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ORTHOGONAL DOUBLE COVERS OF CIRCULANT GRAPHS BY CORONA PRODUCT OF CERTAIN INFINITE GRAPH CLASSES

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The existence of an orthogonal labelling of a graph G with respect to a certain group implies the existence of the cyclic orthogonal double cover of the Circulant graphs on that group. In this article, a technique for orthogonal labelling is produced for the corona product of two finite or infinite graph classes such as path, cycle and star graphs. In addition, the nonexistence of the orthogonal L-labelling is proved for the corona product of K_2 and an infinite cycle.

Key words : Corona product; CODC; orthogonal labelling; circulant graph.

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1. INTRODUCTION

All groups used here are finite. All graphs considered here are undirected and simple graphs $G(V,\ E)$. The order and the size of G are $|V(G)|$ and $|E(G)|$, respectively.

From here on, the usual notations shall be used: K_n for the complete graph of order n, K_1 for an isolated vertex, C_m for a cycle graph of order and size m, S_m for a star graph of order $m+1$ and size m and P_{n+1} for a path graph of order $n+1$ and size n. For notations and definitions not defined here, the reader can refer to [1].

Let G_1 and G_2 be graphs with orders (n_1 and n_2) and sizes (m_1 and m_2), respectively. The corona $G_1 \circ G_2$ is defined as the graph produced by holding one copy of G_1 and n_1 copies of G_2, and then linking the i^{th} vertex of G_1 with an edge to each vertex in the i^{th} copy of G_2 (see e.g. [5] or [9]). It follows from the definition of the corona that the order of $G_1 \circ G_2$ equal to $n_1 + n_1 n_2$ and its size is equal to $m_1 + n_1 m_2 + n_1 n_2$. Figure 1 shows $C_3 \circ K_1$

A collection $\mathcal{C} = \{\mathcal{G}_\S : \S \in \mathcal{V}(\mathcal{X})\}$ of subgraphs of X, all isomorphic to G, is called an *orthogonal double cover* (ODC) of X by G if there is a bijective mapping $\psi : V(X) \to \mathcal{C}$:

(1) every edge of X is found in exactly two members of $\mathcal{C}$ and

(2) for any different vertices x, y from $V(X)$:

$$|\psi(x) \cap \psi(y)| = \begin{cases} 1 \text{ if } \{x,\ y\} \in E(X), \\ 0 \text{ otherwise.} \end{cases}$$

This concept and its earlier version, an ODC for K_n and $K_{n,n}$, have been investigated by several researcher (see the survey [7]). The main question is: what is the pair $(X,\ G)$ for which there is an ODC of X by G?

If the automorphism group (the automorphism group of the graph G) of an ODC, $\mathcal{C} = \{\mathcal{G}_\S : \S \in \mathcal{V}(\mathcal{X})\}$, of X contains a cyclic group with order equal to that of X, then it is called cyclic (CODC) [6].

For combinatorial reasons, the graph X must be regular and its degree is equal to the size of G (see [7]).

Let $L = \{l_1,\ l_2,\ ...,\ l_k\}$ be a sequence of positive integers such that $1 \leq l_1 < l_2 < \cdots < l_k \leq \lfloor n/2 \rfloor$, let X be the circulant graph $Circ(n;\ L)$, then$V(X) = \mathbb{Z}_n = \{0,\ 1,\ 2,\ ...,\ n-1\}$; the edge $\{x,\ y\} \in E(X)$ if and only if $x - y = \pm l_h \pmod{n}$ for some $h,\ h \in \{1,\ 2,\ ...,\ k\}$. For an edge $\{x,\ y\}$ in $Circ(n;\ L)$, the length of $\{x,\ y\}$ is $\min\{|x-y|,\ n-|x-y|\}$ (see e.g. [11]).

Let $e_1 = \{x_1,\ y_1\}$ and $e_2 = \{x_2,\ y_2\}$ be two edges of the same length l in $Circ(n;\ L)$, we define the rotation distance $rd(l)$ between e_1 and e_2 as $rd(l) = \min\{d_1,\ d_2 : \{x_1 + d_1, y_1 + d_1\} = e_2,\ \{x_2 + d_2,\ y_2 + d_2\} = e_1\}$. Note that the edges e_1 and e_2 are adjacent if $rd(l) = l$ and they are non adjacent if $rd(l) \neq l$.

A graph labelling is a giving of values to the vertices or edges, or both, to satisfy certain conditions. There are many contributions and different kinds of labelling (see e.g. [2, 8, 12]).

In [6], considering the graph $K_n = Circ(n;\ \{1,\ 2,\ 3,\ ...,\ \lfloor n/2 \rfloor\}$, the notion of *orthogonal labelling* has been defined as follows.

Let $G = (V, E)$ with $n \geq 1$, a bijection mapping $\varphi : V \to \mathbb{Z}_n$ is an orthogonal labelling of G if: (a) for every $l \in \{1, 2, \ldots, \lfloor n-1/2 \rfloor\}$, G has exactly two edges of length l, and single edge of length $n/2$ if n is even, and (b) $\{rd(l) : l \in \{1, 2, \ldots, \lfloor n-1/2 \rfloor\}\} = \{1, 2, \ldots, \lfloor n-1/2 \rfloor\}$.

In [6], Gronau *et al*. have proved the following theorem.

***Theorem* 1.1** — *([6]). A CODC of $K_n = Circn;\ \{1, 2, 3, ..., \lfloor n/2 \rfloor\}$ by a graph G exists if and only if there exists an orthogonal labelling of G.*

The orthogonal labelling has been called orthogonal $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$-labelling by Sampathkumar *et al*. in [13, 14] and they have generalized it to orthogonal L-labelling where $L = \{l_1, l_2, ..., l_k\}$ such that $1 \leq l_1 < l_2 <\cdots< l_k \leq \lfloor n/2 \rfloor$.

I. If n is odd or n is even and $l_k \neq n/2$:

Assume G is a subgraph of $Circ(n;\ L)$ with size $2k$, a labelling of G, from $\mathbb{Z}_n$, is an orthogonal L-labelling of G if: (1) for every $l \in L$, there exist exactly two edges of length l in G, and (2) $\{rd(l) : l \in L\} = L$.

II. If n is even and $l_k = n/2$.

Assume G is a subgraph of $Circ(n;\ L)$ with $2\,|L| - 1$ edges, a labelling of G, from $\mathbb{Z}_n$, is an orthogonal L-labelling of G if: (1) for every $l \in L - \{n/2\}$, there exist exactly two edges of length l and exactly a single edge with length $n/2$ in G, and (2) $\{rd(l) : l \in L \setminus \{n/2\}\} = L \setminus \{n/2\}$.

In [14], Sampathkumar *et al*. have proved the following theorem.

***Theorem* 1.2** — *(14). A CODC of Circ$(\mathbb{Z}_n;\ L)$ by a graph G exists if and only if there exists an orthogonal L-labelling of G.*

For any graph G, if G has an orthogonal L-labelling, then the CODCs of $Circ(\mathbb{Z}_n;\ L)$ by a graph G are constructed effectively via the idea of translating the graph G by a group acting on $V(G)$.

In [10], an ODC of a graph of small degree, X , by G have been produced by Hartmann *et al*. and by Scapellato, El-Shanawany and Higazy in [15].

The existence of an ODC of Circulants of degree 2 has been proved in [10] and of Circulants of degree 3 in [15]. Using the technique presented in Theorem 1.2, Sampathkumar *et al*. [14] have constructed the pairs $(X,\ G)$ where X is 4-regular circulant graph, and in [3, 4], the CODCs of circulants by some infinite graph classes have been produced. Applying Theorem 1.2, we produce a technique to construct the CODCs of circulants of infinite orders by the corona of two graphs.

In this paper, the main task is to show that the corona of two finite or infinite graph classes has an orthogonal L-labelling which imply the existence of the CODCs of the corresponding Circulant graphs. The following theorem is our main result.

***Theorem* 1.3** — *For $m \geq 1$, let $G \in \{K_3 \circ K_1,\ K_3 \circ K_2,\ C_4 \circ K_2,\ K_1 \circ P_{m+1},\ K_1 \circ C_m,\ K_1 \circ S_m,\ K_2 \circ P_{m+1},\ P_3 \circ P_{m+1},\ K_3 \circ P_{m+1},\ P_{m+1} \circ K_2\}$ and $X = Circ(\mathbb{Z}_n;\ L)$. All the tuples $(n,\ L,\ G,\ X)$, for which there exists a CODC of X by G, are known.*

PROOF : For $s = 1$ and 2, let G_s be two graphs with $(\text{order}(G_s),\ \text{size}(G_s)) = (n_s,\ m_s)$. Then from the definition of the corona $G_1 \circ G_2$, the $(\text{order}(G_1 \circ G_2),\ \text{size}(G_1 \circ G_2)) = (n_1 + n_1 n_2,\ m_1 + n_1 n_2 + n_1 m_2)$.

The graph X is $m_1 + n_1 n_2 + n_1 m_2$-regular $Circ(\mathbb{Z}_n;\ L)$, where $n \geq n_1 + n_1 n_2$.

The size of the graph G is $m_1 + n_1 n_2 + n_1 m_2$.

In Section 2, $3,\ 4$ and 5, the proof of the above result is obtained through Theorem 1.2. Namely, for each G we shall define the pairs $(n,\ L)$ whose corresponding graph has an orthogonal L-labelling by G. Also, the nonexistence of the orthogonal L-labelling of the corona product of K_2 and an infinite cycle is proved in Section 6. □

2. THE ORTHOGONAL L-LABELLING OF THE CORONA OF CERTAIN SMALL GRAPHS

In this section, we shall construct the orthogonal L-labelling of $C_3 \circ K_1$, $C_3 \circ K_2$ and $C_4 \circ K_2$.

Lemma 2.1 — Let l_1, l_2, l_3 and $n \geq 7$ be integers such that $n \neq 2l_1 + l_3$ and $n \neq 2l_1 - l_2 - \dot{l}_3$ and let $n - (l_1 - l_3) = l_2$. There exists a CODC of 6-regular $Circ(\mathbb{Z}_\ltimes;\{l_1,\ l_2,\ l_3\})$ by $C_3 \circ K_1$.

PROOF : The two edges of length l_1 are $\{0,\ l_1\}$ and $\{0,\ n - l_1\}$, the two edges of length l_2 are $\{0, -l_2\}$ and $\{-l_2, -2l_2\}$ and the two edges of length l_3 are $\{l_1, l_1 + l_3\}$ and $\{l_1 - l_3, l_1\}$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in L\} = L$ where $L = \{l_1,\ l_2,\ l_3\}$. Hence $C_3 \circ K_1$ has an orthogonal $\{l_1,\ l_2,\ l_3\}$-labelling. For illustration, see Figure 1.

Lemma 2.2 — Let $n \geq 13$ be a positive integer. There exists a CODC of 12-regular $Circ(\mathbb{Z}_n; \{l_1,\ 2l_1,\ 3l_1,\ 4l_1,\ 6l_1,\ n - 8l_1\})$ by $K_3 \circ K_2$.

PROOF : $K_3 \circ K_2$ can be labelled as follows.

Let $(l_1,\ l_2,\ l_3,\ l_4,\ l_5,\ l_6) = (l_1,\ 2l_1,\ 3l_1,\ 4l_1,\ 6l_1,\ n - 8l_1)$.

The two edges of length l_1 are $\{0\ ,\ l_1\}$ and $\{n - l_1,\ 0\}$; the two edges of length l_2 are $\{n - l_1\ ,\ l_1\}$

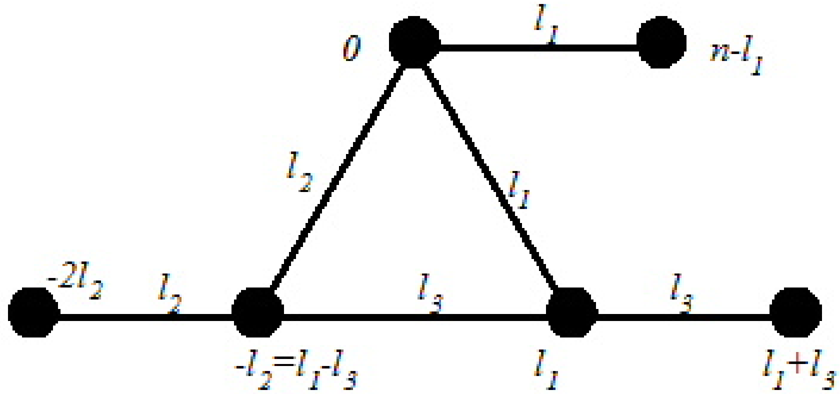

Figure 1: An orthogonal $\{l_1,\ l_2,\ l_3\}$-labelling of $K_3 \circ K_1$.

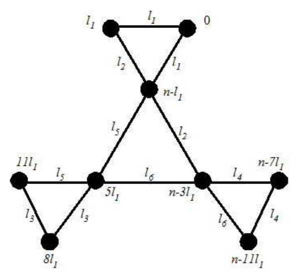

Figure 2: An orthogonal $\{l_1,\ 2l_1,\ 3l_1,\ 4l_1,\ 6l_1,\ n-8l_1\}$-labelling of $C_3 \circ K_2$.

and $\{n-l_1,\ n-3l_1\}$; the two edges of length l_3 are $\{5l_1\ ,\ 8l_1\}$ and $\{8l_1,\ 11l_1\}$; the two edges of length l_4 are $\{n-3l\ _1,\ n-7l_1\}$ and $\{n-7l_1,\ n-11l_1\}$; the two edges of length l_5 are $\{5l_1\ ,\ 11l_1\}$ and $\{n-l_1,\ 5l_1\}$ and the two edges of length l_6 are $\{n-3l_1,\ n-11l_1\}$ and $\{n-3l_1,\ 5l_1\}$.

Because all couples of edges that equally lengthed are connected, then $\{rd(l) : l \in L\} = L$ where $L = \{l_1,\ 2l_1,\ 3l_1,\ 4l_1,\ 6l_1,\ n-8l_1\}$. Hence $K_3 \circ K_2$ has an orthogonal $\{l_1,\ 2l_1,\ 3l_1,\ 4l_1,\ 6l_1,\ n-8l_1\}$-labelling. For illustration, see Figure 2.

Lemma 2.2 — Let $l_1 \geq 1$ be a positive integer and n is chosen such that

$$128l_1 \equiv 26l_1 (\mathrm{mod}\ n).$$

Thete exists a CODC of 16-regular $Circ(\mathbb{Z}_n;\ \{2^x l_1 : 0 \leq x \leq 7\})$ by $C_4 \circ K_2$.

PROOF : Let $(l_1,\ l_2,\ l_3,\ l_4,\ l_5,\ l_6,\ l_7) = (l_1,\ 2l_1,\ 4l_1,\ 8l_1,\ 16l_1,\ 32l_1,\ 64l_1,\ 128l_1)$.

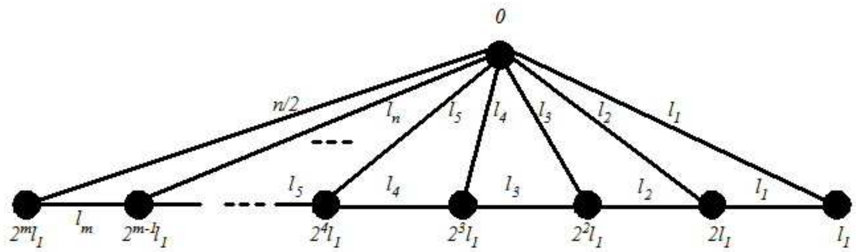

Figure 3: An orthogonal $\{2^x l_1,\ n/2 : 0 \leq x \leq m-1\}$-labelling of $K_1 \circ P_{m+1}$ where $n = 2^{m+1} l_1$.

The two edges of length l_1 are $\{0\,,\ l_1\}$ and $\{l_1,\ 2l_1\}$; the two edges of length l_2 are $\{0\,,\ 2l_1\}$ and $\{0,\ n-2l_1\}$; the two edges of length l_3 are $\{n-2l_1\,,\ n-6l_1\}$ and $\{n-6l_1,\ n-10l_1\}$; the two edges of length l_4 are $\{n-2l\,_1,\ n-10l_1\}$ and $\{n-2l_1,\ 6l_1\}$; the two edges of length l_5 are $\{6l_1\,,\ 22l_1\}$ and $\{22l_1,\ 38l_1\}$ and the two edges of length l_6 are $\{n-26l_1,\ n-90l_1\}$ and $\{n-90l_1,\ 154l_1\}$; the two edges of length l_7 are $\{n-154l_1,\ n-26l_1\}$ and $\{n-26l_1,\ 0\}$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in L\} = L$ where $L = \{l_1,\ 2l_1,\ 4l_1,\ 8l_1,\ 16l_1,\ 32l_1,\ 64l_1,\ 128l_1\}$. Hence $C_4 \circ K_2$ has an orthogonal $\{2^x l_1 : 0 \leq x \leq 7\}$-labelling.

3. The Orthogonal L-labelling of the Corona of K_1 and Infinite: Path, Cycle and Star Graphs

In this section, we shall construct the orthogonal L-labelling of $K_1 \circ P_{m+1},\ K_1 \circ C_m$ and $K_1 \circ S_m$.

Proposition 3.1 — Let $m,\ l_1 \geq 1$ and $n = 2^{m+1} l_1$ be positive integers, There exists a CODC of $2m+1$-regular Circ$(\mathbb{Z}_n;\ \{2^x l_1,\ n/2 : 0 \leq x \leq m-1\})$ by $K_1 \circ P_{m+1}$.

Proof : For all $0 \leq x \leq m-1$, the two edges of length $2^x l_1$ are $\{0\,,\ 2^x l_1\}$ and $\{2^x l_1, 2^{x+1} l_1\}$ and the edge of length $n/2$ is $\{0,\ 2^m l_1\}$

Because all couples of equal length edges are connected, then $\{rd(l) : l \in \{2^x l_1 : 0 \leq x \leq m-1\}\} = L$. Hence $K_1 \circ P_{m+1}$ has an orthogonal $\{2^x l_1,\ n/2 : 0 \leq x \leq m-1\}$-labelling. For illustration, see Figure 3.

Proposition 3.2 — Let $m \geq 3$, $l_1 \geq 1$ and n, such that $2^m l_1 \equiv l_1 (\mathrm{mod} n)$, be positive integers. There exists a CODC of $2m$-regular Circ$(\mathbb{Z}_n;\ \{2^x l_1 : 0 \leq x \leq m-1\})$ by $K_1 \circ C_m$.

Proof : For all $0 \leq x \leq m-1$, the two edges of length $2^x l_1$ are $\{0\,,\ 2^x l_1\}$ and $\{2^x l_1, 2^{x+1} l_1\}$.

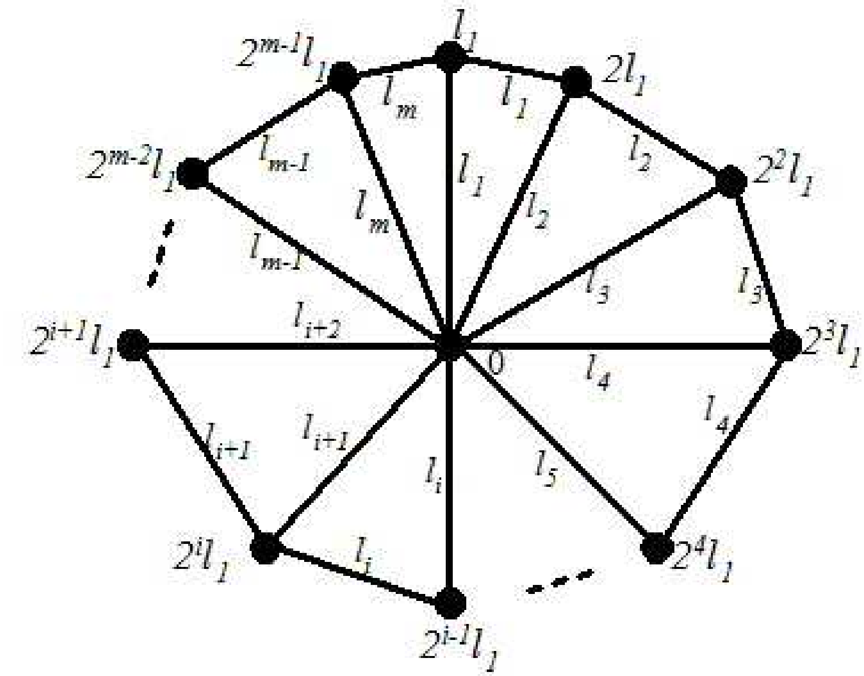

Figure 4: An orthogonal $\{2^x l_1 : 0 \le x \le m-1\}$-labelling of $K_1 \circ C_m$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in L\} = L$. Hence $K_1 \circ C_m$ has an orthogonal $\{2^x l_1 : 0 \le x \le m-1\}$-labelling. For illustration, see Figure 4.

Proposition 3.3 — Let $m \ge 2$ and $n \ge 2m+2$ be positive integers. There exists a CODC of $(2m+1)$-regular Circ$(\mathbb{Z}_n;\ \{l_i,\ n/2 : 1 \le i \le m\})$ by $K_1 \circ S_m$.

PROOF : *Case* 1. m is even.

Assume $L = \{l_x,\ n/2 - l_x,\ n/2 : 0 \le x \le m/2\}$, let $\{l_x,\ n/2 - l_x : 0 \le x \le m/2\} \subseteq \{1,\ 2,\ 3,\ \ldots, n/2-1\}$, the two edges of length l_x are $\{0,\ l_x\}$ and $\{0,\ n/2 - l_x\}$; the two edges of length $n/2 - l_x$ are $\{n/2,\ l_x\}$ and $\{n/2,\ -l_x\}$ and the edge of length $n/2$ is $\{0,\ n/2\}$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in \{l_x,\ n/2 - l_x : 0 \le x \le m/2\}\} = \{l_x,\ n/2 - l_x : 0 \le x \le m/2\}$. Hence $K_1 \circ S_m$ has an orthogonal $\{l_x,\ n/2 - l_x,\ n/2 : 0 \le x \le m/2\}$-labelling. For illustration, see Figure 5.

Case 2. m is odd.

Assume $L = \{l_x,\ n/2 - l_x,\ n/4,\ n/2 : 0 \le x \le (m-1)/2\}$, let $\{l_x : 0 \le x \le (m-1)/2\} \subseteq$

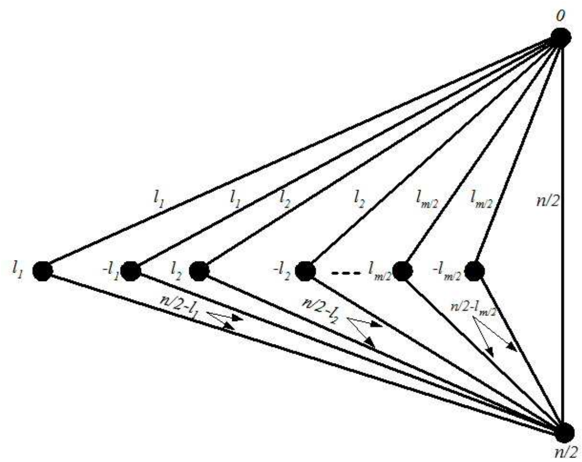

Figure 5: An orthogonal $\{l_x,\ n/2-l_x,\ n/2 : 0 \leq x \leq m/2\}$-labelling of $K_1 \circ S_m$ where m is even, $n \geq 2m+2$.

$\{1,\ 2,\ 3,\ \ldots, n/4-1\}$, the two edges of length l_x are $\{0,\ l_x\}$ and $\{0,\ n/2-l_x\}$; the two edges of length $n/2-l_x$ are $\{n/2,\ l_x\}$ and $\{n/2,\ -l_x\}$; the two edges of length $n/4$ are $\{0,\ n/4\}$ and $\{n/4,\ n/2\}$ and the edge of length $n/2$ is $\{0,\ n/2\}$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in \{l_x,\ n/2-l_x,\ n/4 : 0 \leq x \leq (m-1)/2\} = \{l_x,\ n/2-l_x,\ n/4 : 0 \leq x \leq (m-1)/2\}$, Hence $K_1 \circ S_m$ has an orthogonal $\{l_x,\ n/2-l_x,\ n/4,\ n/2 : 0 \leq x \leq (m-1)/2\}$-labelling. For illustration, see Figure 6.

4. The Orthogonal L-labelling of the Corona of K_1 , P_3, $C_3\}$-Graphs and Infinite Path Graphs

In this section, we shall construct the orthogonal L-labelling of $K_2 \circ P_{m+1}$, $P_3 \circ P_{m+1}$ and $C_3 \circ P_{m+1}$.

Proposition 4.1 — Let $l_1 \geq 1,\ m \geq 1$ and $n = 2^{2m+2} l_1$ be positive integers. There exists a CODC of $4m+3$-regular Circ$(\mathbb{Z}_n;\ \{2^x l_1,\ n/2 : 0 \leq x \leq 2m+1\})$ by $K_2 \circ P_{m+1}$.

Proof : For $0 \leq x \leq m-1$, the two edges of length $2^x l_1$ are $\{0,\ 2^x l_1\}$ and $\{2^x l_1,\ 2^{x+1} l_1\}$; the two edges of length $2^m l_1$ are $\{0,\ 2^m l_1\}$ and $\{0,\ -2^m l_{1\}}$, for $1 \leq y \leq m$; the two edges of length $2^{m+y} l_1$ are $\{-2^m l_1,\ -(2^y+1)2^m l_1\}$ and $\{-(2^y+1)2^m l_1,\ -(2^{y+1}+1)2^m l_1\}$ and the edge of length $n/2$ is $\{-2^m l_1,\ -(2^{m+1}+1)2^m l_1\}$.

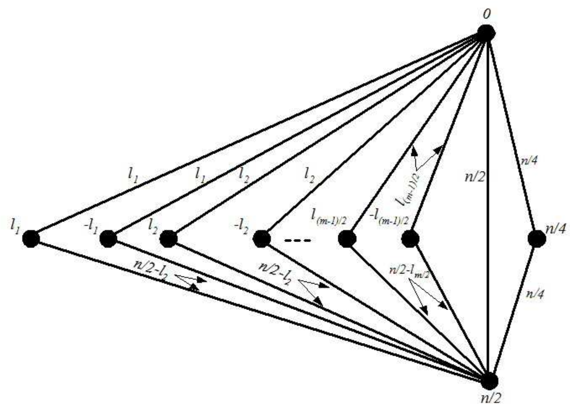

Figure 6: An orthogonal $\{l_x,\ n/2-l_x,\ n/4,\ n/2 : 0 \leq x \leq (m-1)/2\}$-labelling of $K_1 \circ S_m$ where m is odd, $n \geq 2m+2$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in \{2^x l_1 : 0 \leq x \leq 2m+1\}\} = \{2^x l_1 : 0 \leq x \leq 2m+1\}$. Hence $K_2 \circ P_{m+1}$ has an orthogonal $\{2^x l_1,\ n/2 : 0 \leq x \leq 2m+1\}$-labelling, see Figure 7.

Proposition 4.2 — Let $l_1 \geq 1,\ m \geq 1$ and $n = 2^{3m+3} l_1$ be positive integers. There exists a CODC of $6m+5$-regular Circ$(\mathbb{Z}_n;\ \{2^x l_1,\ n/2 : 0 \leq x \leq 3m+2\})$ by $P_3 \circ P_{m+1}$.

PROOF : For $0 \leq x \leq m-1$, the two edges of length $2^x l_1$ are $\{0,\ 2^x l_1\}$ and $\{2^x l_1,\ 2^{x+1} l_1\}$; the two edges of length $2^m l_1$ are $\{0,\ 2^m l_1\}$ and $\{0,\ -2^m l_{1}\}$, for $0 \leq i \leq m-1$; the two edges of length $2^{m+1+i} l_1$ are $\{-2^m l_1,\ -(2^{i+1}+1)2^m l_1\}$ and $\{-(2^{i+1}+1)2^m l_1,\ -(2^{i+2}+1)2^m l_1\}$; the two edges of length $2^{2m+1} l_1$ are $\{-2^m l_1,\ -(2^{m+1}+1)2^m l_1\}$ and $\{-2^m l_1,\ (2^{m+1}-1)2^m l_1\}$; for $0 \leq j \leq m-1$, the two edges of length $2^{2m+2+j} l_1$ are $\{(2^{m+1}-1)2^m l_1,\ (2^{m+1}-1+2^{m+2+j})2^m l_1\}$ and $\{(2^{m+1}-1+2^{m+2+j})2^m l_1,\ (2^{m+1}-1+2^{m+2+j+1})2^m l_1\}\}$; the edge of length $n/2 = 2^{3m+2}$ is $\{(2^{m+1}-1)2^m l_1,\ (2^{m+1}-1+2^{2m+2})2^m l_1\}$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in \{2^x l_1 : 0 \leq x \leq 3m+2\}\} = \{2^x l_1 : 0 \leq x \leq 3m+2\}$. Hence $P_3 \circ P_{m+1}$ has an orthogonal $\{2^x l_1,\ n/2 : 0 \leq x \leq 3m+2\}$-labelling, see Figure 8.

Conjecture 4.3 — There exists a tuple $(l_1,\ m,\ n,\ k)$ of integers for which there exists a CODC of $(2mn+2m+2n+1)$-regular Circ$(\mathbb{Z}_k;\ \{2^x l_1,\ k/2 : 0 \leq x \leq mn+m+n\})$ by $P_{n+1} \circ P_{m+1}$.

Proposition 4.4 — Let $l_1 \geq 1,\ m \geq 1$ and $n > 6m+6$, such that $2^{3m+2} l_1 \equiv (2^{m+1} -$

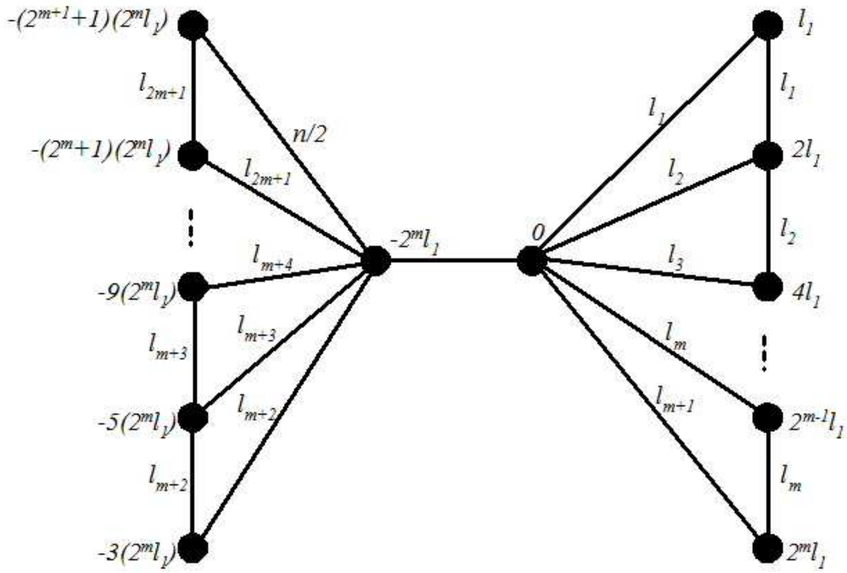

Figure 7: An orthogonal $\{2^x l_1,\ n/2 : 0 \leq x \leq 2m+1\}$-labelling of $K_2 \circ P_{m+1}$.

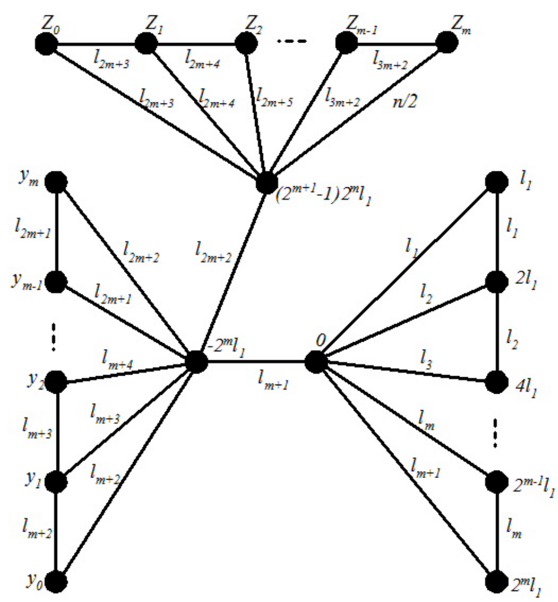

Figure 8: An orthogonal $\{2^x l_1,\ n/2 : 0 \leq x \leq 3m+2\}$-labelling of $P_3 \circ P_{m+1}$, where for $0 \leq i,\ j \leq m-1 :\ y_i = -(2^{i+1}+1)2^m l_1$ and $z_i = (2^{m+1} - 1 + 2^{m+2+j})2^m l_1$.

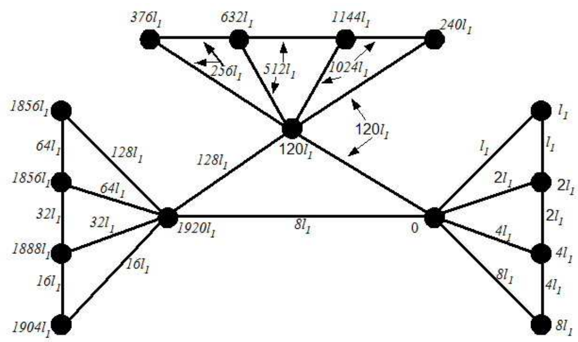

Figure 9: An orthogonal $\{2^x l_1 : 0 \le x \le 11\}$-labelling of $K_3 \circ P_4$ with respect to $\mathbb{Z}_{1928}$.

$1)2^m l_1(\bmod n)$, be positive integers. There exists a CODC of $6m+6$-regular Circ$(\mathbb{Z}_n;\ \{2^x l_1 : 0 \le x \le 3m+2\})$ by $K_3 \circ P_{m+1}$.

PROOF : For all $0 \le x \le m-1$, the two edges of length $2^x l_1$ are $\{0,\ 2^x l_1\}$ and $\{2^x l_1,\ 2^{x+1} l_1\}$; the two edges of length $2^m l_1$ are $\{0,\ 2^m l_1\}$ and $\{0,\ -\ 2^m l_1\}$, for all $1 \le y \le m$, the two edges of length $2^{m+y} l_1$ are $\{-2^m l_1,\ -(2^y+1)2^m l_1\}$ and $\{-(2^y+1)2^m l_1,\ -(2^{y+1}+1)2^m l_1\}$; the two edges of length $2^{2m+1} l_1$ are $\{-(2^{m+1}+1)2^m l_1,\ -2^m l_1\}$ and $\{-2^m l_1,\ (2^{m+1}-1)2^m l_1\}$; for all $1 \le z \le m$, the two edges of length $2^{2m+z+1} l_1$ are $\{(2^{m+1}-1)2^m l_1,\ \left((2^z+1)2^{m+1}-1\right)2^m l_1\}$ and $\{\left((2^z+1)2^{m+1}-1\right)2^m l_1,\ \left((2^{z+1}+1)2^{m+1}-1\right)2^m l_1\}$ and the two edges of length $2^{3m+2} l_1$ are $\{(2^{m+1}-1)2^m l_1,\ \left((2^{m+1}+1)2^{m+1}-1\right)2^m l_1\}$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in \{2^x l_1 : 0 \le x \le 3m+2\}\} = \{2^x l_1 : 0 \le x \le 3m+2\}$. Hence $K_3 \circ P_{m+1}$ has an orthogonal $\{2^x l_1 : 0 \le x \le 3m+2\}$-labelling. For illustration, see Figure 9 in which $K_3 \circ P_4$ has an orthogonal $\{2^x l_1 : 0 \le x \le 11\}$-labelling with respect to $\mathbb{Z}_{1928}$.

5. THE ORTHOGONAL L-LABELLING OF THE CORONA OF AN INFINITE PATH AND K_2

Proposition 5.1 — Let $m \ge 3$ be an integer. Let $\{x_i,\ n = 4x_m,\ y_j = -2\sum_{i=1}^{j} x_i \ (\bmod n) : 0 \le i \le m,\ 1 \le j \le m\}$ be positive integers such that $\{0,\ x_0,\ 2x_0,\ y_i,\ y_i + x_{i+1},\ y_i + 2x_{i+1} : 1 \le i \le m\}$ be a set. There exists a CODC of $4m+3$-regular Circ$(\mathbb{Z}_n;\ \{x_i,\ 2x_i,\ x_m,\ n/2 : 0 \le i \le m-1\})$ by $P_{m+1} \circ K_2$.

PROOF : Let $y_0 = 0$. For all $0 \le i \le m-1$, the two edges of length x_i are $\{y_i,\ y_i + x_i\}$ and $\{y_i + x_i,\ y_i + 2x_i\}$; the two edges of length $2x_i$ are $\{y_i + 2x_i,\ y_i\}$ and $\{y_i,\ y_i - 2x_i\}$; the

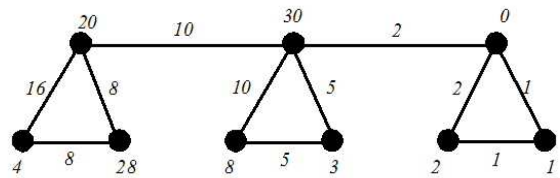

Figure 10: An orthogonal $\{1,\ 2,\ 5,\ 10,\ 8,\ 16\}$-labelling of $P_3 \circ K_2$ with respect to $\mathbb{Z}_{32}$.

two edges of length x_m are $\{y_m,\ y_m + x_m\}$ and $\{y_m + x_m,\ y_m + 2x_m\}$; the edge of length $n/2$ is $\{y_m + 2x_m,\ y_m\}$.

Because all couples of equal length edges are connected, then $\{rd(l) : l \in \{x_i,\ 2x_i,\ x_m : 0 \leq i \leq m-1\}\} = \{x_i,\ 2x_i,\ x_m : 0 \leq i \leq m-1\}$. Hence $P_{m+1} \circ K_2$ has an orthogonal $\{x_i,\ 2x_i,\ x_m,\ n/2 : 0 \leq i \leq m-1\}$-labelling. For illustration, see Figure 10 in which $P_3 \circ K_2$ has an orthogonal $\{1,\ 2,\ 5,\ 10,\ 8,\ 16\}$-labelling with respect to $\mathbb{Z}_{32}$.

6. Nonexistence of the Orthogonal L-labelling of $K_2 \circ C_m$

Proposition 6.1 — Let n, m and $l_1 \geq 1$ be integers such that $n > 4m+1$. There is no CODC of $4m+1$-regular Circ$(\mathbb{Z}_n; \{2^x l_1,\ n\ /2 : 0 \leq x \leq 2m-1\})$ by $K_2 \circ C_m$.

Proof : Conversely, assume that there is a CODC of Circ$(\mathbb{Z}_n; \{2^x l_1,\ n/2 : 0 \leq x \leq 2m-1\})$ by $K_2 \circ C_m$.

Then let $K_2 \circ C_m$ has the following orthogonal $\{2^x l_1,\ n/2 : 0 \leq x \leq 2m-1\}$-labelling.

The edge of length $n/2$ is $\{0,\ n/2\}$; for $0 \leq x \leq m-2$,the two edges of length $2^x l_1$ are $\{0,\ 2^x l_1\}$ and $\{2^x l_1,\ 2^{x+1} l_1\}$; the two edges of length $2^{m-1} l_1$ are $\{0,\ 2^{m-1} l_1\}$ and $\{l_1,\ 2^{m-1} l_1\}$ and this imply that $(2^{m-1} - 1) l_1 \equiv 0(\mathrm{mod} n)$; for $m \leq x \leq 2m-2$, the two edges of length $2^x l_1$ are $\{n/2,\ n/2 + 2^x l_1\}$ and $\{\ n/2 + 2^x l_1,\ n/2 + 2^{x+1} l_1\}$; the two edges of length $2^{2m-1} l_1$ are $\{n/2,\ n/2 + 2^{2m-1} l_1\}$ and $\{n/2 + 2^m l_1,\ n/2 + 2^{2m-1} l_1\}$ and this imply that $(2^{2m-1} - 1) l_1 \equiv 0(\mathrm{mod} n)$. The above labelling imply that l_1 has two inverses in $\mathbb{Z}_n$ and this is a contradiction. Then there is no orthogonal $\{2^x l_1 : 0 \leq x \leq 2m-1\}$-labelling of $K_2 \circ C_m$.

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