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REGULARITY PROPERTIES OF CERTAIN CROSSED PRODUCT C^*-ALGEBRAS

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We show that the following properties of C^*-algebras in a class Ω are inherited by simple unital C^*-algebras in class $\mathrm{TA}\Omega$: (1) m-comparison of positive elements, (2) strong tracial m-comparison of positive elements, and (3) tracial nuclear dimension at most m. As an application, every unital simple C^*-algebra with tracial topological rank at most k has tracial nuclear dimension at most k. Also as an application, let A be an infinite-dimensional simple unital exact C^*-algebra such that A has one of the above-listed properties. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C^*-algebra $\mathrm{C}^*(G, A, \alpha)$ also has the property under consider also.

Key words : C^*-algebras; tracial approximation; Cuntz semgroup.

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1. INTRODUCTION

The Elliott program for the classification of amenable C^*-algebras might be said to have begun with the K-theoretical classification of AF algebras in [7]. Since then, many classes of C^*-algebras have been classified by the Elliott invariant. Among them, one important class is the class of simple unital AH algebras without dimension growth (in the real rank zero case see [10], and in the general case see [6]). To axiomatize Elliott-Gong's decomposition theorem for AH algebras of real rank zero (classified by Elliott-Gong in [10]) and Gong's decomposition theorem [17] for simple AH algebras (classified by Elliott-Gong-Li in [6]), Huaxin Lin introduced the concepts of TAF and TAI [23, 24].

Instead of assuming inductive limit structure, he started with a certain abstract approximation property, and showed that C*-algebras with this abstract approximation property and certain additional properties are AH algebras without dimension growth. More precisely, Lin introduced the class of tracially approximate interval algebras (also called C*-algebras of tracial topological rank one). This axiomatization has proved to be very important in the classification of simple amenable C*-algebras. For example, in the program of classification of simple amenable C*-algebras, it led to the classification of unital simple separable C*-algebras with finite nuclear dimension in the UCT class (see [12, 18, 43]). The isomorphism theorem was established first for those separable amenable C*-algebras with generalized tracial rank at most one (see [18]). Simple C*-algebras with tracial topological rank at most one have good regularity properties. There are three regularity properties of particular interest: tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$, also called $\mathcal{Z}$-stability; finite nuclear dimension; and strict comparison of positive elements. The latter can be reformulated as an algebraic property of the Cuntz semigroup, called almost unperforation. Toms and Winter have conjectured that these three fundamental properties are equivalent for all separable, simple, unital, amenable C*-algebras.

Winter and Zacharias introduce the nuclear dimension for C*-algebras in [47], Winter show that finite nuclear dimension implies $\mathcal{Z}$-stability for separable simple unital C*-algebra in [45]. Recently Winter announce that $\mathcal{Z}$-stability imply finite nuclear dimension in [46] for separable simple unital amenable C*-algebras.

Hirshberg and Orovitz introduce the tracially $\mathcal{Z}$-absorbing in [20], they show that tracially $\mathcal{Z}$-absorbing is equivalence $\mathcal{Z}$-stability for separable simple amenable unital C*-algebra in [20].

Inspired by Hirshberg and Orovitz's tracial $\mathcal{Z}$-absorbing, Fu introduced the finite tracial nuclear dimension in his doctoral dissertation in [16], and he show that finite tracial nuclear dimension implies tracially $\mathcal{Z}$-absorbing for separable, simple C*-algebra. By Winter's, Hirshberg and Orovitz's results, we know that finite tracial nuclear dimension implies finite nuclear dimension for separable simple amenable unital C*-algebra.

The properties of m-comparison of positive elements property and strong tracial m-comparison of positive elements property were introduced by Winter in [45]. Those properties were used to prove that finite nuclear dimension implies tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$.

Inspired by Lin's tracial approximation by interval algebras in [24], Elliott and Niu in [11] considered the natural notion of tracial approximation by other classes of C*-algebras. Let Ω be a class of unital C*-algebras. Then the class of separable C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by $\mathrm{TA}\Omega$, is defined as follows. A simple unital C*-algebra A is said to belong to the class $\mathrm{TA}\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any element $a \geq 0$, there is a projection $p \in A$ and a C*-subalgebra B of A with $1_B = p$ and $B \in \Omega$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$, and

(2) $pxp \in_\varepsilon B$ for all $x \in F$, and

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

The question of which properties pass from a class Ω to the class $\mathrm{TA}\Omega$ is interesting and sometimes important. In fact, the property of being of stable rank one, and the property that the strict order on projections is determined by traces are important in the classification theorem.

In this paper, we show that the following properties of C*-algebras in the class Ω are inherited by simple unital C*-algebras in class $\mathrm{TA}\Omega$: (1) m-comparison of positive elements, (2) strong tracial m-comparison of positive elements, and (3) tracial nuclear dimension at most m. As an application, every unital simple C*-algebra with tracial rank at most k has tracial nuclear dimension at most k.

The Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by Connes in [3]. It was adapted by Herman and Ocneanu for UHF-algebras in [19]. Rørdam [38] and Kishimoto [22] considered the Rokhlin property in a much more general C*-algebra context. More recently, Phillips and Osaka studied finite group and integer group $\mathbb{Z}$ actions on certain simple C*-algebras which satisfy a certain type of Rokhlin property in [30, 33].

In [15], the following result was obtained: Let Ω be a class of unital C*-algebras such that Ω is closed under passing to unital hereditary C*-subalgebras and tensoring by matrix algebras. Let $A \in \mathrm{TA}\Omega$ be an infinite dimensional simple unital C*-algebra. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ belongs to $\mathrm{TA}\Omega$.

Using the results mentioned, we get the following theorems: Let A be an infinite dimensional separable simple unital exact C*-algebra such that A has the m-comparison of positive elements and A has SP property (strong tracial m-comparison of positive elements, tracial nuclear dimension at most m). Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ has the m-comparison of positive elements (strong tracial m-comparison of positive elements, tracial nuclear dimension at most m).

2. Preliminaries and Definitions

Let a and b be positive elements of a C*-algebra A. We write $[a] \leq [b]$ if there is a partial isometry $v \in A^{**}$ such that, for every $0 \leq c \in \mathrm{Her}(a)$, $cv \in A$, $vv^* = P_a$, where P_a is the range projection of a in A^{**}, and $v^*cv \in \mathrm{Her}(b)$ ($[a] \leq [b]$ imply that a is Cuntz subequivalent to b, i.e. $a \lesssim b$). We write $[a] = [b]$ if $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \leq [b]$ if in addition there are n mutually orthogonal positive elements $b_1, b_2, \cdots, b_n \in \mathrm{Her}(b)$ such that $[a] \leq [b_i], i = 1, 2, \cdots, n$ (cf. Definition 1.1 in [31], Definition 3.2 in [29], or Definition 3.5.2 in [25]).

Recall that a C^*-algebra A has SP property, if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection.

Let A be a C*-algebra, and let $\mathrm{M}_n(A)$ denote the C*-algebra of $n \times n$ matrices with entries elements of A. Let $\mathrm{M}_\infty(A)$ denote the algebraic inductive limit of the sequence $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is given by $a \mapsto \mathrm{diag}(a, 0)$. Let $\mathrm{M}_\infty(A)_+$ (resp. $\mathrm{M}_n(A)_+$) denote the positive elements of $\mathrm{M}_\infty(A)$ (resp. $\mathrm{M}_n(A)$). For positive elements a and b of $\mathrm{M}_\infty(A)$, write $a \oplus b$ to denote the element $\mathrm{diag}(a, b)$, which is also positive of $\mathrm{M}_\infty(A)$. Given $a, b \in \mathrm{M}_\infty(A)_+$, we say that a is Cuntz subequivalent to b (written $a \lesssim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \lesssim b$ and $b \lesssim a$. We write $\langle a \rangle$ for the equivalence class of a.

The object $\mathrm{Cu}(A) := \mathrm{W}(A \otimes K)$ will be called the Cuntz semigroup of A, where $\mathrm{W}(A) := \mathrm{M}_\infty(A)_+/\sim$. Observe that $\mathrm{Cu}(A)$ becomes an ordered semigroup when equipped with the addition operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

and the order relation

$$\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \lesssim b.$$

Let $0 < \sigma_1 < \sigma_2 \leq 1$ be two positive numbers. Define

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1 & \text{if } t \geq \sigma_2 \\ \frac{t-\sigma_1}{\sigma_2-\sigma_1} & \text{if } \sigma_1 \leq t \leq \sigma_2 \\ 0 & \text{if } 0 < t \leq \sigma_1. \end{cases}$$

We denote by A_+ the positive cone of A and by $\mathrm{T}(A)$ the set of tracial states of A. We define the dimension function (or rank function) d_τ associated with $\tau \in \mathrm{T}(A)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$, or $d_\tau(a) = \lim_{n\to\infty} \tau(f_\varepsilon(a))$ for $a \in \mathrm{M}_k(A)_+$, where τ is regarded as an unnormalized trace on $\mathrm{M}_k(A)$ and $f_\varepsilon(a) = f_\varepsilon^{2\varepsilon}(a)$. We say that a separable exact C*-algebra A has the strict comparison if for $a,\ b \in \mathrm{M}_k(A)_+$, with $d_\tau(a) < d_\tau(b)$ for any $\tau \in \mathrm{T}(A)$, then we have $a \lesssim b$.

The definition of the m-comparison of positive elements and strong tracial m-comparison of positive elements were introduced by Winter in [45], which used to prove that certain locally finite nuclear dimension implies tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$.

Definition 2.1 — [45]. Let A be a separable, simple, unital C*-algebra and $m \in \mathbb{N}$. We say A has m-comparison of positive elements, if for any nonzero positive contractions $a,\ b_0,\ b_1,\ \cdots,\ b_m \in \mathrm{M}_\infty(A)$, we have $a \lesssim b_0 \oplus b_1 \oplus \cdots \oplus b_m$ whenever $d_\tau(a) < d_\tau(b_i)$ for any $\tau \in \mathrm{T}(A)$ and $i = 0,\ 1,\ \cdots,\ m$.

Definition 2.2 — [45]. Let A be a separable, simple, unital C*-algebra and $m \in \mathbb{N}$. A has strong tracial m-comparison of positive elements, if for any nonzero positive contractions $a,\ b \in \mathrm{M}_\infty(A)$ we have $a \lesssim b$ whenever $d_\tau(a) < \frac{\tau(b)}{m+1}$ for any $\tau \in \mathrm{T}(A)$.

Definition 2.3 — Let A and B be C*-algebras and $\varphi : A \to B$ a completely positive map. Then φ is said to be an order zero map if $\varphi(e)\varphi(f) = 0$, whenever $e, f \in A$ are positive elements such that $ef = 0$.

Winter and Zacharias introduce the nuclear dimension for C*-algebras.

Definition 2.4 — [47]. Let A be a C*-algebra, $m \in \mathbb{N}$. A complete positive compression $\varphi : F \to A$ is m-decomposable (where F is finite dimensional C*-algebra), if there is a decomposition $F = F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)}$ such that the restriction $\varphi^{(i)}$ of φ to $F^{(i)}$ is an order zero map, for each $i \in \{0, \cdots, m\}$, we say φ is m-decomposable with respect to $F = F^{(0)} \oplus F^{(1)} \oplus \cdots \oplus F^{(m)}$. A has nuclear dimension m, write $\dim_{\mathrm{nuc}}(A) = m$, if m is the least integer such that the following holds: For any finite subset $G \subseteq A$ and $\varepsilon > 0$, there is a finite dimension complete positive compression approximation (F, φ, ψ) for G to within ε (i.e., F is finite-dimensional $\psi : A \to F$ and $\varphi : F \to A$ are complete positive and $\|\varphi\psi(b) - b\| < \varepsilon$ for any $b \in G$) such that ψ is complete positive compression, and φ is m-decomposable with complete positive compression order zero components φ^i. If no such m exists, we write $\dim_{\mathrm{nuc}}(A) = \infty$.

Definition 2.5 — [20]. We say a unital C*-algebra A is tracially $\mathcal{Z}$-absorbing, if $A \neq \mathbb{C}$ and for any finite set $F \subseteq A$, $\varepsilon > 0$, non-zero positive element $a \in A$, and $n \in \mathbb{N}$, there is a completely positive order zero contraction $\psi : \mathrm{M}_n \to A$, such that the following properties hold:

(1) $1 - \psi(1) \precsim a$, and

(2) for any normalized element $x \in \mathrm{M}_n$ (i.e., with $\|x\| = 1$), and any $y \in F$ we have $\|\psi(x)y - y\psi(x)\| < \varepsilon$.

Definition 2.6 — [16]. A unital C^*-algebra A is said to have tracial nuclear dimension at most m, denote $\mathrm{Tdim}_{\mathrm{nuc}}(A) \leq m$, if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element a of A, there exist a C^*-subalgebra $B \subset A$ and a contractive completely positive linear map $\varphi : A \to A$ and a contractive completely positive linear map $\psi : A \to B$ with $\dim_{\mathrm{nuc}}(B) \leq m$ such that

(1) $\varphi(1) \lesssim a$, and

(2) $\|x - \varphi(x) - \psi(x)\| < \varepsilon$, for any $x \in F$.

Let A be a stably finite simple unital C^*-algebra. Recall that a positive element $a \in A$ is called purely positive if a is not Cuntz equivalent to a projection. This is equivalent to saying that 0 is an accumulation point of $\sigma(a)$ (Recall that $\sigma(a)$ denotes the spectrum of a).

Given a in $\mathrm{M}_\infty(A)_+$ and $\varepsilon > 0$, we denote by $(a - \varepsilon)_+$ the element of $\mathrm{C}^*(a)$ corresponding (via the functional calculus) to the function $f(t) = \max(0, t - \varepsilon), t \in \sigma(a)$. By the functional calculus, it follows in a straightforward manner that $((a - \varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+$.

***Theorem* 2.7** — *[1, 40]. Let A be a C^*-algebra.*

(1) *Let $a,\ b \in A_+$ and $\varepsilon > 0$, be such that $\|a - b\| < \varepsilon$. Then there is a contraction d in A with $(a - \varepsilon)_+ = dbd^*$.*

(2) *Let $a,\ p$ be positive elements in $\mathrm{M}_\infty(A)$ with p a projection. If $p \lesssim a$, then there is b in $\mathrm{M}_\infty(A)_+$ such that $p \oplus b \sim a$.*

(3) *The following conditions are equivalent: $(1)'$ $a \lesssim b$, $(2)'$ for any $\varepsilon > 0$, $(a - \varepsilon)_+ \lesssim b$, $(3)'$ for any $\varepsilon > 0$ there is $\delta > 0$, such that $(a - \varepsilon)_+ \lesssim (b - \delta)_+$.*

(4) *Let a be a purely positive element of A. Let $\delta > 0$, and let $f \in C_0(0, 1]$ be a non-negative function with $f = 0$ on $(\delta, 1)$, $f > 0$ on $(0, \delta)$, and $\|f\| = 1$. We have $f(a) \neq 0$ and $(a-\delta)_+ + f(a) \lesssim a$.*

Let Ω be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebras in Ω is denoted by $\mathrm{TA}\Omega$.

Definition 2.8 — [11, 24]. A simple unital C^*-algebra A is said to belong to the class $\mathrm{TA}\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \geq 0$, there exist a nonzero projection $p \in A$ and a C*-subalgebra B of A with $1_B = p$ and $B \in \Omega$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in_\varepsilon B$ for all $x \in F$, and

(3) $[1 - p] \leq [a]$.

Let $\mathfrak{I}^k$ denote the class of all unital C*-algebras which are unital hereditary subalgebras of C*-algebras of the form $C(X) \otimes F$, where X is a k-dimensional finite CW complex and F is finite dimension algebras. A is said to have tracial topological rank at most k if $A \in \mathrm{TA}\mathfrak{I}^k$.

Lemma 2.9 — [11]. If the class Ω is closed under tensoring with matrix algebras, or closed under taking unital hereditary C*-subalgebras, then $\mathrm{TA}\Omega$ is closed under passing to matrix algebras or unital hereditary C*-subalgebras.

Definition 2.10 — [33]. Let A be an infinite dimensional separable simple unital C*-algebra, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the tracial Rokhlin property if, for any finite set $F \subseteq A$, any $\varepsilon > 0$, and any nonzero positive element $b \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g,\ h \in G$,

(2) $\|e_g d - d e_g\| < \varepsilon$ for all $g \in G$ and all $d \in F$,

(3) with $e = \Sigma_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary C*-subalgebra of A generated by b, and

(4) $\|ebe\| \geq \|b\| - \varepsilon$.

The following lemma is lemma 5.3 in [26].

***Theorem* 2.11** — *Let A be a* C*-*algebra, $a \in A_+$ with $0 \leq a \leq 1$ and let $p \in A$ be a projection. Let $0 < \varepsilon < 1$. Then $f_\varepsilon(a) \lesssim f_{\varepsilon^2/2^9}(pap + (1-p)a(1-p))$.*

3. The Main Results

***Theorem* 3.1** — *Let Ω be a class of amenable unital stably finite C^*-algebras such that $\mathrm{Tdim}_{\mathrm{nuc}}(B) \leq m$ for any $B \in \Omega$. Then we have $\mathrm{Tdim}_{\mathrm{nuc}}(A) \leq m$ for any infinite dimension simple unital C^*-algebra, which A has* SP *property and $A \in \mathrm{TA}\Omega$ (Furthermore if all C^*-algebras $B \in \Omega$ have* SP *property, then A automatically has* SP *property).*

PROOF : We need to show that for any $\varepsilon > 0$, any finite subset $F = \{a_1, a_2, \cdots, a_n\}$ of A, any nonzero positive element a of A, there exist a contractive completely positive linear map $\varphi : A \to A$ and a contractive completely positive linear map $\psi : A \to D$ with $\dim_{\text{nuc}}(D) \le m$ such that

(1) $\varphi(1) \lesssim a$, and

(2) $\|x - \varphi(x) - \psi(x)\| < 10\varepsilon$, for any $x \in F$.

Since A has SP property. There exist nonzero projections $p_1, p_2 \in A$ such that $p_1 p_2 = 0$ and $[p_1 + p_2] \le a$.

For any $\varepsilon > 0$, since $A \in \mathrm{TA}\Omega$, there exist a sub-C*-subalgebra B of A and a nonzero projection $p \in A$ with $\mathrm{Tdim}_{nuc}(B) \le m$ and $1_B = p$, such that

$(1)'$ $\|p_2 p - p p_2\| < \varepsilon$, $\|a_i p - p a_i\| < \varepsilon$, for all $1 \le i \le n$,

$(2)'$ $p p_2 p \in_\varepsilon B$, $p a_i p \in_\varepsilon B$ for all $1 \le i \le n$, and

$(3)'$ $[1 - p] \le [p_1]$ and $2[1 - p] \le [p_2]$.

By $(1)'$ and $(2)'$, there exist positive elements $a_i' \in B$ and $a_i'' = (1-p)a_i(1-p) \in (1-p)A(1-p)$ and projections $p_2' \in B$ and $p_2'' \in (1 - p)A(1 - p)$ such that $\|p_2 - p_2' - p_2''\| < 2\varepsilon$, $\|p a_i p - a_i'\| < \varepsilon$.

We clear that p_2' is not zero, otherwise $2[1 - p] \le [p_2] = [p_2' + p_2''] = [p_2''] \le [1 - p]$, this contradicts the stable finiteness of A (since C*-algebra in Ω are stably finite).

We define $\varphi'' : A \to A$ by $\varphi''(a) = (1 - p)a(1 - p)$, then φ'' is a contractive completely positive linear map. Since B is a nuclear C*-algebra, by Theorem 2.3.13 in [25], there exist a contractive completely positive linear map $\psi'' : A \to B$ such that $\|\psi''(a_i') - a_i'\| < \varepsilon$ for all $1 \le i \le n$.

Since $\mathrm{Tdim}_{\text{nuc}}(B) \le m$, there exist a C*-subalgebra $D \subset B$ and a contractive completely positive linear map $\varphi' : B \to B$ and a contractive completely positive linear map $\psi' : B \to D$ with $\dim_{\text{nuc}}(D) \le m$ such that

$(1)''$ $\varphi'(p) \lesssim p_2'$, and

$(2)''$ $\|a_i - \varphi'(a_i) - \psi'(a_i)\| < \varepsilon$ for all $1 \le i \le n$.

Write $\varphi : A \to A$ by $\varphi(a) = \varphi''(a) + \varphi'(\psi''(pap))$, $\psi : A \to D$ by $\psi(a) = \psi'(\psi''(pap))$, then φ and ψ are contractive completely positive linear maps.

We have

(1) $\varphi(1) = \varphi''(1) + \varphi'(\psi''(p)) \lesssim 1 - p \oplus \varphi'(p) \lesssim p_1 \oplus p_2 \lesssim a$, and

(2) $\|a_i - \varphi(a_i) - \psi(a_i)\| \leq \|a_i - (1-p)a_i(1-p) - a_i'\| + \|a_i' - \varphi'(\psi''(pap)) - \psi'(\psi''(pap))\| < 2\varepsilon + \|a_i' - \varphi'(a_i') - \psi'(a_i')\| + \|\varphi'(a_i') - \varphi'(\psi''(pap)\| + \|\psi'(a_i') - \psi'(\psi''(pap)\| + \|\psi''(pap) - \psi''(a_i)\| + \|\psi''(a_i) - a_i\| < 2\varepsilon + 2\varepsilon + 2\varepsilon + \varepsilon + \varepsilon < 10\varepsilon$, for all $1 \leq i \leq n$. □

Corollary 3.2 — Every unital simple C*-algebra with tracial topological rank at most k has tracial nuclear dimension at most k.

PROOF : Since unital simple C*-algebra with tracial topological rank at most k has SP property, by Theorem 3.1, we can get this corollary. □

Recently Winter announce that $\mathcal{Z}$-stability imply that finite nuclear dimension in [46] for separable simple unital nuclear C*-algebras. By Winter's and Fu's result in his doctoral dissertation and Winter's, Hirshberg and Orovitz's results, we obtain the following corollary.

Corollary 3.3 — Every unital nuclear simple C*-algebra with tracial topological rank at most k has nuclear dimension at most k.

***Theorem* 3.4** — *Let Ω be a class of unital C^*-algebras which Ω is closed under passing to unital hereditary C^*-subalgebras and closed under passing to tensoring with matrix algebras. Let $A \in \mathrm{TA}\Omega$ be an infinite dimensional simple unital C^*-algebra. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C^*-algebra $\mathrm{C}^*(G, A, \alpha)$ belongs to $\mathrm{TA}\Omega$.*

Corollary 3.5 — Let A be an infinite dimensional separable nuclear unital simple C*-algebra which has tracial nuclear dimension at most m. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ has tracial nuclear dimension at most m.

PROOF : In Theorem 3.4, we take Ω be a class of C*-algebras with tracial nuclear dimensional at most m. Since $A \in \mathrm{TA}\Omega$, by Theorem 3.1, A has tracial nuclear dimensional at most m, by Theorem 3.4, $\mathrm{C}^*(G, A, \alpha) \in \mathrm{TA}\Omega$, then $\mathrm{C}^*(G, A, \alpha)$ has tracial nuclear dimension at most m, by Theorem 3.1.

***Theorem* 3.6** — *Let Ω be a class of stably finite unital exact C^*-algebras such that B has the m-comparison of positive elements for any $B \in \Omega$. Then A has the m-comparison of positive elements for any separable simple unital exact C^*-algebra, which A has SP property and $A \in \mathrm{TA}\Omega$ (Furthermore if all C^*-algebras $B \in \Omega$ have SP property, then A automatically has SP property).*

PROOF : By Lemma 2.4 in [37] and Lemma 2.10, we will show that $f_{2^{11/2}\varepsilon}(a) \lesssim b_0 \oplus b_1 \oplus \cdots \oplus b_m$ for any $\varepsilon > 0$, for any positive elements $a, b_0, b_1, \cdots, b_m$ in A, any $i = 0,\ 1,\ \cdots,\ m$, any $\tau \in \mathrm{T}(A)$

with $d_\tau(a) < d_\tau(b_i)$.

Let $\varepsilon > 0$. We assert that $r = \inf\{d_\tau(b) - d_\tau(f_\varepsilon(a)), \tau \in \mathrm{T}(A)\} > 0$ with $d_\tau(a) < d_\tau(b)$.

If $f_\varepsilon(a)$ is Cuntz equivalent to a, then a is Cuntz equivalent to a projection, $d_\tau(a)$ is continuous on $\mathrm{T}(A)$, and $d_\tau(b)$ is lower semicontinuous on $\mathrm{T}(A)$. Then $d_\tau(b) - d_\tau(a)$ is lower semicontinuous on $\mathrm{T}(A)$, by the property of lower semicontinuous, we know that $d_\tau(b) - d_\tau(a)$ can take the infimum on $\mathrm{T}(A)$. We may assume $A_0 = d_{\tau_0}(b) - d_{\tau_0}(a)$ is the infimum, since $d_{\tau_0}(b) - d_{\tau_0}(a) > 0$, we have $A_0 > 0$, i.e., $r = \inf\{d_\tau(b) - d_\tau(f_\varepsilon(a)), \tau \in \mathrm{T}(A)\} > 0$.

Otherwise, by Theorem 2.7, there exists a nonzero positive element d, such that $f_\varepsilon(a)d = df_\varepsilon(a) = 0$ and $f_\varepsilon(a) + d \precsim a$. We have $d_\tau(b) - d_\tau(a) \geq d_\tau(b) - (d_\tau(f_\varepsilon(a) + d_\tau(d))$, so we have $r = \inf\{d_\tau(b) - d_\tau(f_\varepsilon(a)), \tau \in \mathrm{T}(A)\} > 0$.

By the assertion above, $r'_i = \min\{d_\tau(b_i) - d_\tau(f_\varepsilon(a))\} > 0$. Write $\delta' = \min\{r'_0, r'_1, \cdots, r'_m\}$, then $\delta' > 0$.

Fixed a positive number $\delta_1 < \delta'$, since $d_\tau(b) = \sup\{(d_\tau((b-\sigma)_+)), \sigma > 0\}$, there exists $\varepsilon_1 < \varepsilon$, such that $d_\tau(b) - \delta_1 \leq d_\tau((b - \varepsilon_1)_+)$.

For $i = 0,\ 1,\ \cdots,\ m$, there exist $r''_i = \min\{d_\tau(b_i - \varepsilon_1)_+) - d_\tau(f_\varepsilon(a))\} > 0$, write $\delta'' = \min\{r''_0, r''_1, \cdots, r''_m\}$, we have $\delta'' > 0$.

Since $d_\tau((b_i - \varepsilon_1)_+) \leq \tau(f_{\varepsilon_1/2}(b_i))$ and $\tau(f_\varepsilon(a)) \leq d_\tau(f_\varepsilon(a))$, we have $r_i = \inf\{\tau(f_{\varepsilon_1}(b_i)) - \tau(f_\varepsilon(a)), \tau \in \mathrm{T}(A)\} > 0$, write $\delta = \min\{r_0, r_1, \cdots, r_m\}$, we have $\delta > 0$.

Since A has SP property, there exist projections $e_i \in A$ such that $f_{\varepsilon_1}(b_i)e_i = e_i$, $i = 0,\ 1,\ \cdots,\ m$, and $\tau(e_i) < \delta/12$. Set $b'_i = (1 - e_i)f_{\varepsilon_1}(b_i)(1 - e_i)$. One has $(1 - e_i)f_{\varepsilon_1}(b_i)(1 - e_i) + e_i = f_{\varepsilon_1}(b_i)$ for any $i = 0,\ 1,\ \cdots,\ m$.

Since A is simple and $\mathrm{T}(A)$ is compact, there is $0 < \delta_1 < 1/2$ such that $\frac{1}{3}\delta < \inf\{\tau(f_{\delta_1}(b_i) - \tau(f_\varepsilon(a)) : \tau \in \mathrm{T}(A)\}, i = 0,\ 1,\ \cdots,\ m$.

By Theorem 5.7 in [5], if $\tau(a) = \tau(b)$ for any $\tau \in \mathrm{T}(A)$, there are $y_1,\ y_2,\ \cdots, y_k \cdots \in A$ such that $a = \sum_{j=1}^{\infty} y_j^* y_j, b = \sum_{j=1}^{\infty} y_j y_j^*$. Since $\frac{1}{3}\delta < \inf\{\tau(f_{\delta_1}(b'_i)) - \tau(f_\varepsilon(a)) : \tau \in \mathrm{T}(A)\}$, for any $\varepsilon'' > 0$ with $\varepsilon'' < \delta/12$, there are $x_1^i,\ x_2^i,\ \cdots,\ x_k^i \in A$, and $c \in A_+$ such that

$$\|(f_\varepsilon(a) + c) - \sum_{j=1}^{k} ({x_j}^i)^* x_j^i\| < \varepsilon'',$$

$$\|f_{\delta_1}(b'_i) - \sum_{j=1}^{k} x_j^i ({x_j}^i)^*\| < \varepsilon'',$$

for any $i = 0,\ 1,\ \cdots,\ m$. We have $\tau(c)+\tau(f_\varepsilon(a))+2\varepsilon'' > \tau(\sum_{j=1}^k x_j^i {x_j}^{i*})+\varepsilon'' = \tau(\sum_{j=1}^k {x_j}^{i*} x_j^i)+\varepsilon'' > \tau(f_{\delta_1}(b_i'))$. Therefore $\tau(c) > \delta/6$.

Since $A \in \mathrm{TA}\Omega$, with $F = \{a,\ b_i,\ c, {x_1}^i,\ {x_2}^i,\ \cdots,\ {x_k}^i\}$, $i = 0,\ 1,\ \cdots,\ m$, any $\varepsilon' > 0$ with $\varepsilon' < \frac{\varepsilon}{4(2k+1)}$, there exist a C*-subalgebra $B \subseteq A$ and a projection $p \in A$ with $1_B = p$ and $B \in \Omega$, such that

(1) $\|px - xp\| < \varepsilon'/4$ for any $x \in F$,

(2) $pxp \in_{\varepsilon'/4} B$ for any $x \in F$, and

(3) $[1-p] \leq [e]$.

By (1) and (2) there exist $\{a',\ b_i',\ c',\ {x_1^i}',\ {x_2^i}',\ \cdots,\ {x_k^i}'\} \subseteq B$ and $\{a'',\ b_i'',\ {x_1^i}'',\ {x_2^i}'',\ \cdots,\ {x_n^i}''\} \subseteq (1-p)A(1-p)$, such that

$$\|b_i - b_i' - b_i''\| < \varepsilon', \|c - c' - c''\| < \varepsilon', \|b_i' - pb_ip\| < \varepsilon',$$

$$\|a - a' - a''\| < \varepsilon', \|pap - a'\| < \varepsilon', \|{x_j}^i - {x_j}^{i'} - {x_j}^{i''}\| < \varepsilon',$$

$$\|f_\varepsilon(a) - f_\varepsilon(a') - f_\varepsilon(a'')\| < \varepsilon''/4, \|f_{\delta_1}(b_i) - f_{\delta_1}(b_i') - f_{\delta_1}(b_i'')\| < \varepsilon''/4,$$

for any $1 \leq j \leq k, 0 \leq i \leq m$.

By functional calculate, we have

$$\|(f_\varepsilon(a') + c') - \sum_{j=1}^k {x_j}^{i*'} {x_j}^{i'}\| < (2k+1)\varepsilon' + \varepsilon''/2 < \varepsilon'',$$

$$\|f_{\delta_1}(b_i') - \sum_{j=1}^k {x_j}^{i'} {x_j}^{i*'}\| < (2k+1)\varepsilon' + \varepsilon''/2 < 3\varepsilon''/4.$$

Since $\tau(c')+\tau(c'')+\varepsilon' > \tau(c) > 3\varepsilon''$ and $\tau(c'') < \tau(1-p) < \tau(e) < \frac{1}{12}\delta$, we have $\tau(c') > 2\varepsilon''$ for any $\tau \in \mathrm{T}(B)$.

Since $f_\eta(f_{2\varepsilon}(a') \leq f_\varepsilon(a')$ for any $\eta > 0$, we have $\lim_{\eta\to 0} f_\eta(f_{2\varepsilon}(a')) \leq f_\varepsilon(a')$, therefore, $d_\tau(f_{2\varepsilon}(a')) = \lim_{\eta\to 0} \tau(f_\eta(f_{2\varepsilon}(a'))) \leq \tau(f_\varepsilon(a'))$, there exists sufficiently small η, such that

$$\begin{aligned} d_\tau(f_{2\varepsilon}(a')) &\leq \tau(f_\varepsilon(a')) \\ &\leq \varepsilon'' + \textstyle\sum_{j=1}^k \tau({x_j}^{i'} {x_j}^{i*'}) - \tau(c') \\ &= \varepsilon'' + \textstyle\sum_{j=1}^k \tau({x_j}^{i*'} {x_j}^{i'}) - \tau(c') \\ &\leq 3\varepsilon''/4 + \textstyle\sum_{j=1}^k \tau({x_j}^{i*'} {x_j}^{i'}) \\ &\leq \tau(f_{\delta_1}(b_i')) \leq d_\tau(f_{\delta_1}(b_i')) \end{aligned}$$

for any $\tau \in \mathrm{T}(B)$. Since $f_{\delta_1}(b_i'),\ f_{2\varepsilon}(a') \in B$ and $B \in \Omega$, we have

$$f_{2\varepsilon}(a') \lesssim f_{\delta_1}(b_0') \oplus f_{\delta_1}(b_1') \oplus \cdots \oplus f_{\delta_1}(b_m').$$

By Lemma 2.10, we have

$$\begin{aligned} f_{2^{11/2}\varepsilon}(a) &\lesssim (f_{4\varepsilon}(pap + (1-p)a(1-p)) \lesssim f_{4\varepsilon}(pap) \oplus (1-p) \\ &\lesssim f_{4\varepsilon}(pap) \oplus e \lesssim f_{2\varepsilon}(a') \oplus e \\ &\lesssim f_{\delta_1}(b_0') \oplus f_{\delta_1}(b_1') \oplus \cdots \oplus f_{\delta_1}(b_m') \oplus e \\ &\lesssim f_{\delta_1}(b_0') \oplus e \oplus f_{\delta_1}(b_1') \oplus \cdots \oplus f_{\delta_1}(b_m') \\ &\lesssim f_{\delta_1/2}(pb_0p) \oplus e \oplus f_{\delta_1/2}(pb_1p) \oplus \cdots \oplus f_{\delta_1/2}(pb_mp) \\ &\lesssim b_0 \oplus e \oplus b_1 \oplus \cdots \oplus b_m \\ &\lesssim (1-e)f_{\varepsilon_1}(b_0)(1-e) \oplus e \oplus b_1 \oplus \cdots \oplus b_m \\ &= f_{\varepsilon_1}(b_0) \oplus b_1 \oplus \cdots \oplus b_m \\ &\lesssim b_0 \oplus b_1 \oplus \cdots \oplus b_m \end{aligned}$$

By Lemma 2.4 in [37], we have $a \lesssim b_0 \oplus b_1 \oplus \cdots \oplus b_m$. □

***Theorem* 3.7** — *Let Ω be a class of stably finite unital exact C^*-algebras such that B has the strong tracial m-comparison of positive elements for any $B \in \Omega$. Then A has the strong tracial m-comparison of positive elements for any separable simple unital exact C^*-algebra, which A has* SP *property and $A \in \mathrm{TA}\Omega$ (Furthermore if all C^*-algebras $B \in \Omega$ have* SP *property, then A automatically has* SP *property).*

PROOF : By Lemma 2.4 in [37] and Lemma 2.10, we will show that $f_{2^{11/2}\varepsilon}(a) \lesssim b$ for any $\varepsilon > 0$, for any positive elements a, b in A, any $\tau \in \mathrm{T}(A)$ with $d_\tau(a) < \frac{1}{m+1}\tau(b)$. we may assume that $\|b\| \le 1$.

For any $\tau \in \mathrm{T}(A)$, since $d_\tau(a) < \frac{1}{m+1}\tau(b)$, we have $d_\tau(a) < \frac{1}{m+1}d_\tau(b)$, by the proof of Theorem 3.6, $r = \inf\{\frac{1}{m+1}d_\tau(b) - d_\tau(f_\varepsilon(a)), \tau \in \mathrm{T}(A)\} > 0$ for any $\varepsilon > 0$.

Since $d_\tau(b) = \sup\{(d_\tau((b-\sigma)_+)), \sigma > 0\}$, so for any $\delta' > 0$ with $\delta' < r$, there exists $\varepsilon_1 < \varepsilon$, such that $\frac{1}{m+1}d_\tau((b-\varepsilon_1)_+) - \delta' \le \frac{1}{m+1}d_\tau((b-\varepsilon_1)_+)$, So we have $\delta = \inf\{\frac{1}{m+1}d_\tau((b-\varepsilon_1)_+) - d_\tau(f_\varepsilon(a)), \tau \in \mathrm{T}(A)\} > 0$. Since $d_\tau((b-\varepsilon_1)_+) \le \tau(f_{\varepsilon_1}(b))$ and $\tau(f_\varepsilon(a)) \le d_\tau(f_\varepsilon(a))$, we have $\delta = \inf\{\frac{1}{m+1}\tau(f_{\varepsilon_1}(b)) - \tau(f_\varepsilon(a)), \tau \in \mathrm{T}(A)\} > 0$.

Since A has SP property, there exists projection $e \in A$ such that $be = e$ and $\tau(e) < \delta/12$. Write $b_1 = (1-e)f_{\varepsilon_1}(b)(1-e)$. Since A is simple and $\mathrm{T}(A)$ is compact, there is $0 < \delta_1 < 1/2$ such that $\frac{1}{3}\delta < \inf\{\frac{1}{m+1}\tau(f_{\delta_1}(b_1) - \tau(f_\varepsilon(a)) : \tau \in \mathrm{T}(A)\}$.

By Theorem 5.7 in [5], if $\tau(f_{\delta_1}(b_1)) = \tau(f_\varepsilon(a))$ for any $\tau \in \mathrm{T}(A)$, there are $y_1,\ y_2,\ \cdots,\ y_k \cdots \in A$ such that $f_{\delta_1}(b_1) = \sum_{j=1}^{\infty} y_j^* y_j$, $f_\varepsilon(a) = \sum_{j=1}^{\infty} y_j y_j^*$. Since $\frac{1}{3}\delta < \inf\{\frac{1}{m+1}\tau(f_{\delta_1}(b_1)) - \tau(f_\varepsilon(a)) : \tau \in \mathrm{T}(A)\}$, for any $\varepsilon'' > 0$ with $\varepsilon < \delta/12$, there are $x_1,\ x_2,\ \cdots,\ x_k \in A$ and $c \in A_+$ such that

$$\|(f_\varepsilon(a) + c) - \sum_{j=1}^{k} x_j^* x_j\| < \varepsilon'',$$

$$\|\frac{1}{m+1} f_{\delta_1}(b_1) - \sum_{j=1}^{k} x_j x_j^*\| < \varepsilon''.$$

We have $\tau(c)+\tau(f_\varepsilon(a))+2\varepsilon'' > \tau(\sum_{j=1}^{k} x_j^i x_j{}^{i*})+\varepsilon'' = \tau(\sum_{j=1}^{k} x_j{}^{i*} x_j^i)+\varepsilon'' > \frac{1}{m+1}\tau(f_{\delta_1}(b_i'))$. Therefore $\tau(c) > \delta/6$.

Since $A \in \mathrm{TA}\Omega$, with $F = \{a,\ b_1,\ c,\ x_1,\ x_2,\ \cdots,\ x_k\}$, any $\varepsilon_1 > 0$ with $\varepsilon' < \frac{\varepsilon}{4(2k+1)}$, there exist C*-subalgebra $B \subseteq A$ and projection $p \in A$ with $1_B = p$ and $B \in \Omega$, such that

(1) $\|px - xp\| < \varepsilon'/4$ for any $x \in F$,

(2) $pxp \in_{\varepsilon'/4} B$ for any $x \in F$, and

(3) $[1-p] \leq [e]$.

By (1) and (2) there exist $\{a',\ b_1',\ c',\ x_1',\ x_2',\ \cdots,\ x_k'\} \subseteq B$ and $\{a'',\ b_1'',\ x_1'',\ x_2'',\ \cdots,\ x_n''\} \subseteq (1-p)A(1-p)$, such that

$$\|b_1 - b_1' - b_1''\| < \varepsilon', \|c - c' - c''\| < \varepsilon', \|pb_1p - b_1'\| < \varepsilon'$$

$$\|a - a' - a''\| < \varepsilon', \|pap - a\| < \varepsilon', \|x_i - x_i' - x_i''\| < \varepsilon',$$

$$\|f_\varepsilon(a) - f_\varepsilon(a') - f_\varepsilon(a'')\| < \varepsilon''/4, \|f_{\delta_1}(b_1) - f_{\delta_1}(b_1{}') - f_{\delta_1}(b_1{}'')\| < \varepsilon''/4,$$

for any $1 \leq i \leq k$.

By functional calculate, we have

$$\|(f_\varepsilon(a') + c') - \sum_{j=1}^{k} x_j'^* x_j'\| < (2k+1)\varepsilon_1 + \varepsilon''/2 < \varepsilon'',$$

$$\|\frac{1}{m+1} f_{\delta_1}(b') - \sum_{j=1}^{k} x_j' x_j'^*\| < (2k+1)\varepsilon' + \varepsilon''/2 < 3\varepsilon''/4.$$

Since $\tau(c')+\tau(c'')+\varepsilon' > \tau(c) > 3\varepsilon''$ and $\tau(c'') < \tau(1-p) < \tau(e) < \delta/12$, we have $\tau(c') > 2\varepsilon''$ for any $\tau \in \mathrm{T}(B)$.

Since $f_\eta(f_{2\varepsilon}(a') \leq f_\varepsilon(a')$ for any $\eta > 0$, we have $\lim_{\eta\to 0} f_\eta(f_{2\varepsilon}(a')) \leq f_\varepsilon(a')$, therefore, $d_\tau(f_{2\varepsilon}(a')) = \lim_{\eta\to 0} \tau(f_\eta(f_{2\varepsilon}(a'))) \leq \tau(f_\varepsilon(a'))$, so there exists sufficiently small η, such that

$$\begin{aligned} d_\tau(f_{2\varepsilon}(a')) &\leq \tau(f_\varepsilon(a')) \\ &\leq \varepsilon'' + \textstyle\sum_{j=1}^k \tau(x'_j x_j^{*'}) - \tau(c') \\ &= \varepsilon'' + \textstyle\sum_{j=1}^k \tau(x_j^{'*} x'_j) - \tau(c') \\ &\leq 3\varepsilon''/4 + \textstyle\sum_{j=1}^k \tau(x_j^{'*} x'_j) \\ &\leq \tfrac{1}{m+1}\tau(f_{\delta_1}({b_1}')) \end{aligned}$$

for any $\tau \in \mathrm{T}(B)$. Since $f_\delta({b_1}'),\ f_{2\varepsilon}(a') \in B$ and $B \in \Omega$, we have

$$f_{2\varepsilon}(a') \lesssim f_{\delta_1}({b_1}').$$

We also have

$$\begin{aligned} f_{2^{11/2}\varepsilon}(a) &\lesssim f_{4\varepsilon}(pap + (1-p)a(1-p)) \lesssim f_{4\varepsilon}(pap) \oplus (1-p) \\ &\lesssim f_{2\varepsilon}(a') \oplus e \lesssim f_{\delta_1}({b_1}') \oplus e \lesssim f_{\delta_1/2}(pb_1p) \oplus e \\ &\lesssim b_1 \oplus e = (1-e)f_{\varepsilon_1}(b_1)(1-e) + e = f_{\varepsilon_1}(b_1) \lesssim b \end{aligned}$$

By Lemma 2.4 in [37], we have $a \lesssim b$. □

Corollary 3.8 — Let A be an infinite dimensional separable simple unital exact C^*-algebra has the m-comparison of positive elements and A has SP property (strong tracial m-comparison of positive elements). Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C^*-algebra $\mathrm{C}^*(G, A, \alpha)$ has the m-comparison of positive elements (strong tracial m-comparison of positive elements).

PROOF : This follow from 3.4., 3.6 and 3.7.

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