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SOME CHARACTERIZATION OF HERZ AND HERZ-TYPE HARDY SPACES FOR THE DUNKL OPERATOR

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In this work, new characterizations of Herz and Herz-type Hardy spaces associated with the Dunkl operator on the real line are introduced.

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1. Introduction

The study of Hardy spaces is started by Hardy [1] at the beginning of 20^{th} century in parallel with the setting of one variable Fourier series and complex analysis, providing basic insights into such topics as L^p spaces, maximal functions and atomic decomposition. The works of Elias Stein and Guido Weiss [2] in several complex variables presents an extension of Hardy spaces for Fourier analysis. The important feature of Hardy spaces is that they avoid the collection of undesirable characteristics in L^p spaces if $p \leq 1$. These spaces are much better suited to a host questions in harmonic analysis than are the L^p spaces. Hardy spaces become a centerpiece of harmonic analysis due that they constituted the right domain of study of a large number of linear and nonlinear operators. Furthermore, this spaces are extremely useful tool in mathematical analysis, partial differential equations, probability, as well as in control theory and in scattering theory. The multiple characterization of Hardy spaces in IR^n is discovered by Fefferman and Stein [3]. They present a multiple equivalent definitions of Hardy spaces in IR^n and used them to prove the boundedness of singular integral operator. Strmberg and Torchinsky [4] give the theory of weighted Hardy spaces of tempered distribution on IR^n. They consider properties of weights in a general setting; they derive mean value inequalities for wavelet transforms. In [5-7] the Herz spaces, many applications of some particular cases of the generalized Herz spaces and the duals of the generalized Herz spaces are considered. The weighted Herz spaces and its applications is introduced and a very useful boundedness theorem for a large class of sublinear operators is proved in [8]. Many applications of this theorem are presented in [9, 10]. For certain condition, the boundedness theorem is not holds and an counter example are given in [8]. This leads in [9, 11-16] to consider a new spaces called weighted Herz-type Hardy spaces such that the boundedness theorem is true even when the last certain condition is satisfied. Recently, a family of weight functions invariant under a finite reflection group on IR^n and an a correspondent isometry similar to the classical Fourier transform called Dunkl transform are the goal of many papers. Many definitions and properties in this theory are mainly based on a commutative set of differential-difference operators [17], the analogue of the first-order partial derivatives in the ordinary theory. This theory is enriched and continues to enrich the pure and applied mathematical area by a set of a useful tools that have a significant impact on the study of many problems. For example special functions associated with root systems (see [18-21]), certain representations of degenerated affine Hecke algebras (see [22, 23]) and certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in one dimensional space (see [24, 25]). Trimeche [26, 27] studied the Dunkl intertwining operator and its inversion on spaces of functions and distributions. He defined the integral representation of the dual of this operator and his inverse. Furthermore, he prove in [21] the Paley-Wiener theorems for the Dunkl transform and general translation operator. Many papers addressed a number of issues pertaining to the generalized translation, convolution Operator and maximal function for the Dunkl Transform. Thangavelu and Xu [28] studied the generalized translation and convolution operator and use them to analysis the summability of the inverse Dunkl transform. They found the particular expression of the generalized translation operator in the case where $G = \mathbb{Z}_2$ and showed that it is not positive operator and satisfied some properties of the classical translation operator. Many authors addressed a number of issues relating to the case $G = \mathbb{Z}_2$. Deleaval [29] studied the Dunkl maximal operator and showed the Fefferman-Stein inequalities. Abdelkefi and Sifi [30] give an estimate of the Dunkl translation of the original circle of radius epsilon characteristic function and shows the L^p boudedness of the uncentered Dunkl maximal operator. Mourou [31] extend the Taylor series classical theory of a differential-difference operator on the real line. The following Herz and Herz-type Hardy spaces for the Dunkl operator Λ_α, $\alpha \geq -1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ are introduced by Gasmi *et al.* [32].

- Let $0 < p \leq 1 < q \leq \infty$ and $\beta \geq 1 - 1/q$, the homogeneous weighted Herz space is defined by

$$\dot{K}^{\beta,p}_{\alpha,q} := \Big\{ f \in L^q_{loc}(\mu_\alpha) \ / \ \|f\|_{\dot{K}^{\beta,p}_{\alpha,q}} = \Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \|f\chi_k\|^p_{q,\alpha} \Big]^{1/p} < \infty \Big\},$$

where $L^q_{loc}(\mu_\alpha)$ is the space of functions f such that $|f|^q$ is locally integrable with respect to the measure

$$d\mu_\alpha(x) := \big(2^{\alpha+1}\Gamma(\alpha+1)\big)^{-1}|x|^{2\alpha+1}dx$$

and χ_k is the characteristic function of the set $A_k := \{x \in \mathbb{R} \,/\, 2^{k-1} \leq |x| \leq 2^k\}$, for $k \in \mathbb{Z}$.

- The non-homogeneous weighted Herz space is $K^{\beta,p}_{\alpha,q} := L^q(\mu_\alpha) \cap \dot{K}^{\beta,p}_{\alpha,q}$.

Moreover, $\|f\|_{K^{\beta,p}_{q,\alpha}} := \|f\|_{q,\alpha} + \|f\|_{\dot{K}^{\beta,p}_{\alpha,q}}$.

- Let $N \in \mathbb{N}$, the Herz-type Hardy space is defined by

$$H\dot{K}^{\beta,p,N}_{\alpha,q} := \Big\{ f \in \mathcal{S}'(\mathbb{R}) \ / \ G_{\alpha,N}(f) \in \dot{K}^{\beta,p}_{\alpha,q} \Big\},$$

where $G_{\alpha,N}$ is the α-grand maximal function. Moreover, $\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}} := \|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}}$. Under some hypothesis, they showed an atomic decomposition of Herz-type Hardy spaces. The aim of this work is to establish some new characterizations of these spaces.

The procedure adopted in this study is as follows: in Section 2, some useful results associated with the Dunkl operator on $\mathbb{R}$ are presented. Moreover, a various form of maximal function are defined. In Section 3, a unit decomposition theorem for the Herz spaces associated with the Dunkl operator is given, by which the boundedness of some sublinear operators are proved on these spaces. Section 4 is devoted to some new characterizations for the Herz-type Hardy spaces associated with the Dunkl operator.

In the sequel C represents a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ the Schwartz space on $\mathbb{R}$ and the space of tempered distributions on $\mathbb{R}$ respectively .

2. Preliminaries

In this section we recapitulate some results about harmonic analysis associated with the Dunkl operator which will be used later. For details the reader is referred to [17, 19, 26, 33]. Let $\alpha \geq -1/2$. We consider the differential-difference operator introduced by [17]

$$\Lambda_\alpha f(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\Big[\frac{f(x)-f(-x)}{2}\Big],$$

and call it the *Dunkl operator*.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the Dunkl kernel is defined by

$$E_\alpha(\lambda x) = \mathrm{Im}_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha+1)}\,\mathrm{Im}_{\alpha+1}(\lambda x), \quad x \in \mathbb{R},$$

is the unique solution of the initial differential equation:

$$\Lambda_\alpha f(x) = \lambda f(x), \quad f(0) = 1,$$

where Im_α is the modified Bessel function of order α given by [34, 35]

$$\mathrm{Im}_\alpha(\lambda x) := \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(\lambda x/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}.$$

Let $q \in [1,\infty]$, we write $L^q(\mu_\alpha)$ to be the set of all functions satisfying

$$\|f\|_{q,\alpha} = \Big(\int_{\mathbb{R}} |f(x)|^q d\mu_\alpha(x)\Big)^{\frac{1}{q}}.$$

Like to the Fourier transform, the Dunkl transform of a function $f \in L^1(\mu_\alpha)$ on $\mathbb{R}$ (see [19]) is given by

$$\mathcal{F}_\alpha(f)(y) := \int_{\mathbb{R}} E_\alpha(-ixy) f(x) d\mu_\alpha(x), \quad y \in \mathbb{R}.$$

Here the integral makes sense since $|E_\alpha(ix)| \leq 1$ for every $x \in \mathbb{R}$ ([36], p. 295). If $\alpha = -\frac{1}{2}$, the Dunkl transform coincides with the Fourier transform $\mathcal{F}$, given by:

$$\mathcal{F}(f)(y) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ixy} f(x)\,dx, \quad y \in \mathbb{R}.$$

First, we note (See [21], p. 25, 26) that the Dunkl transform $\mathcal{F}_\alpha(f)$ is a bounded continuous function and for all $f \in L^1(\mu_\alpha)$, we have

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\alpha} \leq \|f\|_{1,\alpha}.$$

Moreover, $\mathcal{F}_\alpha$ is a topological isomorphism on $\mathcal{S}(\mathbb{R})$ which extends to a topological isomorphism on $\mathcal{S}'(\mathbb{R})$ and for each $f \in L^1(\mu_\alpha)$ and $x, y \in \mathbb{R}$ (See [19, 21])

$$\mathcal{F}_\alpha^{-1}(f)(x) = \mathcal{F}_\alpha(f)(-x) \quad \text{and} \quad \mathcal{F}_\alpha(\Lambda_\alpha f)(y) = iy\,\mathcal{F}_\alpha(f)(y).$$

Now we present the definitions of the general translation and the general convolution product and their properties.

For all $x, y, z \in \mathbb{R}$,

$$\sigma_{x,y,z} := \begin{cases} \frac{x^2+y^2-z^2}{2xy}, & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ 0, & \text{otherwise} \end{cases}$$

$$d_\alpha := 2^{1-\alpha}(\Gamma(\alpha+1))^2/\sqrt{\pi}\,\Gamma(\alpha+1/2), \quad A_{x,y} := \Big[\Big||x|-|y|\Big|, |x|+|y|\Big].$$

$$\Delta_\alpha(|x|,|y|,|z|) := \begin{cases} d_\alpha \dfrac{\Big[\Big((|x|+|y|)^2-z^2\Big)\Big(z^2-(|x|-|y|)^2\Big)\Big]^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y} \\ \\ 0, & \text{otherwise ,} \end{cases}$$

$$W_\alpha(x,y,z) := \Big[1-\sigma_{x,y,z}+\sigma_{z,x,y}+\sigma_{z,y,x}\Big]\Delta_\alpha(|x|,|y|,|z|)$$

$\nu_{x,y}$ is a signed measures given by

$$d\nu_{x,y}(z) := \begin{cases} W_\alpha(x,y,z)d\mu_\alpha(z), & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ d\delta_x(z), & \text{if } y = 0 \\ d\delta_y(z), & \text{if } x = 0. \end{cases}$$

The main properties of the signed kernel W_α is

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z),$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\, d\mu_\alpha(z) \leq 4.$$

and the measures $\nu_{x,y}$ have the properties:

$$\text{supp }(\nu_{x,y}) = A_{x,y} \cup (-A_{x,y}), \quad \|\nu_{x,y}\| := \int_{\mathbb{R}} d|\nu_{x,y}|(z) \leq 4.$$

For a continuous function f and $x \in \mathbb{R}$, the Dunkl translation operator is defined by

$$\tau_x f(y) := \int_{\mathbb{R}} f(z)d\nu_{x,y}(z), \quad y \in \mathbb{R}.$$

As in the classical analysis, the Dunkl convolution is defined for all $f, g \in L^1(\mu_\alpha)$ and $x \in \mathbb{R}$ by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y)g(y)d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

Note that the convolution $*_\alpha$ is associative and commutative [36] and $*_{-1/2}$ is the standard convolution.

In the space of C^∞-functions on $\mathbb{R}$, the general Dunkl translation is defined in [36] by

$$\tau_x f(y) = (V_\alpha)_x (V_\alpha)_y \Big((V_\alpha)^{-1}(f)(x+y) \Big)$$

where

$$V_\alpha(f)(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^{1} f(tx)(1+t)(1-t^2)^{\alpha-1/2} dt.$$

On the other hand, [21] shows that τ_x has the following important explicit formula:

$$\begin{aligned} \tau_x f(y) &= \tfrac{1}{2}\int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1+\tfrac{x-y}{\sqrt{x^2+y^2-2xyt}})\phi(t)dt \\ &\quad + \tfrac{1}{2}\int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt})(1-\tfrac{x-y}{\sqrt{x^2+y^2-2xyt}})\phi(t)dt, \end{aligned} \tag{2.1}$$

where $\phi(t) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}(1+t)(1-t^2)^{\alpha-1/2}$
and we have

$$\tau_x f(-y) = \tau_y f(-x), \quad \forall\, x, y \in \mathbb{R}.$$

The Dunkl translation of a nonnegative even function is nonnegative but this operator is not positive.

Proposition 2.1 —

1. The operators τ_x, $x \in \mathbb{R}$, are continuous from $\mathcal{S}(\mathbb{R})$ into itself.
2. For all $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$, we have $\Lambda_\alpha(\tau_x f) = \tau_x(\Lambda_\alpha f)$.
3. For all $f \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi(x) d\mu_\alpha(x) = 1$, we have

$$\lim_{t\to 0} f *_\alpha \phi_t = f, \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

where ϕ_t is the dilation of ϕ given by

$$\phi_t(x) := t^{-2(\alpha+1)}\phi\Big(\frac{x}{t}\Big), \quad x \in \mathbb{R}. \tag{2.2}$$

Proposition 2.2 — Let $y \in \mathbb{R}$ and $f, g \in \mathcal{S}(\mathbb{R})$ then

1. $\displaystyle\int_{\mathbb{R}} \tau_y f(x) d\mu_\alpha(x) = \int_{\mathbb{R}} f(x) d\mu_\alpha(x).$
2. $\displaystyle\int_{\mathbb{R}} \tau_y f(x) g(x) d\mu_\alpha(x) = \int_{\mathbb{R}} f(x) \tau_y g(-x) d\mu_\alpha(x).$

Proposition 2.3 — Let $f \in \mathcal{S}(\mathbb{R})$ be supported in $\{x : |x| \leq B\}$ then $\tau_y f$ is supported in $\{x : |x| \leq B + |y|\}$.

Proposition 2.4 — The following two propositions are shown in [37].

1. For all $x \in \mathbb{R}$ and $f \in L^q(\mu_\alpha)$, $q \in [1, \infty]$:

$$\|\tau_x f\|_{q,\alpha} \leq 4\,\|f\|_{q,\alpha}.$$

2. For all $x \in \mathbb{R}$ and $f \in L^1(\mu_\alpha)$:

$$\mathcal{F}_\alpha(\tau_x f)(\lambda) = E_\alpha(ix\lambda)\,\mathcal{F}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R}.$$

Proposition 2.5 —

1. Assume that $q, q', r \in [1, \infty]$ satisfy the Young condition $1/q + 1/q' = 1 + 1/r$. Then the map $(f, g) \to f *_\alpha g$ extends to a continuous map from $L^q(\mu_\alpha) \times L^{q'}(\mu_\alpha)$ to $L^r(\mu_\alpha)$, and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq 4\,\|f\|_{q,\alpha}\|g\|_{q',\alpha}.$$

2. For all $f \in L^1(\mu_\alpha)$ and $g \in L^2(\mu_\alpha)$, we have

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\,\mathcal{F}_\alpha(g).$$

3. For all $f \in L^1(\mu_\alpha)$ and $g \in L^p(\mu_\alpha)$ and $1 \leq p < \infty$ we have

$$\tau_x(f *_\alpha g) = (\tau_x(f) *_\alpha g) = (f *_\alpha \tau_x(g)), \quad x \in \mathbb{R}.$$

Lemma 2.6 — For every x, y in $\mathbb{R}$ and $r > 0$, we have

1. $\tau_y \chi_{I_r} \leq \tau_x \chi_{I_{\sqrt{2}(r+|y-x|)}}$.
2. $\tau_y \chi_{I_r}(-x) \leq \tau_y \chi_{I_r}(x)$, for $xy > 0$.

Here χ_{I_r} is the characteristic function of the interval $I_r = \{x \in \mathbb{R}/|x| \leq r\}$.

PROOF : From (2.1) and the fact that $f = \chi_{I_r}$ is even implies that

$$\tau_x f(z) = \int_{-1}^{1} f(\sqrt{x^2 + z^2 - 2xzt})\phi(t)dt. \tag{2.3}$$

Using the inclusion

$$\Big\{s \in \mathbb{R}/\sqrt{y^2+z^2-2yzs} \in I_r\Big\} \subset \Big\{s \in \mathbb{R}/\sqrt{x^2+z^2-2xzs} \in I_{\sqrt{2}(r+|y-x|)}\Big\},$$

we get 1.
It is clear that if $S_+ = \{t>0/\sqrt{x^2+y^2-2xyt} \in I_r\}$ and $S_- = \{-t/t<0, \sqrt{x^2+y^2-2xyt} \in I_r\}$ then $S_- \subset S_+$, which implies with equality (2.3),

$$\tau_y \chi_{I_r}(x) = \int_{S_-} (\phi(t)+\phi(-t))dt + \int_{S_+\setminus S_-} \phi(t)dt,$$

from which we conclude 2.

The maximal function of a locally integrable function f on $\mathbb{R}$ is defined by

$$M_\alpha(f)(x) := \sup_{t>0} \Big\{ \frac{1}{\mu_\alpha(]-t,t[)} \int_{-t}^{t} |\tau_x(f)(y)|\, d\mu_\alpha(y)\Big\}, \quad x \in \mathbb{R}. \tag{2.4}$$

This operator satisfies the following properties shown in [32].

Proposition 2.7 — For all $q \in]1,\infty[$, the operator M_α is continuous from $L^q(\mu_\alpha)$ into itself.

For all $N \in \mathbb{N}$, we denote by F_N the subset of $\mathcal{S}(\mathbb{R})$ constituted by all those $\phi \in \mathcal{S}(\mathbb{R})$ such that supp$(\phi) \subset [-1,1]$ and

$$\|\phi\|_{F_N} := \max_{m,n \le N} \rho_{m,n}(\phi) := \max_{m,n\le N} \sup_{x\in\mathbb{R}} (1+|x|)^m |\Lambda_\alpha^n \phi(x)| \le 1.$$

Moreover the system of semi-norms $\{\rho_{m,n}\}_{m,n\in\mathbb{N}}$ generates the topology of $S(\mathbb{R})$ (see [38]). Let $f \in \mathcal{S}'(\mathbb{R})$ and $N \in \mathbb{N}$. We define

$$\phi^*_{+,\alpha}(f)(x) := \sup_{t>0} |\phi_t *_\alpha f(x)|, \quad x \in \mathbb{R}$$

$$M_{\phi,\alpha}(f)(x) := \sup_{t>0} \sup_{|x-y|<t} |\phi_t *_\alpha f(y)|, \quad x \in \mathbb{R},$$

$$M_{\phi,\alpha,N}(f)(x) := \sup_{t>0} \sup_{|x-y|<Nt} |\phi_t *_\alpha f(y)|, \quad x \in \mathbb{R},$$

and the α-grand maximal function of order-N, $G_{\alpha,N}(f)$ of f, by

$$G_{\alpha,N}(f)(x) := \sup_{t>0,\, \phi\in F_N} M_{\phi,\alpha}(f)(x), \quad x \in \mathbb{R},$$

where ϕ_t is the dilation of ϕ given by (2.2).

Lemma 2.8 — If $\phi \in \mathcal{S}(\mathbb{R})$ and $\int_{\mathbb{R}} \phi(x)d\mu_\alpha(x) = 1$. Then for any $\psi \in \mathcal{S}(\mathbb{R})$ there is a sequence $(\eta^{(k)})_k$ in $S(\mathbb{R})$, so that

$$\psi(x) = \sum_{k=0}^{\infty} \eta^{(k)} *_\alpha \phi_{2^{-k}}(x), \quad x \in \mathbb{R}.$$

Moreover, for any $M, N \in \mathbb{N}$ there exists $C > 0$, independent to ψ such that

$$\max_{m,n \le N} \rho_{m,n}(\eta^{(k)}) < C 2^{-Mk} \|\psi\|_{F_N} \ \ as \ \ k \to \infty.$$

PROOF : Since $\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\,\mathcal{F}_\alpha(g)$, $\Lambda_\alpha(f *_\alpha g) = (\Lambda_\alpha f) *_\alpha g$ and $\Lambda_\alpha(\tau_x f) = \tau_x(\Lambda_\alpha f)$, the proof is similar to that of Lemma 2 on p.93 in [39]. □

3. A UNIT DECOMPOSITION THEOREM FOR THE HERZ SPACES $\dot{K}^{\beta,p}_{\alpha,q}$

Let $\beta \in \mathbb{R}$ and $q \in [1, \infty]$. A measurable function a on $\mathbb{R}$ is called a (central) (β, q)-unit if it satisfies $\mathrm{supp}(a) \subset I_r$ and $\|a\|_{q,\alpha} \le r^{-2(\alpha+1)\beta}$, for a certain $r > 0$. The aim of this section is to establish a unit decomposition theorem for the Herz spaces associated with the Dunkl operator. Then, as an application, we establish the boundedness of some sublinear operators on these spaces.

***Theorem* 3.1** — *Let $p \in]0, \infty[$, $1 < q \le \infty$ and $\beta \in \mathbb{R}$. Then $f \in \dot{K}^{\beta,p}_{\alpha,q}$ if and only if there exist, for all $j \in \mathbb{Z}$, an (β, q)-unit a_j and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$. Moreover,*

$$\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}} \sim \inf\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\Big)^{1/p},$$

where the infimum is taken over all unit decompositions of f.

PROOF : First we prove the necessity. Let $f \in \dot{K}^{\beta,p}_{\alpha,q}$, we have

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} f(x)\chi_k \\ &= \sum_{k\in\mathbb{Z}} 2^{2(k+1)(\alpha+1)\beta} \|f\chi_k\|_{q,\alpha} \frac{f(x)\chi_k(x)}{2^{2(k+1)(\alpha+1)\beta}\|f\chi_k\|_{q,\alpha}} \\ &= \sum_{k=-\infty}^{\infty} \lambda_k a_k(x). \end{aligned}$$

Evidently $a_k(x) = \frac{f(x)\chi_k(x)}{2^{2(k+1)(\alpha+1)\beta}\|f\chi_k\|_{q,\alpha}}$ is (β, q)-unit and $\sum_{k=-\infty}^{\infty} \lambda_k^p = c\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}} < \infty$.
Now we prove the inverse, let $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ such that $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ and a_j is an (β, q)-unit for all j with the support $B_j = \{x \in \mathbb{R} / |x| \le 2^j\}$. Then we have

$$\|f\chi_j\|_{\alpha,q} \le \sum_{k \ge j+1} |\lambda_k| \|a_k\|_{\alpha,q} \tag{3.1}$$

For $p \leq 1$, from (3.1) we have

$$\begin{aligned}
\|f\|^p_{\dot{K}^{\beta,p}_{\alpha,q}} &= \sum_{j=-\infty}^{\infty} 2^{2(\alpha+1)\beta jp} \|f\chi_j\|^p_{q,\alpha} \\
&\leq \sum_{j=-\infty}^{\infty} 2^{2(\alpha+1)\beta jp} \sum_{k\geq j+1} |\lambda_k|^p \|a_k\|^p_{\alpha,q} \\
&\leq \sum_{k=-\infty}^{\infty} |\lambda_k| \sum_{j\leq k-1} 2^{-2(k-j)p(\alpha+1)\beta} \\
&\leq C \sum_{k=-\infty}^{\infty} |\lambda_k| < \infty.
\end{aligned}$$

For $p > 1$, we consider p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. By Hölder's inequality and (3.1) we obtain

$$\begin{aligned}
\|f\chi_j\|_{\alpha,q} &\leq \Big(\sum_{k\geq j+1} |\lambda_k|^p \|a_k\|^{\frac{p}{2}}_{\alpha,q}\Big)^{\frac{1}{p}} \Big(\sum_{k\geq j+1} |\|a_k\|^{\frac{p'}{2}}_{\alpha,q}\Big)^{\frac{1}{p'}} \\
&\leq \Big(\sum_{k\geq j+1} |\lambda_k|^p 2^{-kp(\alpha+1)\beta}\Big)^{\frac{1}{p}} \Big(\sum_{k\geq j+1} 2^{-kp'(\alpha+1)\beta}\Big)^{\frac{1}{p'}} \\
&\leq C 2^{-j(\alpha+1)\beta} \Big(\sum_{k\geq j+1} |\lambda_k|^p 2^{-kp(\alpha+1)\beta}\Big)^{\frac{1}{p}}.
\end{aligned}$$

Then

$$\begin{aligned}
\|f\|^p_{\dot{K}^{\beta,p}_{\alpha,q}} &= \sum_{j=-\infty}^{\infty} 2^{2(\alpha+1)\beta jp} \|f\chi_j\|^p_{q,\alpha} \\
&\leq C \sum_{k=-\infty}^{\infty} |\lambda_k|^p \sum_{j\leq k-1} 2^{(j-k)p(\alpha+1)\beta} \\
&\leq C \sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty,
\end{aligned}$$

we conclude that $f \in \dot{K}^{\beta,p}_{\alpha,q}$. □

***Theorem* 3.2** — *Let $0 < p$ and $1 < q \leq \infty$, $\beta \leq 1 - 1/q$. If a sublinear operator T satisfies*

$$|Tf(x)| \leq C\|f\|_{1,\alpha}/|x|^{2(\alpha+1)} \quad if \quad dis(x, supp(f)) > \frac{|x|}{2}$$

for any integrable function f with a compact support and T is bounded on $L^q(\mu_\alpha)$, then T is bounded on $\dot{K}^{\beta,p}_{\alpha,q}$ and $K^{\beta,p,N}_{\alpha,q}$.

PROOF : Let $f \in \dot{K}^{\beta,p}_{\alpha,q}$ by the last theorem we can write $f = \sum_{j=-\infty}^{\infty} \lambda_j b_j$ where b_j is (β, q)-unit and $\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}} \sim \inf\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\Big)^{1/p}$, then

$$\begin{aligned}
\|Tf\|^p_{\dot{K}^{\beta,p}_{\alpha,q}} &= \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \|Tf\chi_k\|^p_{q,\alpha} \\
&\leq C \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \Big[\Big(\sum_{j=-\infty}^{k-2} |\lambda_j| \|Tb_j\chi_k\|_{q,\alpha}\Big)^p + \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \|Tb_j\chi_k\|_{q,\alpha}\Big)^p\Big] \\
&= C(I_1 + I_2).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\Big(\|Tb_j\chi_k\|_{q,\alpha}\Big)^q &= \int_{A_k} |Tb_j(x)|^q d\mu_\alpha(x) \\
&\leq C \int_{C_k} |x|^{-2(\alpha+1)q} \Big(\int_{B_j} |b_j(y)| d\mu_\alpha(y)\Big)^q d\mu_\alpha(x) \\
&\leq C\mu_\alpha(B_j)^{q-1} \int_{B_j} |b_j(y)|^q \mu_\alpha(y) \int_{C_k} |x|^{-(2\alpha+1)qx} \mu_\alpha(x) \\
&\leq C 2^{2(\alpha+1)j(q-1)} 2^{-2j(\alpha+1)\beta q} 2^{(k-1)(2(\alpha+1)-q)},
\end{aligned}$$

then for $p \leq 1$, we have

$$\begin{aligned}
I_1 &\leq \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \sum_{j=1}^{k-2} |\lambda_j|^p \|Tb_j\chi_k\|^p_{q,\alpha} \\
&\leq C \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{2(\alpha+1)jp(1-1/q)} 2^{-2j(\alpha+1)\beta p} 2^{(k-1)(2(\alpha+1)/q-1)p} \\
&\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{2(\alpha+1)p(k-j)(\beta+1/q-1)} \\
&\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}}.
\end{aligned}$$

For $p > 1$, by Hölder inequality

$$\begin{aligned}
I_1 &= \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j|\|Tb_j\chi_k\|_{q,\alpha}\Big)^p \\
&\leq C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{2(\alpha+1)p(k-j)(\beta+1/q-1)}\Big)\Big(\sum_{j=1}^{k-2} 2^{2(\alpha+1)p'(k-j)(\beta+1/q-1)}\Big)^{p/p'} \\
&\leq C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{2(\alpha+1)p(k-j)(\beta+1/q-1)}\Big) = C\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j+2}^{\infty} |\lambda_j|^p 2^{2(\alpha+1)p(k-j)(\beta+1/q-1)}\Big) \\
&\leq C\sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq c\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}}.
\end{aligned}$$

Using the boundedness of T on $L^q(\mu_\alpha)$ we get, if $p \leq 1$

$$\begin{aligned}
I_2 &= \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\|Tb_j\chi_k\|_{q,\alpha}\Big)^p \\
&\leq C\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\sum_{j=k-1}^{\infty} |\lambda_j|^p\|b_j\|_{q,\alpha}^p \\
&\leq C\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\sum_{j=k-1}^{\infty} |\lambda_j|^p 2^{-2j(\alpha+1)\beta p} \\
&\leq C\sum_{j=-\infty}^{\infty} |\lambda_j|^p\sum_{k=-\infty}^{j+1} 2^{2(k-j)(\alpha+1)\beta p} \\
&\leq C\sum_{j=1}^{\infty} |\lambda_j|^p \leq C\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}}.
\end{aligned}$$

If $p > 1$, we have

$$\begin{aligned}
I_2 &= \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\|Tb_j\chi_k\|_{q,\alpha}\Big)^p \\
&\leq C\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|^p\|b_j\|_{q,\alpha}^p\Big)\Big(\sum_{j=k-1}^{\infty} \|b_j\|_{q,\alpha}^{p'}\Big)^{p/p'} \\
&\leq C\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|^p 2^{-2j(\alpha+1)\beta p}\Big)\Big(\sum_{j=k-1}^{\infty} 2^{-2j(\alpha+1)\beta p'}\Big)^{p/p'}
\end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{k=-\infty}^{\infty}2^{2(\alpha+1)\beta kp}\sum_{j=k-1}^{\infty}|\lambda_j|^p2^{-2j(\alpha+1)\beta p}\\
&\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+1}2^{2(k-j)(\alpha+1)\beta p}\\
&\leq C\sum_{j=1}^{\infty}|\lambda_j|^p\leq C\|f\|_{\dot K^{\beta,p}_{\alpha,q}}.
\end{aligned}$$

This finishes the proof. □

4. SOME NEW REAL-VARIABLE CHARACTERIZATION OF THE HERZ-TYPE HARDY SPACES $H\dot K^{\beta,p,N}_{\alpha,q}$

Let $q\in]1,\infty]$ and $\beta\geq 1-1/q$. A measurable function a on $\mathbb{R}$ is called a (central) (β,q)-atom if it satisfies supp$(a)\subset I_r$, $\|a\|_{q,\alpha}\leq r^{-2(\alpha+1)\beta}$, for a certain $r>0$ and $\displaystyle\int_{\mathbb{R}}a(x)x^k d\mu_\alpha(x)=0,\ k=0,1,...,2s+1$, where $s=[(\alpha+1)(\beta-1+1/q)]$ (the integer part of $(\alpha+1)(\beta-1+1/q)$).

In [32], the authors established the following atomic decomposition theorem for the Herz-type Hardy spaces $H\dot K^{\beta,p,N}_{\alpha,q}$.

***Theorem* 4.1** — *Let $0<p\leq 1<q\leq\infty$, $\beta\geq 1-1/q$ and $N\in\mathbb{N}$, $N>2(2s+3+\alpha)$. Then $f\in H\dot K^{\beta,p,N}_{\alpha,q}$ if and only if there exist, for all $j\in\mathbb{N}\backslash\{\not{0}\}$, an (β,q)-atom a_j and $\lambda_j\in\mathbb{C}$, such that $\sum_{j=1}^{\infty}|\lambda_j|^p<\infty$ and $f=\sum_{j=1}^{\infty}\lambda_j a_j$. Moreover,*

$$\|f\|_{H\dot K^{\beta,p,N}_{\alpha,q}}\sim\inf\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p},$$

where the infimum is taken over all atomic decompositions of f.

The aim of this section is to establish the following some new real-variable characterizations theorem of these spaces.

***Theorem* 4.2** — *Let $0<p\leq 1<q\leq\infty$, $\beta\geq 1-1/q$ and $N\in\mathbb{N}$, $N>2(2s+3+\alpha)$. Suppose $\phi\in\mathcal{S}(\mathbb{R})$ such that $\int\phi(x)d\mu_\alpha(x)=1$ and $\phi_t(x)=\frac{1}{t^{2\alpha+1}}\phi(\frac{x}{t}),\quad for\quad t>0$. Then for $f\in\mathcal{S}'(\mathbb{R})$, the following conditions are equivalent.*

(A). $\phi^*_{+,\alpha}(f)\in\dot K^{\beta,p}_{\alpha,q}$.

(B). $M_{\phi,\alpha}(f)\in\dot K^{\beta,p}_{\alpha,q}$.

(C). $M_{\phi,\alpha,N}(f)\in\dot K^{\beta,p}_{\alpha,q}$.

(D). $G_{\alpha,N}(f) \in \dot{K}^{\beta,p}_{\alpha,q}$.

(E). There exist, for all $j \in \mathbb{N}\backslash\{0\}$, *an* (β, q)*-atom* a_j *and* $\lambda_j \in \mathbb{C}$, *such that* $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ *and* $f = \sum_{j=1}^{\infty} \lambda_j a_j$.

PROOF : 1. Theorem 4.1 gives $(D) \Leftrightarrow (E)$.

2. It is easily to verify $M_{\phi,\,\alpha,\,N}(f) \leq M_{\phi,\,\alpha}(f)$ and $M_{\phi,\,\alpha}(f) \leq \phi^*_{+,\,\alpha}(f)$, then $(C) \Rightarrow (B) \Rightarrow (A)$.

3. Since there is a real number C such that $M_{\phi,\,\alpha}(f) \leq CG_{\alpha,N}(f)$, thus $(D) \Rightarrow (B)$.

4. Now we prove that $(A) \Rightarrow (C)$. For $\epsilon > 0$ and $\psi \in \mathcal{S}(\mathbb{R})$, we define

$$M_\psi f(x) = \sup_{|y-x|<Nt<1/\epsilon} |f *_\alpha \psi_t(y)| \Big(\frac{Nt}{Nt+\epsilon}\Big)^l (1+\epsilon N|y|)^{-l},$$

$$T^{l,k}_\psi f(x) = \sup_{y\in\mathbb{R}, Nt<1/\epsilon} |f *_\alpha \psi_t(y)| \Big(\frac{Nt}{Nt+|y-x|}\Big)^k \Big(\frac{Nt}{Nt+\epsilon}\Big)^l (1+\epsilon N|y|)^{-l},$$

$$M^*_\psi f(x) = \sup_{|y-x|<Nt<1/\epsilon} t| \wedge_\alpha f *_\alpha \psi_t(y)| \Big(\frac{Nt}{Nt+\epsilon}\Big)^l (1+\epsilon N|y|)^{-l}.$$

It is clear from the equation (2.4), Proposition 2.2 and Lemma 2.6.

$$\begin{aligned}
M_\alpha(M^r_\psi f)(x) &= \sup_{t>0} \Big\{ \frac{1}{\mu_\alpha(]-t,t[)} \int_{\mathbb{R}} |M^r_\psi f(z) \tau_x(\chi_{]-t,t[}(-z)| d\mu_\alpha(z) \Big\} \\
&\geq \frac{1}{\mu_\alpha(I_{\sqrt{2}(|y-x|+Nt)})} \int_{\mathbb{R}} |M^r_\psi f(z) \tau_x \chi_{I_{\sqrt{2}(|y-x|+Nt)}}(-z)| d\mu_\alpha(z) \\
&\geq \frac{1}{\mu_\alpha(I_{\sqrt{2}(|y-x|+Nt)})} \int_{\mathbb{R}} |M^r_\psi f(z) \tau_y \chi_{I_{Nt}}(-z)| d\mu_\alpha(z) \\
&\geq \frac{1}{\mu_\alpha(I_{\sqrt{2}(|y-x|+Nt)})} \int_{\mathbb{R}_{sig(-y)}} |M^r_\psi f(z) \tau_y \chi_{I_{Nt}}(-z)| d\mu_\alpha(z).
\end{aligned}$$

Since $\mathrm{supp}(\tau_y \chi_{]-Nt,Nt[}) \cap \mathbb{R}_{sig(y)} \subset \{z/|z-y| \leq Nt\}$ then

$$M_\alpha(M^r_\psi f)(x) \geq \frac{(|f *_\alpha \psi_t(y)| \Big(\frac{Nt}{Nt+\epsilon}\Big)^l (1+\epsilon N|y|)^{-l})^r}{\mu_\alpha(I_{\sqrt{2}(|y-x|+Nt)})} \int_{\mathbb{R}_{sig(y)}} |\tau_y \chi_{I_{Nt}}(z)| d\mu_\alpha(z).$$

From Lemma 2.6 and Proposition 2.2, we have

$$\int_{\mathbb{R}_{sig(y)}} \tau_y \chi_{I_{Nt}}(z) d\mu_\alpha(z) \geq \frac{1}{2} \int_{\mathbb{R}_{sig(y)}} \tau_y \chi_{I_{Nt}}(z) d\mu_\alpha(z) + \frac{1}{2} \int_{\mathbb{R}_{sig(-y)}} \tau_y \chi_{I_{Nt}}(z) d\mu_\alpha(z)$$

$$= \frac{1}{2}\int_{\mathbb{R}} \tau_y \chi_{I_{Nt}}(z)d\mu_\alpha(z)$$
$$= \frac{1}{2}\int_{\mathbb{R}} \chi_{I_{Nt}}(z)d\mu_\alpha(z)$$
$$= \frac{1}{2}\mu_\alpha(I_{Nt}).$$

Then

$$\Big(M_\alpha(M_\psi^r f)(x)\Big)^{1/r} \geq \frac{\mu_\alpha(I_{Nt})^{1/r}}{\mu_\alpha(I_{|y-x|+Nt})^{1/r}}|f *_\alpha \psi_t(y)|\Big(\frac{Nt}{Nt+\epsilon}\Big)^l(1+\epsilon N|y|)^{-l}.$$

Then there is a constant C such that

$$T_\psi^{l,(2\alpha+1)/r} f(x) \leq C\Big(M_\alpha(M_\psi^r f)(x)\Big)^{1/r}. \tag{4.1}$$

We now prove that for any $r \in (0,1)$

$$M_\psi^* f(x) \leq C\Big(M_\alpha(M_\psi^r f)(x)\Big)^{1/r}. \tag{4.2}$$

We first note

$$t|\Lambda_\alpha(f *_\alpha \psi_t)(y)| = |f *_\alpha (\Lambda_\alpha\psi)_t(y)|.$$

From Lemma 2.8, we have

$$\begin{aligned}
|f *_\alpha (\Lambda_\alpha\psi)_t(y)| &= \Big|\int_{\mathbb{R}}\sum_{s=0}^{\infty} f *_\alpha \psi_{2^{-s}t}(y-tw)\tau_{y/t}\theta^{(s)}(y/t-w)d\mu_\alpha(y/t-w)ds\Big| \\
&\leq T_\psi^{k,l}f(x)\sum_{s=0}^{\infty}\int_{\mathbb{R}} 2^{sk}2^{sl}\Big(\frac{Nt+|y-tw-x|}{Nt}\Big)^k\Big(\frac{Nt+\epsilon}{Nt}\Big)^l \\
&\quad (1+\epsilon N|y-tw|)^l|\tau_{y/t}\theta^{(s)}(y/t-w)|d\mu_\alpha(y/t-w).
\end{aligned}$$

But, $(1+\epsilon N|y-tw|)^l \leq (1+\epsilon N|y|)^l(1+|w|)^l$ and $\Big(\frac{|y-tw-x|+Nt}{Nt}\Big)^k \leq 2^k(1+|w|)^k$ then

$$\begin{aligned}
|f *_\alpha (\Lambda_\alpha\psi)_t(y)| &\leq T_\psi^{k,l}f(x)\sum_{s=0}^{\infty}\int_{\mathbb{R}} 2^{s(l+k)}2^k(1+|w|)^{l+k}\Big(\frac{Nt+\epsilon}{Nt}\Big)^l \\
&\quad (1+\epsilon N|y|)^l|\tau_{y/t}\theta^{(s)}(y/t-w)|d\mu_\alpha(y/t-w).
\end{aligned}$$

Thus,

$$M_\psi^* f(x) \leq CT_\psi^{k,l}f(x)\sum_{s=0}^{\infty}2^{s(l+k)+k}\int_{\mathbb{R}}(1+|w|)^{l+k}|\tau_{y/t}\theta^{(s)}(y/t-w)|d\mu_\alpha(y/t-w).$$

Since
$$|\theta^{(s)}(x)| \leq \phi^{(s)}(x) = |\theta^{(s)}(x)| + |\theta^{(s)}(-x)|,$$
then from (2.1) we have
$$|\tau_z\theta^{(s)}(x)| \leq \tau_z\phi^{(s)}(x) + \hat{\tau}_z\phi^{(s)}(x), \tag{4.3}$$
where $\hat{\tau}_z\phi^{(s)}(x) = \int_{-1}^{1}\phi^{(s)}(\sqrt{x^2+z^2-2xzt})(1-t^2)^{\alpha-\frac{1}{2}}dt$.

Changing variable t by $-t$ and x by $-x$ in the integrals gives that
$$\int_{\mathbb{R}} f(x)\hat{\tau}_z\phi^{(s)}(x)d\mu_\alpha(x) = \int_{\mathbb{R}}(\frac{1}{2}(f(x)+f(-x))\tau_z\phi^{(s)}(x)d\mu_\alpha(x). \tag{4.4}$$

Then
$$\int_{\mathbb{R}}(1+|w|)^{l+k}|\tau_{y/t}\theta^{(s)}(y/t-w)|d\mu_\alpha(y/t-w)$$
$$\leq \int_{\mathbb{R}}(1+|w|)^{l+k}(\tau_{y/t}\phi^{(s)}(y/t-w)+\hat{\tau}_{y/t}\phi^{(s)}(y/t-w))d\mu_\alpha(y/t-w)$$
$$= 3/2\int_{\mathbb{R}}(1+|y/t-w|)^{l+k}\tau_{y/t}\phi^{(s)}(w)d\mu_\alpha(w) + 1/2\int_{\mathbb{R}}(1+|y/t+w|)^{l+k}\tau_{y/t}\phi^{(s)}(w)d\mu_\alpha(w).$$

From Proposition 2.2 and since for all $w \in \mathbb{R}$ and $h \in [-1,1]$
$$|y/t \pm w| \leq \sqrt{w^2+(y/t)^2-2hwy/t} + 2|y/t|,$$
we obtain
$$\begin{aligned}\int_{\mathbb{R}}(1+|w|)^{l+k}|\tau_{y/t}\theta^{(s)}(y/t-w)|d\mu_\alpha(y/t-w)&\\ \leq C\int_{\mathbb{R}}\tau_{y/t}\Big((1+|z|+2|y/t|)^{l+k}\phi^{(s)}(w)\Big)d\mu_\alpha(w)&\\ = C\int_{\mathbb{R}}(1+|w|+2|y/t|)^{l+k}\phi^{(s)}(w)d\mu_\alpha(w).&\end{aligned}$$

Then
$$M_\psi^* f(x) \leq CT_\psi^{k,l}f(x)\sum_{s=0}^{\infty}2^{s(l+k)+k}\int_{\mathbb{R}}(1+|w|+2|y/t|)^{l+k}\phi^{(s)}(w)d\mu_\alpha(w).$$

Noting that there is a constant C such that
$$(1+|w|+2|y/t|)^{l+k} \leq C\Big((1+|w|)^{l+k}+(1+2|y/t|)^{l+k}\Big),$$
$(1+2|y/t|)^{l+k}\phi^{(s)} \leq \rho_{l+k,n}(\phi^{(s)})$ and $\rho_{l+k,n}(\phi^{(s)}) \leq 2\rho_{l+k,n}(\theta^{(s)})$, then by Lemma 2.8 we have
$$M_\psi^* f(x) \leq CT_\psi^{k,l}f(x). \tag{4.5}$$

From equation (4.1) and (4.5), we deduce (4.2).

Let $E_\epsilon = \Big\{x \in \mathbb{R}/M^*_\psi f(x) \le C_0 M_\psi f(x)\Big\}$, $E^c_\epsilon = \mathbb{R} \setminus E_\epsilon$ and $p_1 \in (0,1)$ such that $0 \le p_1\beta < 1 - 1/q$. From Theorem 3.2 and the definition of E_ϵ, we get

$$\begin{aligned}\|M_\psi f \chi_{E^c_\epsilon}\|_{\dot{K}^{\beta p}_{\alpha,q}} &\le 1/C_0 \|M^*_\psi f\|_{\dot{K}^{\beta p}_{\alpha,q}}\\ &\le C/C_0 \|M\Big((M_\psi f)^{p_1}\Big)\|^{1/p_1}_{\dot{K}^{\beta p_1, p/p_1}_{\alpha,q/p_1}}\\ &\le C/C_0 \|(M_\psi f)^{p_1}\|^{1/p_1}_{\dot{K}^{\beta p_1, p/p_1}_{\alpha,q/p_1}}\\ &= C/C_0 \|(M_\psi f)^{p_1}\|_{\dot{K}^{\beta,p}_{\alpha,q}}.\end{aligned}$$

Therefore

$$\begin{aligned}\|M_\psi f\|_{\dot{K}^{\beta p}_{\alpha,q}} &\le \|M_\psi f\chi_{E_\epsilon}\|_{\dot{K}^{\beta p}_{\alpha,q}} + \|M_\psi f \chi_{E^c_\epsilon}\|_{\dot{K}^{\beta p}_{\alpha,q}}\\ &\le \|M_\psi f\chi_{E_\epsilon}\|_{\dot{K}^{\beta p}_{\alpha,q}} + C/C_0\|M_\psi f\|_{\dot{K}^{\beta p}_{\alpha,q}}.\end{aligned}$$

Taking C_0 such that $C/C_0 \le \frac{1}{2}$, we obtain

$$\|M_\psi f\|_{\dot{K}^{\beta p}_{\alpha,q}} \le 2\|M_\psi f\chi_{E_\epsilon}\|_{\dot{K}^{\beta p}_{\alpha,q}}.$$

Now we prove that

$$\|M_\psi f\|_{\dot{K}^{\beta p}_{\alpha,q}} \le CN^{1/r}\|\psi^*_{+,\alpha}(f)\|_{\dot{K}^{\beta p}_{\alpha,q}}$$

for any $r \in (0,1)$.

Let us first prove that if $x \in E_\epsilon$ then

$$M_\psi f(x) \le CN^{1/r}\Big(M\Big[\Big(\psi^*_{+,\alpha}(f)\Big)^r\Big](x)\Big)^{1/r}.$$

By the definition of $M_\psi(x)$, there exist $y \in \mathbb{R}$ and $t \in \mathbb{R}_+$ such that $|x-y| < Nt < 1/\epsilon$ and

$$|(f *_\alpha \psi_t)(y)|\Big(\frac{Nt}{Nt+\epsilon}\Big)^l (1+\epsilon N|y|)^{-l} \ge M_\psi f(x)/2.$$

If $x \in E_\epsilon$ and $|x-z| < Nt$ then

$$t|\Lambda_\alpha((f *_\alpha \psi_t)(z)| \le C_0\Big(\frac{Nt}{Nt+\epsilon}\Big)^{-l}(1+\epsilon N|y|)^l M_\psi f(x).$$

Thus if $x \in E_\epsilon$, $|x-y| < Nt$ and $|x-z| < Nt$ then there is a constant C_1

$$t|\Lambda_\alpha((f *_\alpha \psi_t)(z)| \le C_1|f *_\alpha \psi_t(y)|.$$

From Theorem 1 in [31] and the mean value theorem, there exists ζ between x and y such that $f(x) - f(y) = \Lambda_\alpha f(\zeta)(x-y)$.

It follows then, if $w \in [x-Nt,\ x+NT] \cap [y-t/2C_1,\ y+t/2C_1]$ there exists $z = \theta w + (1-\theta)y$, where $\theta \in (0,1)$ such that

$$|(f *_\alpha \psi_t)(w) - (f *_\alpha \psi_t)(y)| = |\Lambda_\alpha (f *_\alpha \psi_t)(z)||w-y| \leq |\frac{1}{2}(f *_\alpha \psi_t)(y)|.$$

Thus

$$|(f *_\alpha \psi_t)(w)| \geq \frac{1}{2}|(f *_\alpha \psi_t)(y)| \geq M_\psi f(x)/4.$$

For any $r \in (0,1)$, we have

$$\begin{aligned} M_\alpha\Big[\Big(\psi^*_{+,\alpha}(f)\Big)^r\Big](x) &\geq \frac{1}{\mu_\alpha(I_{\sqrt{2}Nt})}\int_{I_{\sqrt{2}Nt}} |\tau_x\Big(\psi^*_{+,\alpha}(f)\Big)^r|(w)d\mu_\alpha(w) \\ &\geq \frac{1}{\mu_\alpha(I_{\sqrt{2}Nt})}\int_{\mathbb{R}} |\Big(\psi^*_{+,\alpha}(f)\Big)^r|(w)\tau_x\Big(\chi_{I_{\sqrt{2}Nt}}\Big)(-w)|d\mu_\alpha(w). \end{aligned}$$

Let $\kappa > 0$ such that $\kappa \leq t/2C_1$ and $\kappa + |y-x| \leq Nt$, similar to Lemma 2.6, we get

$$\begin{aligned} \tau_y\Big(\chi_{I_\kappa}\Big) &\leq \tau_x\Big(\chi_{I_{\sqrt{2}(\kappa+|y-x|)}}\Big) \\ &\leq \tau_x\Big(\chi_{I_{\sqrt{2}NT}}\Big). \end{aligned}$$

Then

$$\begin{aligned} M_\alpha\Big[\Big(\psi^*_{+,\alpha}(f)\Big)^r\Big](x) &\geq \frac{1}{\mu_\alpha(I_{\sqrt{2}Nt})}\int_{\mathbb{R}} |\Big(\psi^*_{+,\alpha}(f)\Big)^r|(w)\tau_y\Big(\chi_{I_\kappa}\Big)(-w)|d\mu_\alpha(w) \\ &\geq \frac{C}{N}\Big(M_\psi f(x)\Big)^r. \end{aligned}$$

Choosing r sufficiently small so that $0 < r\beta < (1-1/q)$ then

$$\begin{aligned} \|M_\psi f \chi_{E_\epsilon}\|_{\dot{K}^{\beta p}_{\alpha,q}} &\leq CN^{1/r}\|\Big(M\Big(\psi^*_{+,\alpha}(f)\Big)^r\Big)^{1/r}\|_{\dot{K}^{\beta p}_{\alpha,q}} \\ &= CN^{1/r}\|M\Big(\psi^*_{+,\alpha}(f)\Big)^r\|^{1/r}_{\dot{K}^{\beta r, p/r}_{\alpha,q/r}} \\ &\leq CN^{1/r}\|\psi^*_{+,\alpha}(f)\|_{\dot{K}^{\beta p}_{\alpha,q}}. \end{aligned}$$

This finish the proof of $(A) \Longrightarrow (C)$ and we have

$$\|M_{\phi,\,\alpha,\,N}(f)(x)\|_{\dot{K}^{\beta,p}_{\alpha,q}} \leq CN^{1/r}\|\psi^*_{+,\alpha}(f)\|_{\dot{K}^{\beta p}_{\alpha,q}},$$

where r is sufficiently small and C is independent of N and f.

5. In the following we demonstrate $(A) \Longrightarrow (D)$, for this we define the function $\psi_s^{**}(x)$ by

$$\psi_s^{**}(x) = \sup_{(y,t)\in\mathbb{R}_+^2} |f *_\alpha \psi_t(y)| \Big(\frac{t}{t+|x-y|}\Big)^s.$$

By Lemma 2.8 we have,

$$\begin{aligned}
|f *_\alpha \Phi_t(y)| &\leq \sum_{k=0}^{\infty} |f *_\alpha \psi_{2^{-k}t} *_\alpha \eta_t^{(k)}(y)| \\
&\leq \sum_{k=0}^{\infty} \int_{\mathbb{R}} |f *_\alpha \psi_{2^{-k}t}(z)\tau_y\eta_t^{(k)}(-z)| d\mu_\alpha(z) \\
&\leq \sum_{k=0}^{\infty} \int_{\mathbb{R}} t^{-2(\alpha+1)} |f *_\alpha \psi_{2^{-k}t}(z)\tau_{y/t}\eta^{(k)}(-z/t)| d\mu_\alpha(z) \\
&\leq \sum_{k=0}^{\infty} \int_{\mathbb{R}} |f *_\alpha \psi_{2^{-k}t}(y-tw)\tau_{y/t}\eta^{(k)}(y/t-w)| d\mu_\alpha(y/t-w) \\
&\leq \psi_s^{**}(x) \sum_{k=0}^{\infty} \int_{\mathbb{R}} |\Big(\frac{2^{-k}t}{t+|y-tw-x|}\Big)^{-s} \tau_{y/t}\eta^{(k)}(y/t-w)| d\mu_\alpha(y/t-w).
\end{aligned}$$

Note that if $|y-x| \leq t$, then $\left(\frac{t+|y-tw-x|}{t}\right)^s \leq 2^s(1+w)^s$. By (4.3), (4.4) and similar to (4.5) we get if $N \geq s+1$

$$\begin{aligned}
|f *_\alpha \Phi_t(y)| &\leq C\psi_s^{**}(x) \sum_{k=0}^{\infty} 2^{(k+1)s} \int_{\mathbb{R}} \tau_{y/t}\Big((1+|z|+2|y/t|)^s\phi^{(k)}\Big)(w) d\mu_\alpha(w) \\
&\leq C\psi_s^{**}(x).
\end{aligned}$$

We obtain then

$$G_{\alpha,N}(f)(x) \leq C\psi_s^{**}(x).$$

It is easy to verify that

$$\psi_s^{**}(x) \leq M_{\psi,\alpha}(f)(x) + \sum_{k=0}^{\infty} 2^{-ks} M_{\psi,\alpha,2^{k+1}}(f)(x).$$

Therefore we have

$$\begin{aligned}
\|G_{\alpha.N}(f)\|_{\dot{K}_{\alpha,q}^{\beta,p}} &\leq C\|M_{\psi,\alpha}(f)\|_{\dot{K}_{\alpha,q}^{\beta,p}} + C\sum_{k=0}^{\infty} 2^{-ks} \|M_{\psi,\alpha,2^{k+1}}(f)\|_{\dot{K}_{\alpha,q}^{\beta,p}} \\
&\leq C\sum_{k=0}^{\infty} 2^{-k(s-1/r)} \|\psi_{+,\alpha}^*(f)\|_{\dot{K}_{\alpha,q}^{\beta p}} \\
&\leq C\|\psi_{+,\alpha}^*(f)\|_{\dot{K}_{\alpha,q}^{\beta p}}
\end{aligned}$$

if $s > 1/r$. This finishes the proof of $(A) \Longrightarrow (D)$ and hence the proof of Theorem 4.2. $\square$

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