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IDENTIFICATION OF A TIME DEPENDENT SOURCE FUNCTION IN A PARABOLIC INVERSE PROBLEM VIA FINITE ELEMENT APPROACH

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In the present paper, we study one-dimensional inverse source problem of identifying a time-dependent source function from the over-specification condition. The unknown source term is recovered by solving an operator integral equation of the first kind. Due to the ill-posedness of this operator integral equation, the Tikhonov regularization approach of the 1st order is applied in order to acquire a stable approximation. The stable solution is defined by minimization of the Tikhonov functional. The value of the regularization parameter in this method is obtained as a function of error in input data. The finite element method is applied for numerical simulation based on obtained results. The illustrative example is presented to show the accuracy and applicability of the proposed method.

Key words : Inverse source problem; ill-posed operator equation; Tikhonov regularization method; finite element method; regularization parameter.

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1. Introduction

Inverse problems in parabolic partial differential equations of source identification type attracted more attention during the last two decades in various fields of applications in engineering and basic science such as: heat conduction problems, controls problems, thermoelasticity and etc. These types of problems have been one of the growing areas in Mathematics and engineering due to its variety of applications and have been studied by many authors [1, 2, 4, 5, 9, 10, 12, 13, 21-23, 25]. Many real world phenomena can be modeled from the mathematical point of view as an inverse source problem, e.g. the specification of air or water pollution source in the living environment, the determination of unknown heat sources and etc. Determination of unknown parameters or coefficients in the parabolic problems has been considered widely in many practical applications when the unknown source is a space-dependent function in [6, 11, 14, 15] or time-dependent in [18, 17] or dependent both in space and time. Inverse problems basically are ill-posed problems from a mathematical point of view and so by applying standard numerical techniques, a stable approximation can not be achieved. The objective is to improved more accurate and stable methods both analytical and numerical for finding a stable approximation solution. Our aim in this paper is to recover the time-dependent source term in the governing heat equation from overdetermination condition which is given by heat flux history at over the boundary $x = 0$. This inverse problem can be reformulated as an operator equation of the form $Ah = G$. It is well known that the operator equation $Ah = G$ is an ill-posed operator, so the solution is sensitive to small variations in input data. Since in the usual case where only a measured or computation approximation G_δ is available, employing regularization methods is required to obtain a reasonable stable approximation h^δ to h. The Tikhonov regularization technique is applied for operator equation $Ah = G$ for retrieving solution in a stable manner. Applying general results in the theory of Tikhonov regularization method which consists of solving the unconstrained minimization problem, we can find a stable solution. Since the regularization parameter plays an important role in applying the Tikhonov regularization method to the operator equation, we obtain this parameter based on the error in input data directly, which can be one of the advantages of our method for selecting a regularization parameter with respect to other methods [7, 16].

The organization of the present paper is as follows, In forthcoming Section, the mathematical formulation of this inverse problem is introduced and the problem is reformulated as an operator integral equation. In Section 3, the Tikhonov regularization approach is stated and a stable approximation is constructed for the ill-posed operator. Some theoretical results will be proved about the approximate solution and the choice of regularization parameter based the noise level in input data. In Section 4, the numerical approach based on the Finite element method for discretizing of the Euler-Lagrange equation is given to find a stable approximation based on the results obtained in Section 3. Numerical experiments are presented and discussed in Section 5. Finally, the conclusions are given in Section 6.

2. STATEMENT OF THE PROBLEM

This section is devoted to considering the following inverse parabolic problem of finding an unknown source function $h(t)$ and temperature distribution $u(x,t)$ from the following nonhomogeneous heat equation in a quarter plane

$$u_t = u_{xx} + h(t), \qquad 0 < x,\ 0 < t < T, \tag{1}$$

with initial condition

$$u(x,0) = f(x), \qquad 0 < x, \tag{2}$$

and boundary condition

$$u(0,t) = p(t), \qquad 0 < t < T, \tag{3}$$

and an overdetermination condition $g(t)$, which is given by

$$u_x(0,t) = g(t), \qquad 0 < t < T, \tag{4}$$

where $f(x)$, $p(t)$ and $g(t)$ are piecewise known continuous functions. We assume that all the functions appearing in heat equation (1) and initial and boundary conditions and overdetermination condition are measurable in order to obtain uniqueness solution.

The forward problem (1)-(2) and (4), has the unique bounded solution $u(x,t)$ in the form [3].

$$\begin{aligned} u(x,t) = &\int_0^t h(\tau)d\tau - 2\int_0^t K(x,t-\tau)g(\tau)d\tau \\ &+ \int_0^{+\infty} (K(x+\xi,t) + K(x-\xi))f(\xi)d\xi, \end{aligned} \tag{5}$$

where $K(x,t)$ is a fundamental solution of heat equation which is given by

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

Due to the setting, $x = 0$ and using (5) and (3) we obtain

$$u(0,t) = \int_0^t h(\tau)d\tau - \frac{1}{\sqrt{\pi}}\int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}d\tau + \frac{1}{\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{\xi^2}{4t}} f(\xi)d\xi, \tag{6}$$

which is written in the form

$$\int_0^t h(\tau)d\tau = p(t) + \frac{1}{\sqrt{\pi}}\int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}d\tau - \frac{1}{\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{\xi^2}{4t}} f(\xi)d\xi. \tag{7}$$

Now, we reformulate (7) in term of an operator integral equation:

$$(Ah)(t) = G(t), \qquad 0 < t < T, \tag{8}$$

where the operator A is defined by

$$(Ah)(t) = \int_0^t h(\tau)d\tau, \;\; 0 < t < T,$$

and

$$G(t) = p(t) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} d\tau - \frac{1}{\sqrt{\pi t}} \int_0^{+\infty} e^{-\frac{\xi^2}{4t}} f(\xi) d\xi, \;\; 0 < t < T.$$

The integral equation (8) cannot be reduced into an integral equation of the second kind because the operator is inherently ill-posed. The operator A in $L_p[0,T]$ is a compact linear operator for $1 \le p < \infty$. Zero is not belong to the eigenvalues set of A, thus A^{-1} exists and is unbounded. If $G(t)$ is in the range of A, then $h(t)$ is uniquely estimated from $h = A^{-1}G$. In real cases, if the right member of equation (8) is an element G_δ that differs from the exact right-hand member G_T by no more than δ, that is, if $\|G_T(t) - G_\delta(t)\| \le \delta$. It is obvious that the approximate solution h_δ or $h_{\alpha(\delta)}$ of equation (8) cannot be defined as the exact solution of the equation with approximate right-hand member G_δ, that is, according to the formula $h = A^{-1}G_\delta$, because small errors in $G(t)$ lead to large variations in $h(t)$ since A^{-1} is unbounded. It is natural to define h_δ or $h_{\alpha(\delta)}$ with the aid of an operator depending on a parameter having a value chosen in accordance with the error δ in the input data G, if $\delta \longrightarrow 0$ the approximate solution $h_{\alpha(\delta)}$ must approach the exact solution h_T, so the approximate solution $h_{\alpha(\delta)}$ is defined as $h_{\alpha(\delta)} = R(G_\delta, \alpha(\delta))$, where $R(G,\alpha)$ is a regularizing operator for the equation (8). In the next section, the concept of regularizing operator is introduced and Tikhonov regularization method for finding an approximate solution of $h(t)$ for ill-posed integral equation (8) is described.

3. Regularization Approach for Operator Integral Equation

It is well known that the Tikhonov regularization technique is one of methods for solving an ill-posed problem of the form (8) [7, 16, 26]. Since in practical applications, the input data are non-smooth, we apply the Tikhonov regularization method to construct a stable solution for solving ill-posed equation (8). Based on the main concept of the Tikhonov regularization method, we introduce smoothing functional $M^\alpha[h,G]$ which is called also Tikhonov functional and stabilizing functional $\Omega(h)$ as follows

$$M^\alpha[h,G] = \|Ah - G\|^2_{L^2[0,T]} + \alpha\Omega(h),$$

where α is regularization parameter and $\Omega(h)$ is a stabilizer of the 1st order constant coefficients which is defined as:

$$\Omega(h) = \| h(t) \|^2_{W_2^1[0,T]} = \int_0^T (h^2(\tau) + h'^2(\tau))d\tau.$$

We construct a regularize solution $h_{\alpha(\delta)}$ for the integral equation (8) by using the following minimization problem of the functional $M^\alpha[h, G]$ and to find an element $h_{\alpha(\delta)}$ for minimizing it by the following Theorem

***Theorem* 1** — *For every function $G(t) \in L_2[0, T]$, and every positive parameter α, there exists an element $h_{\alpha(\delta)}(t) \in W_2^1$ for which the smoothing functional $M^\alpha[h, G]$ attain it greatest lower bound.*

$$inf M^\alpha[h, G] = M^\alpha[h_{\alpha(\delta)}, G].$$

PROOF : The proof of the existence of a minimizer for the functional $M^\alpha[h, G]$ can be found with more details in [18]. We seek a regularized solution $h_{\alpha(\delta)}(t)$ in the Sobolov space W_2^1, it minimizes the functional

$$M^\alpha[h, G] = \int_0^T (\int_0^s h(\lambda)d\lambda - G(s))^2 ds + \alpha \int_0^T (h^2(\tau) + h'^2(\tau))d\tau.$$

A condition for a minimum of this functional is vanishing of its first variation. This condition is written in the form

$$\frac{1}{2}\frac{d}{d\varepsilon} M^\alpha[h + \varepsilon\eta, G]|_{\varepsilon=0} = 0.$$

By calculating the left-hand side of the above relation and setting $\varepsilon = 0$, we have:

$$\int_0^T \Big(\int_0^s \eta(\lambda)d\lambda\Big)\Big[\int_0^s h(\lambda)d\lambda - G(s)\Big]ds + \alpha \int_0^T \Big(h(\tau)\eta(\tau) + h'(\tau)\eta'(\tau)\Big)d\tau = 0.$$

By changing the order of integration in double integral and using integration by part for the second integral, we have

$$\int_0^T \eta(\lambda)[\int_\lambda^T \int_0^s a(\lambda)d\lambda ds - \int_\lambda^T G(s)ds - \alpha(h''(\lambda) - h(\lambda))]d\lambda + \alpha h'(\lambda)\eta(\lambda)|_0^T = 0, \quad (9)$$

where, $\eta(\tau)$ is an arbitrary variation of $h(\tau)$ such that both $h(\tau)$ and $h(\tau) + \varepsilon\eta(\tau)$ belong to the class of admissible functions.

Condition (9) will be satisfied for every $\eta(\lambda)$ if

$$\int_\lambda^T \int_0^s h(\lambda)d\lambda ds - \int_\lambda^T G(s)ds = \alpha(h''(\lambda) - h(\lambda)), \quad (10)$$

subject to the following consistency conditions

$$h'(0) = h'(T) = 0. \tag{11}$$

The equation (10) is called Euler-Lagrange equation corresponding to functional $M^\alpha[h, G]$ and therefore an operator $R(G, \alpha)$ is defined on the set of pairs (G, α), where $G \in L_2[0, T]$ and $\alpha > 0$, so that the element $h_\alpha = R(G, \alpha)$ minimizes the functional $M^\alpha[h, G]$. The problem of finding minimizer h_α thus reduces to find a solution of the integro differential equation (10) satisfying the conditions (11). The solution of (10) with boundary conditions (11) is unique by classical theorems in ODE.

Definition 1 — An operator $R(G, \alpha)$ depending on a parameter α is called a regularizing operator for the equation $Ah = G$ in a neighborhood of $G = G_T$ if it satisfies the following properties

1- There exists a positive number δ_1 such that the operator $R(G, \alpha)$ is defined for every $\alpha > 0$ and every G in U for which

$$\|G - G_T\|_U \leq \delta \leq \delta_1,$$

and

2- There exists a function $\alpha = \alpha(\delta)$ of δ such that, for every $\varepsilon > 0$, there exists a number $\delta(\varepsilon) \leq \delta_1$ such that the inclusion $G_\delta \in U$ and the inequality

$$\|G_T - G_\delta\|_U \leq \delta(\varepsilon),$$

imply

$$\|h_T - h_{\alpha(\delta)}\|_H \leq \varepsilon,$$

where $h_\alpha = R(G_\delta, \alpha(\delta))$.

In the following theorem, the regularization parameter α is defined as a function depending on δ ($\alpha = \alpha(\delta)$) such that the operator $R(G, \alpha(\delta))$ will be a regularizing operator for equation (8).

***Theorem* 2** — *Let δ is the noise level in the measured data $g(t)$, so by equation (8) the function $G(t)$ contains an error which is represented by G_δ. If the regularization parameter α is selected as a function of δ by $\alpha(\delta) = \delta^\gamma$ where $0 \leq \gamma < 2$, then the operator $R(G, \alpha(\delta))$ in previous section is a regularizing operator for equation (8) that is $h_\alpha = R(G_\delta, \alpha(\delta))$ and $h_{\alpha(\delta)}$ be the solution of $Ah_{\alpha(\delta)} = G_\delta$.*

PROOF : It will be sufficient to show that the operator $R(G, \alpha(\delta))$ possesses the properties of the definition of regularizing operator. For property 1 of the regularizing operator, it is clear that there exists a positive number δ_1 such that the operator $R(G, \alpha(\delta))$ is defined for every $\alpha > 0$ and every $G \in L_2[0, T]$ for which

$$\|G(t) - G_T(t)\|_{L_2[0,T]} \leq \delta < \delta_1.$$

For property 2, let h_T denote exact solution of equation (8) in accordance with the exact right-hand member G_T such that $Ah_T = G_T$. Let δ denote a fixed positive number. For any G_δ of $L_2[0, T]$ such that

$$\|G_T(t) - G_\delta(t)\|_{L_2[0,T]} \leq \delta,$$

we have

$$\|h_T(t) - h_{\alpha(\delta)}(t)\|_{C[0,T]} = \|A^{-1}Ah_T - A^{-1}Ah_{\alpha(\delta)}\| \leq \|A^{-1}\|\|Ah_T - Ah_{\alpha(\delta)}\|,$$

on the other hand

$$\begin{aligned}\|Ah_T - Ah_{\alpha(\delta)}\|_{L_2[0,T]} &\leq \|Ah_T - G_\delta\|_{L_2[0,T]} + \|Ah_{\alpha(\delta)} - G_\delta\|_{L_2[0,T]} \\ &\leq \|G_T - G_\delta\|_{L_2[0,T]} + \|Ah_{\alpha(\delta)} - G_\delta\|_{L_2[0,T]}.\end{aligned}$$

Now, by selecting regularization parameter $\alpha(\delta) = \delta^\gamma$ where $0 \leq \gamma < 2$, since h_α is a minimizer of functional M^α, we have

$$M^{\alpha(\delta)}[h_\alpha, G_\delta] \leq M^{\alpha(\delta)}[h_T, G_\delta],$$

therefore,

$$\begin{aligned}&\|Ah_\alpha - G_\delta\|^2_{L_2[0,T]} \leq M^{\alpha(\delta)}[h_\alpha, G_\delta] \\ &\leq \int_0^T (Ah_T(s) - G_\delta(s))^2 ds + \alpha(\delta) \int_0^T (h_T^2(\tau) + h_T'^2(\tau)) d\tau \\ &= \int_0^T (G_T(s) - G_\delta(s))^2 ds + \delta^\gamma \int_0^T (h_T^2(\tau) + h_T'^2(\tau)) d\tau \\ &\leq \delta^2 + \delta^\gamma \int_0^T (h_T^2(\tau) + h_T'^2(\tau)) d\tau \\ &\leq \delta^\gamma (1 + \int_0^T (h_T^2(\tau) + h_T'^2(\tau)) d\tau) \\ &= \delta^\gamma N,\end{aligned}$$

with $N = 1 + \int_0^T (h_T^2(\tau) + h_T'^2(\tau)) d\tau$.

Hence, $\| \ h_{\alpha(\delta)} \ \|^2_{W_2^1[0,T]} \leq N$ and $\|Ah_\alpha - G_\delta\|_{L_2[0,T]} \leq \delta^{\frac{\gamma}{2}}\sqrt{N}$ and so

$$\|Ah_T - Ah_{\alpha(\delta)}\|_{L_2} \leq \delta^{\frac{\gamma}{2}}(1+\sqrt{N}).$$

The above inequality implies that

$$\|h_T(t) - h_{\alpha(\delta)}(t)\|_{C[0,T]} \leq \|A^{-1}\|\|Ah_T - Ah_{\alpha(\delta)}\| \leq \|A^{-1}\|\delta^{\frac{\gamma}{2}}(1+\sqrt{N}).$$

Consequently, both elements $h_T(t)$ and $h_{\alpha(\delta)}(t)$ belong to the compact subset E of element h of $C[0,T]$ such that

$$E = \{h(s) | \ \|h\|^2_{W_2^1} \leq N\}.$$

Since E is compact in $C[0,T]$, and the operator A is continuous, the mapping $A : E \to AE$ is continuous and one to one, therefore the inverse mapping $A^{-1} : AE \to E$ is also continuous. So, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for G_T and G_δ with $Ah_T = G_T$ and $Ah_\alpha = G_\delta$, the inequality

$$\|G_T - G_\delta\| \leq \delta(\varepsilon),$$

implies the inequality

$$\|h_T(t) - h_\alpha(t)\|_{C[0,T]} \leq \varepsilon.$$

The above results show that if $\delta(\varepsilon)$ was chosen in the form

$$\delta(\varepsilon) \leq [\frac{\varepsilon}{\|A^{-1}\|(1+\sqrt{N})}]^{\frac{2}{\gamma}},$$

then for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that, for $\delta \leq \delta(\epsilon) \leq \delta_1$ and $\alpha(\delta) = \delta^\gamma$ where $0 \leq \gamma < 2$, the regularizing operator $R(G, \alpha(\delta))$ possesses property 2 of definition on regularizing operator and so the Theorem is satisfied. □

4. Numerical Approach Based on Finite Element Method

In this section in order to investigate the applicability of the regularization method which is described in Section 3, the Euler-Lagrange equation (10) subject to conditions (11) is discretized numerically by finite element approach. The equation (10) is replaced with its finite element approximation. Let

$$V = H^0([0,T]) = L_2[0,T].$$

The variational form of Eq. (10) can be shown as [24],

$$B(h,v) = L(v), \qquad \forall v \in V,$$

where

$$\begin{aligned} B(h,v) &= \alpha\Big(-\int_0^T h'(\lambda)v'(\lambda)d\lambda - \int_0^T h(\lambda)v(\lambda)d\lambda\Big) \\ &\quad - \int_0^T \Big(\int_\lambda^T \int_0^s h(\lambda)d\lambda ds\Big)v(\lambda)d\lambda, \\ L(v) &= \int_0^T \Big(-\int_\lambda^T G(s)ds\Big)v(\lambda)d\lambda, \end{aligned}$$

where $v(\lambda) \in V$ is an arbitrary function.

Since the space V is an infinite dimensional space, we select a subspace $\widetilde{V}$ of V with finite dimension and so the problem is converted to find $\widetilde{h} \in \widetilde{V}$ such that

$$\begin{aligned} B(\widetilde{h},v) &= \alpha\Big(-\int_0^T \widetilde{h}'(\lambda)v'(\lambda)d\lambda - \int_0^T \widetilde{h}(\lambda)v(\lambda)d\lambda\Big) \\ &\quad - \int_0^T \Big(\int_\lambda^T \int_0^s \widetilde{h}(\lambda)d\lambda ds\Big)v(\lambda)d\lambda, \\ L(v) &= \int_0^T \Big(-\int_\lambda^T G(s)ds\Big)v(\lambda)d\lambda, \end{aligned}$$

where $v(\lambda) \in \widetilde{V}$.

In the FEM, the basis functions are chosen as piecewise polynomials with arbitrary degree in $[0,\, T]$. In this work, we have used the linear B-spline functions as the basis functions of $\widetilde{V}$ subspace. Let

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_{\mathcal{M}} = T,$$

are mesh points in the time interval $[0,\, T]$, so that $\lambda_i = i\,\kappa$, $i = 0, 1, ..., \mathcal{M}$; $\lambda_0 = 0$, $\lambda_{\mathcal{M}} = T$, $\kappa = \frac{1}{\mathcal{M}}$.

The linear B-splines $\phi_i(\lambda)$ are defined as follows

$$\phi_0(\lambda) = \frac{1}{\kappa}\begin{cases} (\lambda_1 - \lambda), & \lambda \in [\lambda_0, \lambda_1], \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_i(\lambda) = \frac{1}{\kappa}\begin{cases} (\lambda - \lambda_{i-1}), & \lambda \in [\lambda_{i-1}, \lambda_i], \\ (\lambda_{i+1} - \lambda), & \lambda \in [\lambda_i, \lambda_{i+1}], \qquad i = 1, ..., \mathcal{M} - 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{\mathcal{M}}(\lambda) = \frac{1}{\kappa}\begin{cases} (\lambda - \lambda_{\mathcal{M}-1}), & \lambda \in [\lambda_{\mathcal{M}-1}, \lambda_{\mathcal{M}}], \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that $\phi_i(\lambda_i) = 1$ and zero at the other nodes. Hence, we have $\widetilde{h} = \sum_{j=0}^{M} \beta_j \phi_j(\lambda)$ and for $i = 0, \ldots, M$, we can take $v = \phi_i(\lambda)$. By substituting $\widetilde{h}$ and v in the variational formulation, we have

$$\begin{aligned} \sum_{j=0}^{M} \beta_j \Bigg(\alpha \Big(- \int_0^T \phi_j'(\lambda)\phi_i'(\lambda) d\lambda - \int_0^T \phi_j(\lambda)\phi_i(\lambda) d\lambda \Big) \\ - \int_0^T \Big(\int_\lambda^T \int_0^s \phi_j(\lambda) d\lambda ds \Big) \phi_i(\lambda) d\lambda \Bigg) \\ = \int_0^T \Big(- \int_\lambda^T G(s) ds \Big) \phi_i(\lambda) d\lambda. \end{aligned}$$

This yields the following system for the weight β

$$\mathcal{C}\beta = F, \tag{12}$$

where

$$\mathcal{C} = (c_{ji}), \qquad c_{ji} = \alpha \Big(- \int_0^T \phi_i'(\lambda)\phi_j'(\lambda) d\lambda - \int_0^T \phi_j(\lambda)\phi_i(\lambda) d\lambda \Big) - \int_0^T \Big(\int_\lambda^T \int_0^s \phi_j(\lambda) d\lambda ds \Big) \phi_i(\lambda) d\lambda,$$

$$F = (f_i), \qquad f_i = \int_0^T \Big(- \int_\lambda^T G(s) ds \Big) \phi_i(\lambda) d\lambda,$$

and $\beta = (\beta_0, \beta_1, \ldots, \beta_M)^T$, where "$T$" means transpose, is the vector of unknown functions β_j, $j = 0, \ldots, M$.

The unknown coefficients β_j are obtained by solving the linear system (12) and with these coefficients, the approximate solution can be obtained.

5. Numerical Results

In this section in order to show the efficiency of the proposed method described in the current paper, we employ the scheme presented in solving a test example. The illustrative example investigates the inverse problem (1)-(4) by taking $T = 1$, subject to conditions (2) and (3) given by

$$\begin{aligned} f_T(x) &= 0.0001x(1-x^2), \\ p_T(t) &= \frac{t^3}{3} - \frac{t^4}{2} + 0.0002\sqrt{\frac{t}{\pi}}(1+4t(e^{-\frac{1}{4t}}-1)), \end{aligned}$$

and the over-specified condition (4) given by

$$g_T(t) = \frac{256t^{4.5}}{315\sqrt{\pi}}.$$

The exact smooth source term of this example is defined by

$$h_T(t) = t^2(t-1)^2. \tag{13}$$

By the above data we have $G_T(t) = \frac{t^3}{3} - \frac{t^4}{2} + \frac{t^5}{5}$ in Eq. (8).

Since exact smooth data are not given exactly so, we investigate the stability of the numerical approximation by adding the noise to the exact data (2)-(4) defined as follows

$$\begin{aligned} f_\delta(x) &= f_T(x) + \epsilon, \\ p_\delta(t) &= p_T(t) + \epsilon, \\ g_\delta(t) &= g_T(t) + \epsilon, \end{aligned}$$

where ϵ is a Gaussian random variable with mean zero and standard deviation σ, which is defined for $p(t)$ as follows

$$\sigma = \delta \times max_{t\in[0,T]}|p(t)|,$$

where δ represents the percentage of noise level and simultaneously for $f(x)$ and $g(t)$. The random variables are generated using the random command in MATLAB. The numerical results for comparison between the exact source term $h_T(t)$ given by Eq. (13) and the approximate solution $h_\delta(t)$ for different values of noise level δ and known parameter γ are shown in given Tables with $\mathcal{M} = 20$ and $\mathcal{M} = 10$, respectively. The obtained numerical results for a large noise level are not presented.

The Fig. 1 and Fig. 2 show that when noise level goes to the zero and the parameter γ tends to 2, then the numerical approximation converges to the exact solution. Furthermore, the numerical solutions are stable with respect to the noise in input data. Also, we present the root mean square error (RMSE) defined by

$$RMSE(h(t)) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[h_T(t_i) - h_{num}(t_i)]^2},$$

for the numerical results illustrated in Table 1, Table 2 and Table 3.

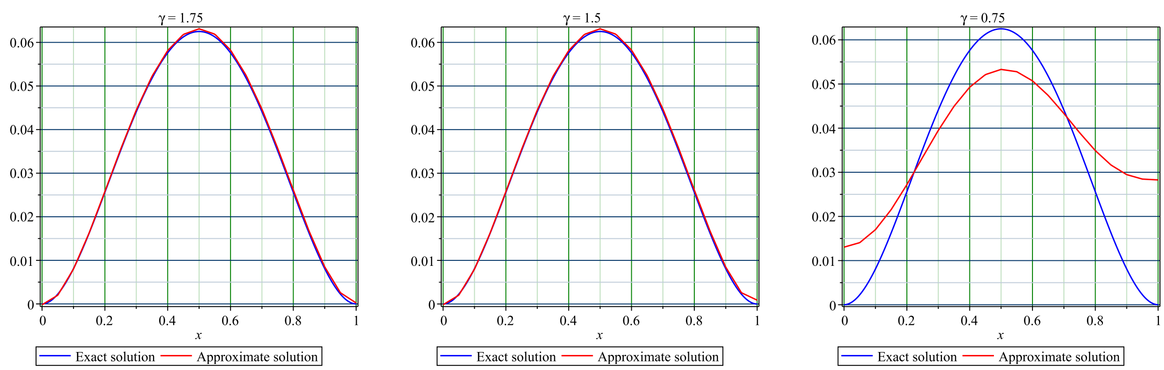

Figure 1: The plots of approximate and exact solutions of $h(t)$ for different values of γ and $\delta = 0.0001$, $\mathcal{M} = 20$.

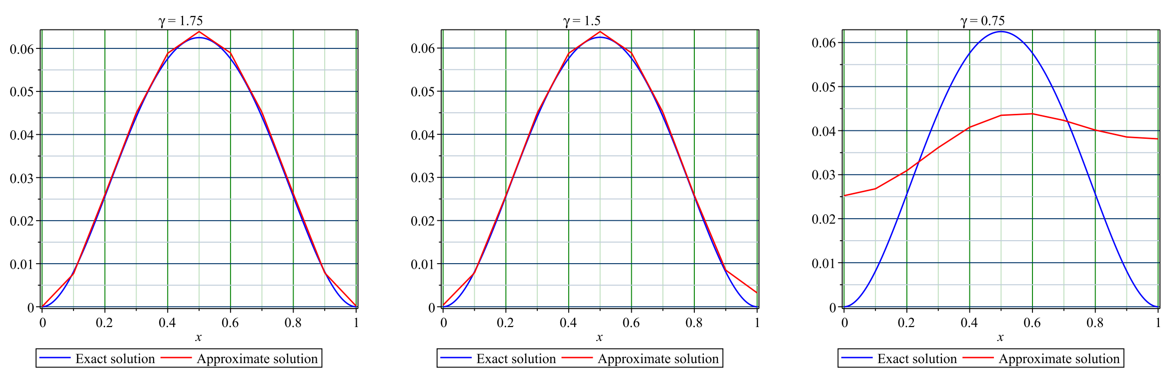

Figure 2: The plots of approximate and exact solutions of $h(t)$ for different values of γ and $\delta = 0.001$, $\mathcal{M} = 10$.

t	Exact solution	present method γ=1.75	present method γ=1.5	present method γ=0.75
0.1	0.008100	0.008003	0.007985	0.017003
0.2	0.025600	0.025773	0.025770	0.027189
0.3	0.044100	0.044484	0.044481	0.039475
0.4	0.057600	0.058129	0.058126	0.049269
0.5	0.062500	0.063114	0.063112	0.053299
0.6	0.057600	0.058242	0.058240	0.050710
0.7	0.044100	0.044715	0.044712	0.043318
0.8	0.025600	0.026134	0.026131	0.034914
0.9	0.008100	0.008501	0.008464	0.029471
$RMSE$		0.000454	0.000488	0.012241

Table 1: The comparison between the values of the exact and numerical solutions of $h(t)$ for different values of γ and $\delta = 0.0001$, $\mathcal{M} = 20$.

t	Exact solution	present method γ=1.75	present method γ=1.5	present method γ=0.75
0.1	0.008100	0.007435	0.007340	0.016740
0.2	0.025600	0.025870	0.025763	0.027082
0.3	0.044100	0.044877	0.044803	0.039554
0.4	0.057600	0.058762	0.058692	0.049489
0.5	0.062500	0.063804	0.063743	0.053568
0.6	0.057600	0.058859	0.058803	0.050928
0.7	0.044100	0.045091	0.045038	0.043421
0.8	0.025600	0.026157	0.026112	0.034903
0.9	0.008100	0.007933	0.007877	0.029397
$RMSE$		0.000883	0.000902	0.012736

Table 2: The comparison between the values of exact and numerical solutions of $h(t)$ for different values of γ and $\delta = 0.0001$, $\mathcal{M} = 10$.

6. CONCLUSIONS

In this work, the finite element method based upon the Tikhonov regularization method is applied for solving an inverse parabolic problem with an unknown time-dependent source function. We show

t	Exact solution	present method γ=1.75	present method γ=1.5	present method γ=0.75
0.1	0.008100	0.007643	0.007855	0.026794
0.2	0.025600	0.026011	0.025805	0.030919
0.3	0.044100	0.045017	0.044947	0.036147
0.4	0.057600	0.058868	0.058792	0.040769
0.5	0.062500	0.063904	0.063825	0.043482
0.6	0.057600	0.058948	0.058883	0.043830
0.7	0.044100	0.045169	0.045100	0.042315
0.8	0.025600	0.026257	0.025848	0.040128
0.9	0.008100	0.007871	0.008495	0.038548
$RMSE$		0.000867	0.001256	0.020242

Table 3: The comparison between the values of exact and numerical solutions of $h(t)$ for different values of γ and $\delta = 0.001$, $\mathcal{M} = 10$.

that in the present work the value of regularization parameter can be selected directly in accordance with the error in input data to find a stable approximation. The proposed method is employed to solve a one-dimensional inverse source problem. The numerical simulations show that there exists a good agreement between exact and numerical solutions. The numerical results suggest that the method is an accurate technique for solving the inverse parabolic problems. The method is very efficient and easy from the computational point of view since it produces accurate results even for randomly perturbed data and so the approximate solution is stable.

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