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RELATIVE IDEALS IN ORDERED SEMIGROUPS

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In this paper, after introducing the notion of relative ideals in ordered semigroups, some important properties of these ideals are studied. Then notions of relatively prime and relatively weakly prime ideals are defined and some vital results have been proved. We also find some conditions under which relative ideals become left, right and two-sided ideals in any regular ordered semigroup. Finally, the notion of maximal and minimal relative ideals are introduced and some of their properties are studied.

Key words : Ordered semigroups; ideals; relative ideals; relatively prime and relatively weakly prime ideals; maximal and minimal relative ideals.

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1. Introduction and Preliminaries

In 1962, Wallace [13], introduced the notion of relative ideals on semigroups and defined left (right) H-ideals of a semigroup as follows: Let S be a semigroup and H, a non-empty subset of S. Then a non-empty subset A of S is called a left (resp. right) H-ideal of S if $HA \subseteq A$ (resp. $AH \subseteq A$). Further A is called an H-ideal of S if it is both a left H-ideal and a right H-ideal of S. In 1967, Hrmová [5] generalized the notion of H-ideal by introducing the notion of an (H_1, H_2)-ideal of a semigroup S, $H_1, H_2 \subseteq S$.

The notions of prime and weakly prime ideals in a semigroup have been considered by Szász in [12]. He proved that the ideals of a semigroup S are prime if and only if S is intra-regular [2] and the ideals of S form a chain. He also showed that an ideal of a semigroup S is prime if and only if it is both semiprime and weakly prime. In 1992, Kehayopulu proved the aforesaid results in [6] for ordered semigroups.

An ordered semigroup is an ordered set S which is also a semigroup such that:

$$(\forall s_1, s_2 \in S)(\forall x \in S)(s_1 \leq s_2 \ \Rightarrow \ s_1x \leq s_2x \text{ and } xs_1 \leq xs_2).$$

A non-empty subset T of an ordered semigroup S is said to be a left (resp. right) ideal of S if $ST \subseteq T$ (resp. $TS \subseteq T$) and for all $t \in T$, $S \ni s \leq t \ \Rightarrow \ s \in T$. Further T is called an ideal of S if it is both a left and a right ideal of S. The ordered semigroup S is called simple if it has no proper ideal in S. An ideal T of S is called prime if for any $C, D \subseteq S,\ CD \subseteq T \ \Rightarrow$ either $C \subseteq T$ or $D \subseteq T$. The ideal T is said to be weakly prime if for ideals $C, D \subseteq S,\ CD \subseteq T \ \Rightarrow$ either $C \subseteq T$ or $D \subseteq T$ and semiprime if for $C \subseteq S,\ C^2 \subseteq T \ \Rightarrow \ C \subseteq T$.

A proper ideal T of S is called maximal if for any ideal A of S such that $T \subseteq A \subseteq S$ implies either $A = T$ or $A = S$. The ideal T is called minimal if there is no proper ideal A of S such that $A \subseteq T$.

For more details of ordered semigroups and their related notions, the reader is referred to [1, 3, 4, 7, 8].

In the present paper, in Section 2, left(right) T-ideals, T-ideal, minimal and maximal T-ideals are defined. Then we define the notion of an (H_1, H_2)-ideal of S (it is also called a relative ideal of S) and prove that intersection and union of (H_1, H_2)-ideals of S are again (H_1, H_2)-ideals of S. Further some properties of minimal (H_1, H_2)-ideals are obtained. We also prove that for any ideal I of any regular ordered semigroup S, (H_1, H_2)-ideals of I are ideals of S. At the end of this section, after defining the relations ${}_{H_1}\mathcal{J}, \mathcal{J}_{H_2}, {}_{H_1}\mathcal{J}_{H_2}, {}_{H_1}\mathcal{H}_{H_2}$ and ${}_{H_1}\mathcal{D}_{H_2}$, some properties of these relations are obtained.

In Section 3, we define the (H_1, H_2)-prime, (H_1, H_2)-weakly prime and (H_1, H_2)-semiprime ideals of S and prove that an H-ideal of S is H-prime if and only if it is both H-semiprime and H-weakly prime. Further, we define H-intra regular ordered semigroup and prove that H-ideals of S are H-prime if and only if they form a chain and S is H-intra-regular. Finally, after defining (H_1, H_2)-simple ordered semigroups, characterizations of (H_1, H_2)-simple ordered semigroups have been discussed.

In Section 4, after definig maximal and minimal (H_1, H_2)-ideals of S, it has been shown that every proper (H_1, H_2)-ideal of S is minimal if and only if the intersection of any two distinct proper

(H_1, H_2)-ideals of S is empty. It has been also proved that any proper (H_1, H_2)-ideal A of S is a maximal ideal of S if and only if for every $a \in S \setminus A$, $I_R(A \cup \{a\}) = S$ $(H = H_1 \cup H_2)$. Finally, we show that every proper (H_1, H_2)-ideal of S is maximal if and only if the union of any two distinct proper (H_1, H_2)-ideals coincides S.

2. Relative Ideals and Green's Relations

Definition 2.1 — Let S be an ordered semigroup and $A,\ T$ be the non-empty subsets of S. Then A is said to be a left T-ideal of S if $TA \subseteq A$ and $T \ni x \leq y \in A$ implies $x \in A$. Dually we may define a right T-ideal of S. Further A is said to be a T-ideal of S if it is both a left T-ideal and a right T-ideal of S. If $T = S$, then left T-ideals, right T-ideals and T-ideals of S will simply be called as left ideals, right ideals and ideals of S respectively.

Remark 1 : Every ideal A of an ordered semigroup S is always a T-ideal of S for each subset T of S, but the converse is not true in general, as illustrated by the following example:

Example 1 : Let $S = \{a, b, c, d, \}$. Define a binary operation (.) on S as follows:

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	a	a	b	a

Define an order as $\leq = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (a, c)\}$. Then S is clearly an ordered semigroup. If $A = \{a, b\}, B = \{a, d\}$ and $C = \{c, d\}$, then it is easy to see that A is a B-ideal of S, but not an ideal of S.

Definition 2.2 — Let A and T be the non-empty subsets of an ordered semigroup S. Let

$$(A]_T = \{t \in T \mid t \leq a, \text{ for some } a \in A\}.$$

The following Lemma easily follows.

Lemma 2.3 — Let S be an ordered semigroup. Then

(1) $A \subseteq (A]_T$ for all $A \subseteq T$.

(2) If $A \subseteq B \subseteq T$, then $(A]_T \subseteq (B]_T$.

(3) $(A]_T(B]_T \subseteq (AB]_T$.

(4) $((A]_T]_T = (A]_T$.

(5) For every T-ideal $A \subseteq T$, we have $(A]_T = A$.

(6) If A, B are T-ideals of S, then $(AB]_T, A \cap B$ are T-ideals of S.

(7) If T is subsemigroup of S, then $(TaT]_T$ is a T-ideal of S for each $a \in S$.

Definition 2.4 — A subsemigroup T of an ordered semigroup S is called T-simple if it has no proper T-ideal.

Definition 2.5 — Let T be a non-empty subset of an ordered semigroup S. A T-ideal A of S is called minimal if there does not exist $B \subseteq A$ such that B is a T-ideal of S.

Definition 2.6 — Let T be a non-empty subset of an ordered semigroup S. A left (right, two-sided) T-ideal A of semigroup S is called maximal if there does not exist a T-ideal B of S such that $A \subset B \subset S$; or equivalently, if for each T-ideal B of S such that $T \subseteq B$, implies either $B = T$ or $B = S$.

Definition 2.7 — Let S be an ordered semigroup and let $A_1,\ A_2$ be non-empty subsets of S. A non-empty subset A of S is said to be an (A_1, A_2)-ideal of S if $A_1 A \subseteq A,\ AA_2 \subseteq A$ and $A_1 \cup A_2 \ni x \leq y$ for some $y \in A$ implies $x \in A$. Such ideals are also called relative ideals of S. If $A_1 = \phi$ or $A_2 = \phi$, then an (A_1, A_2)-ideal of S becomes one sided relative ideal of S. The set of all (A_1, A_2)-ideals of S will be denoted, in whatever follows, by $I(A_1, A_2)$.

In example 1, A is a (B, C)-ideal of S.

Example 2 : Let $S = \{0, a, b\}$. Define a binary operation $'\cdot'$ and an order "$\leq$" by following way:

$\cdot$	0	a	b
0	0	0	0
a	0	b	0
b	0	0	0

$$\leq = \{(0,0), (a,a), (b,b), (0,a), (0,b)\}.$$

Then S is an ordered semigroup. Let $A_1 = \{0, a\}$, $A_2 = \{a, b\}$ and $A = \{0, b\}$. Then it is easy to verify that A is an (A_1, A_2)-ideal of S.

Definition 2.8 — Let S be an ordered semigroup and let $H_1,\ H_2$ be non-empty subsets of S. An (H_1, H_2)-ideal T of S is said to be a minimal (H_1, H_2)-ideal of S if there exists no $B \in I(H_1, H_2)$ such that $B \subseteq T$.

The set of all minimal (H_1, H_2)-ideals of S shall be denoted, in the sequel, by $I_m(H_1, H_2)$.

Remark 2 : Clearly the concept of (A_1, A_2)-ideals of S is a generalization of left, right and two sided T-ideals of S.

Lemmas 2.9 and 2.10 easily follow.

Lemma 2.9 — Let S be an ordered semigroup and let $A_1, A_2, A_1', A_2' \subseteq S$. Then

(1) If $A_1 \subseteq A_1', A_2 \subseteq A_2'$, then $I(A_1', A_2') \subseteq I(A_1, A_2)$.

(2) $I(A_1, A_2) = I(A_1, \phi) \cap I(\phi, A_2)$.

(3) $\phi \in I(A_1, A_2)$ if and only if $A_1 = \phi$ and $A_2 = \phi$.

(4) $I(\phi, \phi) = \{A \mid A \subseteq S\}$.

Lemma 2.10 — Let S be an ordered semigroup and let H_1, H_2 be subsemigroups of S. Then

(1) $(H_1 a]_H \in I(H_1, \phi)$ for each $a \in S$ and $H = H_1 \cup H_2$.

(2) $(aH_2]_H \in I(\phi, H_2)$ for each $a \in S$.

(3) $(H_1 a H_2]_H \in I(H_1, H_2)$ for each $a \in S$.

(4) If $L \in I(H_1, \phi)$ and $R \in I(\phi, H_2)$, then $(LR]_H \in I(H_1, H_2)$.

(5) If $A, B \in I(H_1, H_2)$, then $(AB]_H, A \cap B \in I(H_1, H_2)$.

Lemma 2.11 — Let S be an ordered semigroup and let $H_1,\ H_2$ be subsemigroups of S. Then $\bigcup\limits_{A \in I(H_1,H_2)} A \in I(H_1, H_2)$. Also if $\bigcap\limits_{A \in I(H_1,H_2)} A \neq \phi$, then $\bigcap\limits_{A \in I(H_1,H_2)} A \in I(H_1, H_2)$.

PROOF : Since each (H_1, H_2)-ideal of S is nonempty, $\bigcup\limits_{A \in I(H_1,H_2)} A \neq \phi$. Now

$$H_1\Big(\bigcup_{A \in I(H_1,H_2)} A\Big) = \bigcup_{A \in I(H_1,H_2)} H_1 A \subseteq \bigcup_{A \in I(H_1,H_2)} A$$

and

$$\Big(\bigcup_{A \in I(H_1,H_2)} A\Big)H_2 = \bigcup_{A \in I(H_1,H_2)} A H_2 \subseteq \bigcup_{A \in I(H_1,H_2)} A.$$

Let $x \in \bigcup_{A \in I(H_1,H_2)} A$ and $y \in H = H_1 \cup H_2$ such that $y \leq x$. Then we have to show that $y \in \bigcup_{A \in I(H_1,H_2)} A$. Since $x \in \bigcup_{A \in I(H_1,H_2)} A$, $x \in A$ for some $A \in I(H_1, H_2)$. Since A is an (H_1, H_2)-ideal of S, $y \in A$. Therefore $y \in \bigcup_{A \in I(H_1,H_2)} A$. Hence $\bigcup_{A \in I(H_1,H_2)} A$ is an (H_1, H_2)-ideal of S.

Similarly we may show that $\bigcap_{A \in I(H_1,H_2)} A$ is an (H_1, H_2)-ideal of S.

***Theorem* 2.12** — *Let S be an ordered semigroup, $H_1,\ H_2$ be the subsemigroups of S and $A \subseteq H_1$, $B \subseteq H_2$. If $A \in I(H_1, \phi),\ B \in I(\phi, H_2)$ and $C \in I_m(H_1, H_2)$, then $C = (AcB]_H$ for each $c \in C$, where $H = H_1 \cup H_2$.*

PROOF : Since $A \in I(H_1, \phi)$ and $B \in I(\phi, H_2)$, we have $H_1(AcB]_H \subseteq (H_1]_H(AcB]_H \subseteq (H_1AcB]_H \subseteq (AcB]_H$ and $(AcB]_H H_2 \subseteq (AcB]_H(H_2]_H$. Let $x \in H$ such that $x \leq y$ for some $y \in (AcB]_H$. As $y \in (AcB]_H, y \leq acb$, for some $a \in A$ and $b \in B$. Therefore $x \leq acb$. This implies that $x \in (AcB]_H$. Therefore $(AcB]_H$ is an (H_1, H_2)-ideal of S. Also $(AcB]_H \subseteq (ACB]_H \subseteq (H_1CH_2]_H \subseteq (C]_H \subseteq C$. Since C is a minimal (H_1, H_2)-ideal of S, we have $C = (AcB]_H$, as required. □

Below, the above theorem is illustrated with the help of an example.

Example 3 : Let $S = \{a, b, c, d\}$. Define a binary operation $(\cdot)$ on S as follows:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Define an order as $\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$. Then S is clearly an ordered semigroup. Let $H_1 = \{a, b, c\}$ and $H_2 = \{a, c\}$. Then H_1 and H_2 are subsemigroups of S. Take $A = \{a, b\} \subseteq H_1$ and $B = \{c\} \subseteq H_2$. Then, clearly $A \in I(H_1, \phi)$ and $B \in I(\phi, H_2)$. Now, for $H = H_1 \cup H_2$, it is easy to verify that $C = \{a\} \in I_m(H_1, H_2)$ and $C = (AcB]_H$ for each $c \in C$.

Corollary 2.13 — Let S be an ordered semigroup and let $H_1,\ H_2$ be the subsemigroups of S. If $A \in I_m(H_1, H_2)$, then $A = (H_1aH_2]_H$ for each $a \in A$, where $H = H_1 \cup H_2$.

Lemma 2.14 — Let S be an ordered semigroup and let H_1, H_2 be the subsemigroups of S. If $A \in I_m(\phi, H_2)$ ($A \in I_m(H_1, \phi)$), then $(Ah_2]_{H_2} \in I_m(\phi, H_2)$ ($(h_1A]_{H_1} \in I_m(H_1, \phi)$) for each $h_2 \in H_2$ ($h_1 \in H_1$).

PROOF : Let $A \in I_m(\phi, H_2)$. By Corollary 2.13, we have $A = (aH_2]_{H_2}$ for each $a \in A$. On contrary suppose that $(Ah_2]_{H_2} \notin I_m(\phi, H_2)$. Then there exists $B \in I(\phi, H_2)$ such that $B \subseteq (Ah_2]_{H_2}$. Now take any $b \in B$, then $b \le ch_2$ for some $c \in A$. Since $A = (aH_2]_{H_2}$, we have $c \le ah'$ for some $h' \in H_2$. Therefore $b \le ch_2 \le ah'h_2 \in AH_2H_2 \subseteq AH_2 \subseteq A$. It follows that $B \subseteq A$, which is a contradiction. Hence $(Ah_2]_{H_2} \in I_m(\phi, H_2)$. The dual statement may dually be proved. $\square$

***Theorem* 2.15** — *Let S be an ordered semigroup and let $H_1,\ H_2$ be the subsemigroups of S. Let $A \in I_m(H_1, \phi)$ $(A \subseteq H_1)$ and $B \in I_m(\phi, H_2)$ $(B \subseteq H_2)$. Then $(AcB]_H \in I_m(H_1, H_2)$ for each $c \in S$.*

PROOF : To show that $(AcB]_H \in I_m(H_1, H_2)$, take any $C \in I(H_1, H_2)$ such that $C \subseteq (AcB]_H$. Let $s \in C$. Then $s = acb$, for some $a \in A$, and $b \in B$. By Theorem 2.12, $(AcB]_H = ((H_1a]c(bH_2]]_H \subseteq ((H_1a]_H(c]_H(bH_2]_H]_H \subseteq (H_1acbH_2]_H \subseteq (H_1sH_2]_H \subseteq (H_1CH_2]_H \subseteq (C]_H \subseteq C$. Therefore $C = (AcB]_H$. Hence $(AcB]_H \in I_m(H_1, H_2)$.

***Theorem* 2.16** — *Let S be a commutative ordered semigroup and let $H_1,\ H_2$ be the subsemigroups of S such that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. Let $B \subseteq H$ and $A \in I(H_1, H_2)$-ideal of S. Then*

$$(A : B) = \{x \in S \mid bx \in A \textit{ for each } b \in B\}$$

is an (H_1, H_2)-ideal of S.

PROOF : Let $x \in (A : B), h_1 \in H_1$ and $h_2 \in H_2$. Then $bx \in A$ for each $b \in B$. Therefore $b(h_1x) = h_1(bx) \in A$. Thus $h_1x \in (A : B)$. Again, as $b(xh_2) = (bx)h_2 \in A$, $xh_2 \in (A : B)$.

Let $h \in H$ and $y \in (A : B)$ such that $h \le y$. Therefore $bh \le by$ for each $b \in B$. As H is subsemigroup of S, $bh \in H$ and $A \in I(H_1, H_2)$, we have $bh \in A$. Thus $h \in (A : B)$, as required.

***Theorem* 2.17** — *Let I be any left (resp. right) ideal of a left (resp. right) regular ordered semigroup S and $H_1, H_2 \subseteq S$ such that $I \subseteq H_1$(resp. $I \subseteq H_2$). Then any (H_1, H_2)-ideal of I is a left (resp. right) ideal of S.*

PROOF : Clearly I is an ordered subsemigroup of S. Take any $a \in I \subseteq S$. Then there exists $x \in S$ such that $a \le xaa \le x(xaa)a = (xxa)aa$. As $xxa \in SSI \subseteq I$, $a \le caa$ for some $c \in I$, which shows that I is a left regular ordered subsemigroup of S. Let $B \in I(H_1, H_2)$ such that $B \subseteq I$. For any $b \in B \subseteq I$, $s \in S$, we have $sb \in I$ (resp. $bs \in I$). As I is left regular, there exists $t \in I$ such that $sb \le stbb \in (SSI)B \subseteq IB \subseteq H_1B \subseteq B$ (as $I \subseteq H_1$). Now let $e \in B \subseteq I$, $S \ni f \le e \Rightarrow f \in I \subseteq H_1 \subseteq H_1 \cup H_2 \Rightarrow f \in B$. Hence B is a left ideal of S. The dual statement may dually be proved.

Example 4 : [14]. Let $S = \{a, b, c, d, e\}$ be an ordered semigroup defined by the multiplication and the order as given below:

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}$. Then S is a left-regular ordered semigroup. Let $I = \{a, b, d\}$. Then I is a left ideal of S. Let $H_1 = \{a, b, c, d\}$ and $H_2 = \{b, d\}$. Then $I \subseteq H_1$. Take $A = \{a, d\}$. Clearly A is an (H_1, H_2)-ideal of I. So, by Theorem 2.17, A is a left ideal of S.

The following theorem now easily follows.

***Theorem* 2.18** — *Let I be any ideal of a regular ordered semigroup S and $H_1, H_2 \subseteq S$ such that $I \subseteq H_1 \cap H_2$. Then any (H_1, H_2)-ideal of I is an ideal of S.*

Definition 2.19 — Let S be an ordered semigroup and $H_1,\ H_2$ be two subsets of S. Let $a \in S$. Then the intersection of all (H_1, H_2)-ideals of S containing the element a is called the principal (H_1, H_2)-ideal of S generated by the element a. It is denoted by $I_R(a)$. If $H_2 = \phi$, then the intersection of all (H_1, ϕ)-ideals of S containing the element a is called the principal (H_1, ϕ)-ideal of S generated by the element a and it is denoted by $H_1(a)$. Dually we can define the principal (ϕ, H_2)-ideal of S generated by the element a and it is denoted by $(a)H_2$. If $H_1 = H_2 = H$, then the notion of H-ideals and (H, H)-ideals coincide. The H-ideal generated by a is denoted by $(a)_H$.

Lemma 2.20 — Let S be an ordered semigroup, let $H_1,\ H_2$ be two subsemigroups of S and let $a \in S$. Then $I_R(a) = (a \cup H_1 a \cup a H_2 \cup H_1 a H_2]_H$, where $H = H_1 \cup H_2$.

PROOF : Now $H_1 I_R(a) = H_1(a \cup H_1 a \cup a H_2 \cup H_1 a H_2]_H \subseteq (H_1]_H (a \cup H_1 a \cup a H_2 \cup H_1 a H_2]_H \subseteq ((H_1)(a \cup H_1 a \cup a H_2 \cup H_1 a H_2))]_H \subseteq (H_1 a \cup H_1 a \cup H_1 a H_2 \cup H_1 a H_2]_H = (H_1 a \cup H_1 a H_2]_H \subseteq (a \cup H_1 a \cup a H_2 \cup H_1 a H_2]_H \subseteq I_R(a)$. Similarly $I_R(a) H_2 \subseteq I_R(a)$. Since for each subset A of S, $((A]]_H = (A]_H$, therefore $(I_R(a)]_H = I_R(a)$. Hence $I_R(a)$ is an (H_1, H_2)-ideal of S.

To prove that $I_R(a)$ is the smallest (H_1, H_2)-ideal of S containing the element a, take any (H_1, H_2)-ideal B of S containing a. Now $I_R(a) = (a \cup H_1 a \cup a H_2 \cup H_1 a H_2]_H \subseteq (B \cup H_1 B \cup B H_2 \cup H_1 B H_2]_H \subseteq (B]_H \subseteq B$. Hence $I_R(a)$ is the smallest (H_1, H_2)-ideal of S. $\square$

Corollary 2.21 — Let S be an ordered semigroup, let $H_1,\ H_2$ be two subsemigroups of S, and let $a \in S$. Then $H_1(a) = (a \cup H_1 a]_{H_1}$ and $(a)H_2 = (a \cup aH_2]_{H_2}$.

Let S be an ordered semigroup and H_1, H_2 be two subsemigroups of S. From Lemma 2.20 and Corollary 2.21, we have

$$\begin{aligned} H_1(a) &= \{t \in S \mid t \leq a \text{ or } t \leq h_1 a \text{ for some } h_1 \in H_1\}; \\ (a)H_2 &= \{t \in S \mid t \leq a \text{ or } t \leq a h_2 \text{ for some } h_2 \in H_2\}; \\ \text{and } \ I_R(a) &= \{t \in S \mid t \leq a \text{ or } t \leq h_1 a \text{ or } t \leq a h_2 \text{ or } t \leq h_1 a h_2 \\ &\qquad \text{for some } h_1 \in H_1, h_2 \in H_2\}. \end{aligned}$$

Define relations ${}_{H_1}\mathcal{J}_{\phi},\ {}_{\phi}\mathcal{J}_{H_2},\ {}_{H_1}\mathcal{J}_{H_2},\ {}_{H_1}\mathcal{H}_{H_2}$ and ${}_{H_1}\mathcal{D}_{H_2}$ as follows:

$$\begin{aligned} a\,{}_{H_1}\mathcal{J}_{\phi}\,b &\iff H_1(a) = H_1(b); \\ a\,{}_{\phi}\mathcal{J}_{H_2}\,b &\iff (a)H_2 = (b)H_2; \\ a\,{}_{H_1}\mathcal{J}_{H_2}\,b &\iff I_R(a) = I_R(b); \\ {}_{H_1}\mathcal{H}_{H_2} &\iff {}_{H_1}\mathcal{J}_{\phi} \cap {}_{\phi}\mathcal{J}_{H_2}; \\ {}_{H_1}\mathcal{D}_{H_2} &\iff {}_{H_1}\mathcal{J}_{\phi} \cdot {}_{\phi}\mathcal{J}_{H_2}. \end{aligned}$$

For simplicity, we denote the relations ${}_{H_1}\mathcal{J}_{\phi}$ and ${}_{\phi}\mathcal{J}_{H_2}$ by ${}_{H_1}\mathcal{J}$ and $\mathcal{J}_{H_2}$ respectively. We denote the classes of the relations ${}_{H_1}\mathcal{J}, \mathcal{J}_{H_2}, {}_{H_1}\mathcal{J}_{H_2}, {}_{H_1}\mathcal{H}_{H_2}$ and ${}_{H_1}\mathcal{D}_{H_2}$ containing an element a by ${}_{H_1}J^a, J^a_{H_2}, {}_{H_1}J^a_{H_2}, {}_{H_1}H^a_{H_2}$ and ${}_{H_1}D^a_{H_2}$ respectively.

Proposition 2.22 — Let S be an ordered semigroup and let H_1, H_2 be two subsemigroups of S. Then ${}_{H_1}\mathcal{J}$ and $\mathcal{J}_{H_2}$ are right and left congruences on H_1 and H_2 respectively.

PROOF : Clearly ${}_{H_1}\mathcal{J}$ is an equivalence relation on H_1. Now we show that ${}_{H_1}\mathcal{J}$ is right compatible. Let $a, b, c \in H_1$ such that $a\,{}_{H_1}\mathcal{J}\,b$. For any $t \in H_1(ac)$, either $t \leq ac$ or $t \leq h_1 ac$ for some $h_1 \in H_1$. Since $a \in H_1(a) = H_1(b)$, either $a \leq b$ or $a \leq h_1' b$ for some $h_1' \in H_1$. If $a \leq b$, then either $t \leq bc$ or $t \leq h_1 bc$ i.e. $t \in H_1(bc)$. If $a \leq h_1' b$, then either $t \leq h_1' bc$ or $t \leq h_1 h_1' bc = (h_1 h_1')bc$. Therefore $H_1(ac) \subseteq H_1(bc)$. Similarly $H_1(bc) \subseteq H_1(ac)$. Hence ${}_{H_1}\mathcal{J}$ is a right congruence on H_1.

Dually we may show that $\mathcal{J}_{H_2}$ a left congruence on H_2. $\square$

***Theorem* 2.23** — *Let S be an ordered semigroup and let H_1, H_2 be two subsemigroups of S. Then ${}_{H_1}\mathcal{J} . \mathcal{J}_{H_2} = \mathcal{J}_{H_2} . {}_{H_1}\mathcal{J}$, where ${}_{H_1}\mathcal{J} . \mathcal{J}_{H_2}$ is defined as, ${}_{H_1}\mathcal{J} . \mathcal{J}_{H_2} = \{(a,b) \in H \times H \mid$ there exist $c \in H$ such that $(a,c) \in {}_{H_1}\mathcal{J}$ and $(c,b) \in \mathcal{J}_{H_2}\}$, here $H = H_1 \cap H_2 \neq \emptyset$.*

PROOF : Let $(a,b) \in {}_{H_1}\mathcal{J}.\mathcal{J}_{H_2}$. Then there exists $c \in H$ such that $(a,c) \in {}_{H_1}\mathcal{J}$ and $(c,b) \in \mathcal{J}_{H_2}$.

Case 1 : If $a = c$. Then $(a,b) \in \mathcal{J}_{H_2}$ and as ${}_{H_1}\mathcal{J}$ is reflexive, $(b,b) \in {}_{H_1}\mathcal{J}$. Therefore $(a,b) \in \mathcal{J}_{H_2} \cdot {}_{H_1}\mathcal{J}$.

Case 2 : If $b = c$. Then $(a,a) \in \mathcal{J}_{H_2}$ and $(a,b) \in {}_{H_1}\mathcal{J}$. Therefore $(a,b) \in \mathcal{J}_{H_2} \cdot {}_{H_1}\mathcal{J}$.

Case 3 : If $a \neq b$ and $b \neq c$. By supposition, we have either $a \leq c$ or $a \leq h_1 c$ and either $b \leq c$ or $b \leq ch_2$ for some $h_1 \in H_1, h_2 \in H_2$.

Case 3(i) : $a \leq c$ and $b \leq c$. Now $a \in (c]_{H_2} \subseteq (c \cup cH2]_{H_2}$ which implies that $(a,c) \in \mathcal{J}_{H_2}$ and $b \leq c \in (c \cup H_1 c]_{H_1}$ implies $(b,c) \in {}_{H_1}\mathcal{J} \Rightarrow (c,b) \in {}_{H_1}\mathcal{J}$(since ${}_{H_1}\mathcal{J}$ is an equivalence relation). Therefore $(a,b) \in \mathcal{J}_{H_2} \cdot {}_{H_1}\mathcal{J}$.

Case 3(ii) : $a \leq c$ and $b \leq ch_2$. Now $b \in (c \cup cH2]_{H_2} \Rightarrow (b,c) \in \mathcal{J}_{H_2} \Rightarrow (c,b) \in \mathcal{J}_{H_2}$ and $(a,c) \in \mathcal{J}_{H_2}$ which implies $(a,b) \in \mathcal{J}_{H_2}$. By reflexivity, $(b,b) \in {}_{H_1}\mathcal{J}$. Therefore $(a,b) \in \mathcal{J}_{H_2} \cdot {}_{H_1}\mathcal{J}$.

Case 3(iii) : $a \leq h_1 c$ and $b \leq c$. The proof goes on the lines similar to the proof of Case 3(ii).

Case 3(iv) : $a \leq h_1 c$ and $b \leq ch_2$. As $(c,b) \in \mathcal{J}_{H_2}$ and $\mathcal{J}_{H_2}$ is a left congruence, we have $(h_1 c, h_1 b) \in \mathcal{J}_{H_2}$. Thus either $h_1 c \leq h_1 b$ or $h_1 c \leq h_1 b h_2'$ for some $h_2 \in H_2$. If $h_1 c \leq h_1 b$, then $a \leq h_1 c \leq h_1 b \leq h_1 c h_2 \Rightarrow a \leq h_1 c h_2$. If $h_1 c \leq h_1 b h_2'$, then $a \leq h_1 b h_2' \leq h_1 c h_2 h_2'$. Therefore $(a, h_1 c h_2) \in \mathcal{J}_{H_2}$. On the other hand as $(a,c) \in {}_{H_1}\mathcal{J}$ and ${}_{H_1}\mathcal{J}$ is a right congruence, we have $(ah_2, ch_2) \in {}_{H_1}\mathcal{J}$. Thus either $ch_2 \leq ah_2$ or $ch_2 \leq h_1' a h_2$ for some $h_1' \in H_1$. If $ch_2 \leq ah_2$, then $b \leq ch_2 \leq ah_2 \leq h_1 c h_2$. If $ch_2 \leq h_1' a h_2$, then $b \leq h_1' a h_2 \leq h_1' h_1 c h_2$ which implies $(b, h_1 c h_2) \in {}_{H_1}\mathcal{J} \Rightarrow (h_1 c h_2, b) \in {}_{H_1}\mathcal{J}$. Therefore $(a,b) \in \mathcal{J}_{H_2} \cdot {}_{H_1}\mathcal{J}$. Hence ${}_{H_1}\mathcal{J}.\mathcal{J}_{H_2} \subseteq \mathcal{J}_{H_2} \cdot {}_{H_1}\mathcal{J}$. Similarly we may prove that $\mathcal{J}_{H_2 \cdot H_1} \subseteq {}_{H_1}\mathcal{J}.\mathcal{J}_{H_2}$. Therefore ${}_{H_1}\mathcal{J}.\mathcal{J}_{H_2} = \mathcal{J}_{H_2 \cdot H_1}$, as required.

3. RELATIVELY PRIME AND WEAKLY PRIME IDEALS

Now we define relatively prime and relatively weakly prime ideals in ordered semigroups.

Definition 3.1 — Let S be an ordered semigroup and let H, T be any non-empty subsets of S. Then T is said to be an H-prime ideal of S if

(1) T is an H-ideal of S; and

(2) For any $A, B \subseteq H$ such that $AB \subseteq T$ implies either $A \subseteq T$ or $B \subseteq T$.

In example 1, if we choose $T = \{a,b\}, H = \{a,d\}$ and $A = \{a\}, B = \{d\} \subseteq H$, then it is easy to check that T is an H-prime ideal of S.

Definition 3.2 — Let S be an ordered semigroup and let $H_1,\ H_2$ be any non-empty subsets of S. Then a non-empty subset T of S is said to be an (H_1, H_2)-prime ideal of S if

(1) T is an (H_1, H_2)-ideal of S; and

(2) For any $A, B \subseteq H_1 \cup H_2$ such that $AB \subseteq T$ implies either $A \subseteq T$ or $B \subseteq T$.

In the sequel, the set of all (H_1, H_2)-prime ideals of S will be denoted by $I_p(H_1, H_2)$.

Definition 3.3 — Let S be an ordered semigroup and let $H,\ T$ be any non-empty subsets of S. Then T is said to be an H-weakly prime ideal of S if

(1) T is an H-ideal of S, and

(2) For all H-ideals $A, B \subseteq H$ such that $AB \subseteq T$ implies either $A \subseteq T$ or $B \subseteq T$.

In example 1, if we choose $T = \{a, b, c\}, H = \{a, d\}$, then T is an H-weakly prime ideal of S as for $A = \{a, b\}$ and $B = \{a, b, d\}$ are the H-ideals of S such that $AB \subseteq T$ implies $A \subseteq T$.

Definition 3.4 — Let S be an ordered semigroup and let $H_1,\ H_2$ be any non-empty subsets of S. Then a non-empty subset T of S is said to be an (H_1, H_2)-weakly prime ideal of S if

(1) T is an (H_1, H_2)-ideal of S, and

(2) For all (H_1, H_2)-ideals $A, B \subseteq H_1 \cup H_2$ such that $AB \subseteq T$ implies either $A \subseteq T$ or $B \subseteq T$.

Definition 3.5 — Let S be an ordered semigroup and let $H,\ T$ be any non-empty subsets of S. Then T is said to be an H-semiprime ideal of S if

(1) T is an H-ideal of S.

(2) $A \subseteq H$ such that $A^2 \subseteq T$ implies $A \subseteq T$.

Definition 3.6 — Let S be an ordered semigroup and let $H_1,\ H_2$ be any non-empty subsets of S. A non-empty subset T of S is said to be an (H_1, H_2)-semiprime ideal of S if

(1) $T \in I(H_1, H_2)$; and

(2) $A \subseteq H_1 \cup H_2$ such that $A^2 \subseteq T$ implies $A \subseteq T$.

Proposition 3.7 — Let S be an ordered semigroup and let H be a subsemigroup of S. An H-ideal of S is H-prime if and only if it is both H-semiprime and H-weakly prime. In commutative ordered semigroups, H-prime and H-weakly prime ideals coincide.

PROOF : Let S be an ordered semigroup and let H and T be a subsemigroup and an H-ideal of S respectively. If T is H-prime, then clearly T is an H-semiprime and an H-weakly prime.

Conversely suppose T is an H-semiprime and an H-weakly prime. Suppose $a, b \in H$ such that $ab \in T$. Then

$$(bHa]_H^2 = (bHa]_H(bHa]_H \subseteq (bHabHa]_H \subseteq (H(ab)H]_H \subseteq (HTH]_H \subseteq (T]_H = T.$$

Since T is H-semiprime, $(bHa]_H \subseteq T$. Again

$$(HbH]_H(HaH]_H \subseteq (HbHHaH]_H \subseteq (H(bHa)H]_H \subseteq (H(bHa]_H H]_H \subseteq (HTH]_H \subseteq T.$$

Since $(HbH]_H, (HaH]_H \in I(H, H)$ and T is H-weakly prime, either $(HbH]_H \subseteq T$ or $(HaH]_H \subseteq T$. Suppose $(HaH]_H \subseteq T$. Then

$$\begin{aligned}((a)_H)^3 &= (a \cup Ha \cup aH \cup HaH]_H^3 \\ &\subseteq ((a \cup Ha \cup aH \cup HaH)^2]_H(a \cup Ha \cup aH \cup HaH]_H \\ &\subseteq (Ha \cup HaH]_H(a \cup Ha \cup aH \cup HaH]_H \\ &\subseteq ((Ha \cup HaH)(a \cup Ha \cup aH \cup HaH)]_H \\ &\subseteq (HaH]_H \subseteq T.\end{aligned}$$

Hence, we have

$$(((a)_H)^2](a)_H = (((a)_H)^2]((a)_H] \subseteq (((a)_H)^3]_H \subseteq (T]_H = T.$$

Since T is H-weakly prime and $(((a)_H)^2]_H$ is an H-ideal, then $(((a)_H)^2]_H \subseteq T$ or $(a)_H \subseteq T$. If $(a)_H \subseteq T$, then $a \in (a)_H \subseteq T$. Let $(((a)_H)^2]_H \subseteq T$. Then $((a)_H)^2 \subseteq T$. Since T is H-semiprime, $(a)_H \subseteq T$ and $a \in T$. Simlarly we may prove that if $(HbH]_H \subseteq T$, then $b \in T$. Now suppose that S is commutative and T is H-weakly prime. Let $a, b \in H$ such that $ab \in T$. Then

$$\begin{aligned}(a)_H(b)_H &= (a \cup Ha \cup aH \cup HaH]_H(b \cup Hb \cup bH \cup HbH]_H \\ &= ((a \cup Ha \cup aH \cup HaH)(b \cup Hb \cup bH \cup HbH)]_H \\ &= (ab \cup Hab]_H \subseteq_H (ab)_H \subseteq T.\end{aligned}$$

Since T is H-weakly prime, $a \in (a)_H \subseteq T$ or $b \in (b)_H \subseteq T$, as required. $\square$

Lemma 3.8 — Let S be an ordered semigroup and let H be a subsemigroup of S. Then the following are equivalent:

(1) Each relative H-ideal $A(A \subseteq H)$ of S is idempotent;

(2) $A \cap B = (AB]_H$ for all H-ideals of $A, B \subseteq H$;

(3) $(A)_H \cap (B)_H = ((A)_H(B)_H]_H$ for all $A, B \subseteq H$;

(4) $(A)_H = ((A)_H(A)_H]_H$ for all $A \subseteq H$;

(5) $A \subseteq (HAHAH]_H$ for all $A \subseteq H$.

PROOF : $(1) \Rightarrow (2)$. Let A, B be H-ideals of H. Then $(AB]_H \subseteq (AH]_H \subseteq (A]_H = A$ and $(AB]_H \subseteq (HB]_H \subseteq (B]_H = B$. Therefore $A \cap B \supseteq (AB]_H$. As $A \cap B$ is an H-ideal of S, by assumption, $A \cap B = ((A \cap B)(A \cap B)]_H \subseteq (AB]_H$. Hence $A \cap B = (AB]_H$.

$(2) \Rightarrow (3)$. Take any $A, B \subseteq H$. Then $(A)_H, (B)_H$ are H-ideals of S. So, by assumption $(A)_H \cap (B)_H = ((A)_H(B)_H]_H$.

$(3) \Rightarrow (4)$. As for every $A \subseteq H$, $(A)_H$ is an H-ideal of S, by assumption, we have $(A)_H = ((A)_H(A)_H]_H$.

$(4) \Rightarrow (5)$. Let $(A)_H = ((A)_H(A)_H]_H$ for all $A \subseteq H$. Therefore
$(A)_H = (((A)_H(A)_H](A)_H] = (((A)_H(A)_H]((A)_H]]_H = ((A)^3_H]_H = ((A)^4_H]_H = ((A)^5_H]_H$. Now

$$
\begin{aligned}
(A)^2_H &= (A)_H(A)_H = (A \cup HA \cup AH \cup HAH]_H(A \cup HA \cup AH \cup HAH]_H \\
&\subseteq (HAH \cup HA]_H \\
(A)^3_H &= (A)^2_H(A)_H \subseteq (HAH \cup HA]_H(A \cup HA \cup AH \cup HAH]_H \subseteq (HAH]_H \\
(A)^4_H &= (A)^3_H(A)_H \subseteq (HAH]_H(A \cup HA \cup AH \cup HAH]_H \subseteq (HAHA \cup HAHAH]_H \\
(A)^5_H &= (A)^4_H(A)_H \subseteq (HAHA \cup HAHAH]_H(A \cup HA \cup AH \cup HAH]_H \\
&\subseteq (HAHAH]_H.
\end{aligned}
$$

Therefore $(A)_H = ((A)^5_H]_H \subseteq ((HAHAH]]_H = (HAHAH]_H$. As $A \subseteq (A)_H$, $A \subseteq (HAHAH]_H$.

$(5) \Rightarrow (1)$. Let $A \subseteq (HAHAH]_H$ for all $A \subseteq H$. Let $B \subseteq H$ be an H-ideal of S. Now $(B^2] \subseteq (BH] \subseteq (B] \subseteq B$. Again by hypothesis, $B \subseteq (HBHBH]_H \subseteq (BB] = (B^2]$. Thus $B = (B^2]$. Hence every relative H-ideal of S is idempotent.

For an ordered semigroup S and $(\phi \neq)H_1,\ H_2 \subseteq S$, we say that any (H_1, H_2)-ideals $A,\ B$ of S form a chain if either $A \subseteq B$ or $B \subseteq A$ with respect to the inclusion relation " $\subseteq$ " defined on the set of all (H_1, H_2)-ideals of S.

***Theorem* 3.9** — *Let S be an ordered semigroup and let H be a subsemigroup of S. Then H-ideals of S contained in H are H-weakly prime if and only if they form a chain and one of the five equivalent conditions of Lemma 3.8 holds in S.*

PROOF : Let A, B be H-ideals of S such that $A, B \subseteq H$. Since $(AB]_H$ is an H-ideal of S and $AB \subseteq (AB]_H$, we have

$$A \subseteq (AB]_H \subseteq (HB]_H \subseteq (B]_H = B \ \textit{ or } \ B \subseteq (AB]_H \subseteq (AH]_H \subseteq (A]_H = A.$$

As $A^2 \subseteq (A^2]_H$ and $(A^2]_H$ is an H-ideal of S (cf.6 of Lemma 2.3), we have $A \subseteq (A^2]_H$. If $t \in (A^2]_H$, then $t \leq ab$ for some $a, b \in A$. As $ab \in AB \subseteq AH \subseteq A$ and $t \leq ab$, we have $t \in A$.

Conversely assume that A, B, T be H-ideals of S such that $A, B \subseteq H$ and $AB \subseteq T$. By hypothesis, we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$, then by (cf.2 of Lemma 3.8), $A = A \cap B = (AB]_H \subseteq (T]_H = T$. If $B \subseteq A$, then $B = A \cap B = (AB]_H \subseteq T$. □

Definition 3.10 — Let S be an ordered semigroup and let H be any non-empty subset of S. Then S is called H-intra-regular if

$$\forall\ h \in H\ \exists\ h_1, h_2 \in H \text{ such that } h \leq h_1 h^2 h_2.$$

i.e. $h \in (Hh^2H]_H\ \forall\ \ h \in H$ or $A \subseteq (HA^2H]_H\ \ \forall\ \ A \subseteq H$.

***Theorem* 3.11** — *Let S be an ordered semigroup and let H be any subsemigroup of S. Then H-ideals of S are H-prime if and only if they form a chain and S is H-intra-regular.*

PROOF : $\Rightarrow$ Suppose that H-ideals of S are H-prime. By Proposition 3.7, they are H-weakly prime and H-semiprime. So, by Theorem 3.9, they form a chain. Since $h^4 \in (Hh^2H]_H$ and $(Hh^2H]_H$ is an H-ideal of S (cf.7 of Lemma 2.3), we have $h^2 \in (Hh^2H]_H$. Thus $h \in (Hh^2H]_H$.

$\Leftarrow$ Since S is H-intra-regular, H-ideals of S are H-semiprime. To show that H-ideals are H-prime, take any H-ideal T of S and $h \in H$ such that $h^2 \in T$. Then $h \in (Hh^2H]_H \subseteq (HTH]_H \subseteq (T]_H = T$.

Now we claim:

(1) $(x)_H = (HxH]_H,\ \forall\ \ x \in H$. As $x^4 \in (HxH]_H$ implies $x^2 \in (HxH]_H$ and, so, $x \in (HxH]_H$. Thus $(x)_H \subseteq (HxH]_H$. Also, $(HxH]_H \subseteq (x \cup Hx \cup xH \cup HxH]_H = (x)_H$. Therefore $(x)_H = (HxH]_H\ \forall\ x \in H$

(2) $(xy)_H = (x)_H(y)_H) \ \forall\, x, y \in H$. For this

$$xy \in (x)_H H \subseteq \ (x)_H \implies (xy)_H \subseteq \ (x)_H,$$

$$xy \in \ H(x)_H \subseteq \ (y)_H \implies (xy)_H \subseteq \ (y)_H$$

i.e. $(xy)_H \subseteq (x)_H \cap (y)_H$. Take $h \in (xy)_H \subseteq (x)_H \cap (y)_H$. Thus $h \in (HxH]_H$ and $h \in (HyH]_H$. This implies $h \leq axb$ and $h \leq cyd$ for some $a, b, c, d \in H$. Thus $h^2 \leq cydaxb$. On the other hand, $ydax \in \ _H(xy)_H$. For this

$$(ydax)^2 = ydaxydax \in HxyH \subseteq \ (HxyH]_H = (xy)_H.$$

As $(xy)_H$ is H-semiprime, $ydax \in (xy)_H$. Hence $cydaxb \in (xy)_H$. So $h^2 \in (xy)_H$. Therefore $h \in (xy)_H$.

Again, let T be an H-ideal of S and $a, b \in H$ such that $ab \in T$. By chain condition, we have either $(a)_H \subseteq (b)_H$ or $(b)_H \subseteq (a)_H$. If $(a)_H \subseteq (b)_H$, then $a \in (a)_H = (a)_H \cap (b)_H = (ab)_H \subseteq T$ implies $a \in T$. If $(b)_H \subseteq (a)_H$, then $b \in (b)_H = (a)_H \cap (b)_H = (ab)_H \subseteq T$ implies $b \in T$. □

Definition 3.12 — Let S be an ordered semigroup and let $H_1, \ H_2$ be non-empty subsets of S. Then S is said to be (H_1, H_2)-simple if S has no proper (H_1, H_2)-ideal of S.

Lemma 3.13 — Let S be an ordered semigroup and let $H_1, \ H_2$ be the subsemigroups of S. Then the following are equivalent:

(1) S is (H_1, H_2)-simple;

(2) $(H_1 a H_2]_H = S$ for each $a \in S$;

(3) $I_R(a) = S$ for each $a \in S$.

PROOF : (1) ⇒ (2). Let S be (H_1, H_2)-simple and $a \in S$. Since $(H_1 a H_2]_H$ is an (H_1, H_2)-ideal of S, $(H_1 a H_2]_H = S$.

(2) ⇒ (3). Suppose for each $a \in S$, $(H_1 a H_2]_H = S$. Now $S = (H_1 a H_2]_H \subseteq (a \cup H_1 a \cup aH_2 \cup H_1 a H_2]_H = I_R(a)$. Therefore $I_R(a) = S$.

(3) ⇒ (1). Assume that $I_R(a) = S$ for each $a \in S$. To show that S is (H_1, H_2)-simple, let A be any (H_1, H_2)-ideal of S. Take any $a_1 \in A$. Therefore $I_R(a_1) \subseteq A \subseteq S$. By hypothesis, $I_R(a_1) = S$. Therefore $A = S$. Hence S is (H_1, H_2)-simple. □

4. MAXIMAL AND MINIMAL RELATIVE IDEALS

Definition 4.1 — Let S be an ordered semigroup and let $H_1,\ H_2$ be non-empty subsets of S. An (H_1, H_2)-ideal T of S is said to be maximal (H_1, H_2)-ideal of S if for any $B \in I(H_1, H_2)$ such that $T \subseteq B \subseteq S$, either $T = B$ or $B = S$.

***Theorem* 4.2** — *Let S be an ordered semigroup and let $H_1,\ H_2$ be non-empty subsets of S. Then every proper (H_1, H_2)-ideal of S is minimal if and only if the intersection of any two distinct proper (H_1, H_2)-ideals is empty.*

PROOF : Let A and B be two distinct (H_1, H_2) ideals of S. If $A \cap B \neq \phi$, then $A \cap B$ is an (H_1, H_2)-ideal of S. Since $A \cap B \subseteq A$ and A is minimal, $A \cap B = A$. Also as $A \cap B \subseteq B$ and B is minimal, $A \cap B = B$. Therefore $A = B$, which is a contradiction. Hence $A \cap B = \phi$.

Conversely assume that A be any proper (H_1, H_2)-ideal of S and B be any other proper (H_1, H_2)-ideal of S such that $B \subseteq A$. If $B \neq A$, then $B = A \cap B = \phi$, which is a contradiction. Therefore $A = B$. Hence A is a minimal (H_1, H_2)-ideal of S.

***Theorem* 4.3** — *Let S be an ordered semigroup, let $H_1,\ H_2$ be the subsemigroups of S and let $A \in I(H_1, H_2)$. If $S \setminus A \subseteq (H_1 x H_2]_H$ for each $x \in S \setminus A$, then A is a maximal (H_1, H_2)-ideal of S $(H = H_1 \cup H_2)$.*

PROOF : Let $B \in I(H_1, H_2)$ such that $A \subseteq B$. Suppose $S \setminus A \subseteq (H_1 x H_2]_H$ for each $x \in S \setminus A$. Let $b \in B \setminus A \subseteq S \setminus A$. By hypothesis, $S \setminus A \subseteq (H_1 b H_2]_H \subseteq (H_1 B H_2]_H \subseteq (B]_H \subseteq B$, implies $S \setminus A \subseteq B$. Therefore $S = A \cup (S \setminus A) \subseteq A \cup B = B$. Hence A is a maximal (H_1, H_2)-ideal of S. □

Proposition 4.4 — Let S be an ordered semigroup, let $H_1,\ H_2$ be the subsemigroups of S and let A be any proper (H_1, H_2)-ideal of S. Then A is a maximal ideal of S if and only if for every $a \in S \setminus A$, $I_R(A \cup \{a\}) = S$.

PROOF : Let A be any proper (H_1, H_2)-ideal of S such that A is maximal. Let $a \in S \setminus A$. Since $A \subset I_R(A \cup \{a\})$ and $I_R(A \cup \{a\})$ is an (H_1, H_2)-ideal of S, by maximality of (H_1, H_2)-ideal A of $S,\ I_R(A \cup \{a\}) = S$.

Conversely assume that B is any (H_1, H_2)-ideal of S such that $A \subset B$. Let $b \in B, b \notin A$. By hypothesis, $I_R(A \cup \{b\}) = S$. Since $(A \cup \{b\}) \subseteq B$. Therefore

$$\begin{aligned} S &= I_R(A \cup \{b\}) \\ &= ((A \cup \{b\}) \cup H_1(A \cup \{b\}) \cup (A \cup \{b\})H_2 \cup H_1(A \cup \{b\})H_2]_H \end{aligned}$$

$$\subseteq ((B \cup H_1B \cup BH_2 \cup H_1BH_2]_H \subseteq (B]_H \subseteq B.$$

Hence A is a maximal (H_1, H_2)-ideal of S. □

***Theorem* 4.5** — *Let S be an ordered semigroup and let $H_1,\ H_2$ be non-empty subsets of S. Then every proper (H_1, H_2)-ideal of S is maximal if and only if the union of any two distinct proper (H_1, H_2)-ideals coincides S.*

PROOF : Let A and B be two distinct (H_1, H_2)-ideals of S. Clearly $A \cup B$ is an (H_1, H_2)-ideal of S and $A \subset A \cup B \subseteq S$. Since A is maximal and A is properly contained in $A \cup B$, we have $A \cup B = S$.

Conversely assume that A be any proper (H_1, H_2)-ideal of S and B be any other proper (H_1, H_2)-ideal of S such that $A \subset B$. Then, clearly $B = A \cup B = S$. Hence A is a maximal (H_1, H_2)-ideal of S.

5. PROBLEMS

(1) Let S be an ordered semigroup and let H_1, H_2 be the subsemigroups of S. An (H_1, H_2)-ideal of S is (H_1, H_2)-prime if and only if it is both (H_1, H_2)-semiprime and (H_1, H_2)-weakly prime?

(2) Let S be an ordered semigroup and let (H_1, H_2) be any subsemigroup of S. Then (H_1, H_2)-ideals of S are (H_1, H_2)-prime if and only if they form a chain and S is (H_1, H_2)-intra-regular?

6. CONCLUSION

In ordered semigroups ideals play an important role to discuss the nature of the structure of ordered semigroups. Nowadays, the ideal theory has been extensively studied by several authors. In ordered semigroups, different types of ideals such as bi-ideals, quasi-ideals, and cover ideals had been studied. These notions had been widely studied by several authors in different algebraic structures (see [9-11]). In this paper, we enhance the understanding of ordered semigroups by introducing the concept of relative ideals in ordered semigroups. We also introduce the concept of relatively prime and relatively weakly prime ideals and study some properties of these notions. Moreover, the concept of maximal and minimal relative ideals are introduced and some of their properties are studied.

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