$

Similarly, using  $2j = 1 \circ j = j \circ 1$ ,  $2k = 1 \circ k = k \circ 1$ , (50) and (51), we have

$$\begin{aligned} a_f^{(j)} &= -c_f^{(1)}, & b_f^{(j)} &= d_f^{(1)}, & c_f^{(j)} &= a_f^{(1)}, & d_f^{(j)} &= -b_f^{(1)}, \\ a_f^{(k)} &= -d_f^{(1)}, & b_f^{(k)} &= -c_f^{(1)}, & c_f^{(k)} &= b_f^{(1)}, & d_f^{(k)} &= a_f^{(1)}. \end{aligned} \quad (53)$$

Also, in view of (50)-(53), for  $F = g$  and  $h$ ,

$$\begin{aligned} a_F^{(i)} &= -b_F^{(1)}, & b_F^{(i)} &= a_F^{(1)}, & c_F^{(i)} &= -d_F^{(1)}, & d_F^{(i)} &= c_F^{(1)}, \\ a_F^{(j)} &= -c_F^{(1)}, & b_F^{(j)} &= d_F^{(1)}, & c_F^{(j)} &= a_F^{(1)}, & d_F^{(j)} &= -b_F^{(1)}, \\ a_F^{(k)} &= -d_F^{(1)}, & b_F^{(k)} &= -c_F^{(1)}, & c_F^{(k)} &= b_F^{(1)}, & d_F^{(k)} &= a_F^{(1)}. \end{aligned} \quad (54)$$

Since  $i \circ j = 0$ ,

$$b_g^{(j)} = -c_h^{(i)}, \quad a_g^{(j)} = -d_h^{(i)}, \quad d_g^{(j)} = a_h^{(i)}, \quad c_g^{(j)} = b_h^{(i)}. \quad (55)$$

Again, by (54) and (55),  $x_g^{(1)} = x_h^{(1)}$ , for all  $x = a, b, c, d$ . Therefore,  $g = h$ . Let  $q \in \mathbb{H}_{\mathbb{R}}$ . Then  $q = a + bi + cj + dk$ , where  $a, b, c, d \in \mathbb{R}$ . Hence, by (49) and (52)-(55),

$$\begin{aligned} g(q) &= (aa_g^{(1)} - bb_g^{(1)} - cc_g^{(1)} - dd_g^{(1)}) + (ab_g^{(1)} + ba_g^{(1)} + cd_g^{(1)} - dc_g^{(1)})i \\ &\quad + (ac_g^{(1)} - bd_g^{(1)} + ca_g^{(1)} + db_g^{(1)})j + (ad_g^{(1)} + bc_g^{(1)} - cb_g^{(1)} + da_g^{(1)})k, \\ f(q) &= 2g(q). \end{aligned} \quad (56)$$

Moreover, the required number of elements from  $\mathbb{R}$  to express  $g$  and  $f$  is 4.

Conversely, let  $g$  and  $f$  be of the form (56), for all  $q \in \mathbb{H}_{\mathbb{R}}$ , where  $q = a + bi + cj + dk$  and  $a, b, c, d \in \mathbb{R}$ . Then  $g(q) = q\alpha$  and  $f(q) = 2q\alpha$ , where  $\alpha = a_g^{(1)} + b_g^{(1)}i + c_g^{(1)}j + d_g^{(1)}k$ . Now, let  $p \in \mathbb{H}_{\mathbb{R}}$ . Then  $p = x + yi + zj + tk$ , where  $x, y, z, t \in \mathbb{R}$ . Also, by direct computation,

$$f(pq + qp) = 2(pg(q) + qg(p)).$$

The last part can easily be derived from the converse part.

Below is an example of a Jordan left  $\{g, g\}$ -derivation on  $\mathbb{H}_{\mathbb{R}}$  which is not a left  $\{g, g\}$ -derivation. Also, the condition of Theorem 21 is not satisfied in this case.

*Example 23 :* Let  $q = a + bi + cj + dk \in \mathbb{H}_{\mathbb{R}}$ , where  $a, b, c, d \in \mathbb{R}$ . Define  $f, g : \mathbb{H}_{\mathbb{R}} \rightarrow \mathbb{H}_{\mathbb{R}}$  as  $f(q) = 2q$  and  $g(q) = q$ , respectively. Then  $f, g$  are linear maps and  $f$  is also a Jordan left  $\{g, g\}$ -derivation. But,  $f(ij) = f(k) = 2k \neq 0 = ij + ji = ig(j) + jg(i)$ , therefore,  $f$  is not a left  $\{g, g\}$ -derivation.

Our next result shows that there is no nonzero left  $\{g, g\}$ -derivation on  $\mathbb{H}_{\mathbb{R}}$ .

**Theorem 24** — *Let  $f$  be a left  $\{g, g\}$ -derivation over  $\mathbb{H}_{\mathbb{R}}$ . Then  $g$  and  $f$  are identically zero.*

PROOF : Let  $g$  and  $f$  be of the form (48). Since every left  $\{g, g\}$ -derivation over  $\mathbb{H}_{\mathbb{R}}$  is a Jordan left  $\{g, g\}$ -derivation, so (54) and (56) hold. By Theorem 21,  $f(i) = f(j) = f(k) = 0$ . Since  $i = 1i = i1$ , using (48),

$$a_g^{(i)} = b_g^{(1)}, b_g^{(i)} = -a_g^{(1)}, c_g^{(i)} = d_g^{(1)}, d_g^{(i)} = -c_g^{(1)}. \tag{57}$$

Since  $ij = k$  and  $f(k) = 0$ ,

$$b_g^{(j)} = -c_g^{(i)}, a_g^{(j)} = -d_g^{(i)}, d_g^{(j)} = a_g^{(i)}, c_g^{(j)} = b_g^{(i)}. \tag{58}$$

Now, by (54), (57) and (58),  $a_g^{(1)} = b_g^{(1)} = c_g^{(1)} = d_g^{(1)} = 0$ . So, by (56),  $g = 0 = f$ .

Finally, we give an example of a Jordan left  $\{g, g\}$ -derivation  $f$  on  $\mathbb{H}_{\mathbb{R}}$  where  $g$  and  $f$  are not left centralizers.

*Example 25 :* Suppose  $q = a + bi + cj + dk \in \mathbb{H}_{\mathbb{R}}$  and  $g, f : \mathbb{H}_{\mathbb{R}} \rightarrow \mathbb{H}_{\mathbb{R}}$  are defined by  $g(q) = (a - 2b - 3c - 4d) + (2a + b + 4c - 3d)i + (3a - 4b + c + 2d)j + (4a + 3b - 2c + d)k$  and  $f(q) = 2g(q)$ , respectively. Then  $f$  is a Jordan left  $\{g, g\}$  derivation on  $\mathbb{H}_{\mathbb{R}}$ . Further, if  $p = 5 + 6i + 7j + 8k$  and  $q = 9 + 10i + 11j + 12k$ , then it can be seen that  $g(pq) \neq g(p)q$  and  $f(pq) \neq f(p)q$ . Hence,  $g$  and  $f$  are not left centralizers.

*Remark 26 :* By Frobenius theorem, every finite dimensional noncommutative division algebra over  $\mathbb{R}$  is isomorphic to  $\mathbb{H}_{\mathbb{R}}$ . Therefore, Theorem 24 is true for every noncommutative finite dimensional division algebra over  $\mathbb{R}$ .

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