Indian J. Pure Appl. Math., **51**(4): 1603-1614, December 2020
 DOI: 10.1007/s13226-020-0484-7

NEW CHARACTERIZATIONS OF RESTRICTED INJECTIVE DIMENSIONS FOR COMPLEXES

Dejun Wu and Zhuang Miao

Department of Applied Mathematics, Lanzhou University of Technology,
Lanzhou, 730050, *Gansu, China*
e-mail: wudj@lut.cn 2374035552@qq.com

(*Received* 6 *November* 2018; *after final revision* 26 *June* 2019;
accepted 23 *October* 2019)

In this paper, we study the restricted injective dimensions of complexes. Some new characterizations of the restricted injective dimensions are obtained. In particular, it is shown that the restricted injective dimensions can be computed in terms of the restricted injective resolutions. As applications, we get some results on the behavior of the restricted injective dimensions under change of rings. In addition, a characterization of almost Cohen-Macaulay ring is obtained.

Key words : Restricted injective dimension; Bass formula; Cohen-Macaulay ring.

2010 Mathematics Subject Classification : 13D05, 18G10, 18G35.

1. Introduction

Throughout this paper all rings are non-trivial, commutative, and noetherian. A complex

$$X: \quad \cdots \longrightarrow X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \longrightarrow \cdots$$

is called *acyclic* if the homology complex $\mathrm{H}(X)$ is the zero-complex. If n is an integer and X is a complex, then $\Sigma^n X$ denotes the complex X shifted n degrees (to the left); see for example Christensen [3, A.1.3]. We use the notations $\mathrm{Z}_n(X)$ for the kernel of differential ∂^X_n and $\mathrm{C}_n(X)$ for the cokernel of the differential ∂^X_{n+1}. The notations $\sup X$ and $\inf X$ of a complex X are used for the supremum and infimum of the set $\{n \in \mathbb{Z} \mid \mathrm{H}_n(X) \neq 0\}$, with the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. The derived category is written $\mathcal{D}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote boundedness conditions. The projective, injective and flat dimensions are abbreviated as pd, id, and fd, respectively. The full subcategories $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ of $\mathcal{D}_\square(R)$ consist of complexes of finite, respectively, projective, injective, and flat dimensions. We use the superscript f to denote finite homology and the subscript 0 to denote modules. For example, $\mathcal{P}^f_0(R)$ denotes the category of finite modules of finite projective dimension. We use the standard notations $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes^{\mathbf{L}}_R -$ for the derived Hom and derived tensor product of complexes.

(1.1) *Depth* : Let $(R, \mathfrak{m}, k)$ be a local ring and X an R-complex. The *depth* of X is defined as

$$\operatorname{depth}_R X = -\sup \mathbf{R}\mathrm{Hom}_R(k, X).$$

By Foxby and Iyengar [7, 1.5.(3)], for every R-complex X one has

$$(1.1.1)\ \operatorname{depth}_R X \geqslant -\sup X. \tag{1}$$

If $\sup X = s$ is finite, then equality holds if and only if $\mathfrak{m}$ is an associated prime of the top homology module $\mathrm{H}_s(X)$.

(1.2) *Width* : Let $(R, \mathfrak{m}, k)$ be a local ring. The *width* of an R-complex Y is defined as

$$\operatorname{width}_R Y = \inf\,(k \otimes^{\mathbf{L}}_R Y).$$

By [7, 1.5.(1)], for every R-complex Y there is an inequality

$$\operatorname{width}_R Y \geqslant \inf Y, \tag{2}$$

and equality holds if $\mathrm{H}(Y)$ is bounded below and degreewise finite by Nakayama lemma.

(1.3) *Resolutions* : A morphism of R-complexes that induces an isomorphism in homology is called a *quasi-isomorphism (* denoted by"$\simeq$"*)*. An R-complex I is called *semi-injective* if each module I_i is injective, and the functor $\mathrm{Hom}_R(-, I)$ preserves quasi-isomorphisms (equivalently, it preserves acyclicity). Every bounded above complex of injective modules is semi-injective. Note that in [1] Avramov and Foxby use "DG-injective" in place of "semi-injective". Every R-complex X has a semi-injective resolution. That is, there is a quasi-isomorphism $\iota\colon X \to I$, where I is a semi-injective complex with $I_i = 0$ for all $i > \sup X$; see [1].

It is well known that Gorenstein homological dimensions are refinements of the classical homological dimensions. In [3], Christensen gave some characterizations of Gorenstein projective, injective and flat dimensions over a Cohen-Macaulay local ring with a dualizing module and such results have been generalized in the more general setting of complexes over associative rings (see Christensen, Frankild and Holm [5]).

As a refinement of injective dimension over almost Cohen-Macaulay ring (see [4, Proposition 5.13]), Christensen, Foxby and Frankild in [4, Definition 5.10] defined the *large restricted injective dimension* of a homologically bounded above complex Y by

$$\mathrm{Rid}_R Y = \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(T, Y) | T \in \mathcal{P}_0(R)\}.$$

For an R-module N we get

$$\mathrm{Rid}_R N = \sup\{m \in \mathbb{N}_0 | \mathrm{Ext}^m_R(T, N) \neq 0 \text{ for some } T \in \mathcal{P}_0(R)\}.$$

Note that the number $\mathrm{Rid}_R X$ was denoted by $\mathrm{Ed}_R X$ in Sharif and Yassemi [13].

Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring and $X \in \mathcal{D}^f_\square(R)$. It was shown that in [15] the large restricted injective dimension satisfies the following Bass formula

$$\mathrm{Rid}_R\, X = \operatorname{depth} R - \inf X. \tag{3}$$

The *small restricted injective dimension* [4, Definition 5.1] of a homologically bounded above complex Y is

$$\mathrm{rid}_R Y = \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(T,Y)) | T \in \mathcal{P}^f_0(R)\}.$$

It was shown that the restricted injective dimensions are finite for any homologically bounded complex over commutative noetherian rings of finite Krull dimension and the small restricted injective dimension satisfies the following Bass formula

$$\mathrm{rid}_R\, Y = \operatorname{depth} R - \inf Y, \tag{4}$$

whenever R is local and $Y \in \mathcal{D}^f_\sqsubset(R)$ (see [4, Corollary 5.5]).

In this paper, we further study the properties of the restricted injective dimensions. Firstly, we give some new characterizations of the small restricted injective dimension as follows, where the concept of restricted injective module is Definition 2.1.

Theorem A — *Let $X \in \mathcal{D}(R)$ be a homologically bounded above complex and $n \in \mathbb{Z}$. Consider the following conditions:*

(i) $\mathrm{rid}_R\, X \leq n$.

(ii) *X is equivalent to a bounded complex I of restricted injective R-modules with* $\inf\{i \in \mathbb{Z} | I_i \neq 0\} \geq -n$.

(iii) $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(T,X)) = 0$ *for any $i < -n$ and $T \in \mathcal{P}^f_0(R)$.*

(iv) *$n \geq -\inf X$ and $\mathrm{Z}_{-n}(I)$ is a restricted injective R-module whenever $X \simeq I$, where $I \in \mathcal{D}(R)$ is a bounded above complex of restricted injective modules.*

(v) *$n \geq -\sup U - \inf \mathbf{R}\mathrm{Hom}_R(U,X)$ for any non-acyclic complex $U \in \mathcal{P}^f(R)$.*

Then we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftarrow$ (v). Furthermore, if X is a homologically bounded complex, then all the above statements are equivalent. When condition (ii) is satisfied, I can be chosen such that $I_l = 0$ for $l > \sup X$.

Recall that *the small restricted flat dimension* [4, Definition 2.9] of a homologically bounded below complex X is

$$\mathrm{rfd}_R X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X) | T \in \mathcal{P}^f_0(R)\}.$$

As an application of the above theorem, we get the following result on the behavior of the restricted injective dimension under change of rings.

Theorem B — *Let $\varphi : R \to S$ be a module-finite ring homomorphism of finite flat dimension. Assume that R has finite Krull dimension. If $X \in \mathcal{P}^f(S)$ and $Y \in \mathcal{D}_{\square}(R)$, then the following inequalities hold,*

(i) $\operatorname{rid}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) \leq \operatorname{rfd}_R X + \operatorname{rid}_R Y$.
(ii) $\operatorname{rid}_S(\mathbf{R}\mathrm{Hom}_R(X,Y)) \leq \operatorname{rfd}_S X + \operatorname{rid}_R Y$.

The *Cohen-Macaulay defect* $\operatorname{cmd} R$ of a local ring R is the difference $\dim R - \operatorname{depth} R$. For a non-local ring R the Cohen-Macaulay defect is the supremum of the defects at all prime ideals $\mathfrak{p} \in \operatorname{Spec} R$. Recall that a local ring R is called *almost Cohen-Macaulay* if $\operatorname{cmd} R \leq 1$. There are a lot of characterizations of almost Cohen-Macaulay rings (see [4] and Kang [10]). Another main result in this paper gives a characterization of almost Cohen-Macaulay rings.

Theorem C — *Let R be a local ring. Then the following conditions are equivalent.*

(i) $\operatorname{cmd} R \leq 1$.
(ii) $\operatorname{rfd}_R X = \operatorname{fd}_R X$ *for any complex* $X \in \mathcal{F}(R)$.
(iii) $\operatorname{rfd}_R M = \operatorname{fd}_R M$ *for any R-module M of finite flat dimension.*

2. Measuring Restricted Injective Dimension

Firstly, we introduce the following definition.

Definition 2.1 — An R-module M is *restricted injective* if $\operatorname{Ext}^i_R(T,M) = 0$ for any finite R-module T of finite projective dimension and $i > 0$.

Remark 2.2 : Let $M \neq 0$ be an R-module. Then M is restricted injective if and only if $\operatorname{rid}_R M = 0$.

Lemma 2.3 — Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If two modules in the sequence have finite small restricted injective dimension, then so has the third. In particular, the following statements hold.

(i) If $\operatorname{rid}_R M'' < \operatorname{rid}_R M$, then $\operatorname{rid}_R M' = \operatorname{rid}_R M$.
(ii) If $\operatorname{rid}_R M'' > \operatorname{rid}_R M$, then $\operatorname{rid}_R M' = \operatorname{rid}_R M'' + 1$.
(iii) If $\operatorname{rid}_R M'' = \operatorname{rid}_R M$, then $\operatorname{rid}_R M' \leq \operatorname{rid}_R M'' + 1$.
(iv) If M' is restricted injective, then $\operatorname{rid}_R M = \operatorname{rid}_R M''$.
(v) If the sequence splits, then $\operatorname{rid}_R M = \max\{\operatorname{rid}_R M', \operatorname{rid}_R M''\}$.

Proof : Straightforward.

Lemma 2.4 — If $X \in \mathcal{D}(R)$ is a bounded above acyclic complex of restricted injective modules and P is a bounded complex of modules in $\mathcal{P}^f_0(R)$, then $\mathrm{Hom}_R(P,X)$ is an acyclic complex.

Proof : We may assume that P is non-zero. Without loss of generality, we may assume that $P_l = 0$ for $l < 0$. Let $s = \sup\{i \in \mathbb{Z} | P_i \neq 0\}$. We proceed by induction on s.

If $s = 0$, then $P \in \mathcal{P}_0^f(R)$. It is easy to see that $\mathrm{Ext}_R^i(P, X_l) = 0$ for any $i > 0$ and $l \in \mathbb{Z}$. Hence $\mathrm{Hom}_R(P, X)$ is acyclic.

Let $s > 0$ and assume that $\mathrm{Hom}_R(P, X)$ is acyclic for any bounded complex P of modules in $\mathcal{P}_0^f(R)$ with $\sup\{i \in \mathbb{Z} | P_i \neq 0\} \leq s - 1$ and $P_l = 0$ for $l < 0$. Consider the degree-wise split exact sequence

$$0 \to {}_{\sqsubset s-1}P \to P \to \Sigma^s P_s \to 0$$

of complexes, where the hard truncation ${}_{\sqsubset s-1}P$ is the complex

$$0 \to P_{s-1} \to P_{s-2} \to P_{s-3} \to \cdots$$

Thus we have the following exact sequence

$$0 \to \mathrm{Hom}_R(\Sigma^s P_s, X) \to \mathrm{Hom}_R(P, X) \to \mathrm{Hom}_R({}_{\sqsubset s-1}P, X) \to 0.$$

Since $\mathrm{Hom}_R(\Sigma^s P_s, X)$ and $\mathrm{Hom}_R({}_{\sqsubset s-1}P, X)$ are acyclic, $\mathrm{Hom}_R(P, X)$ is also acyclic.

Lemma 2.5 — Let $X \simeq I$ and $U \simeq P$, where $I \in \mathcal{D}(R)$ is a bounded above complex of restricted injective modules and P is a bounded complex of modules in $\mathcal{P}_0^f(R)$. Then $\mathbf{R}\mathrm{Hom}_R(U, X) \simeq \mathrm{Hom}_R(P, I)$.

PROOF : Take a semi-injective resolution $X \to J$, where $J \in \mathcal{D}(R)$ is a bounded above complex of injective modules. Then $\mathbf{R}\mathrm{Hom}_R(U, X) \simeq \mathrm{Hom}_R(U, J)$. Since $I \simeq J$, there exists a quasi-isomorphism $\alpha : I \to J$. Hence we have a morphism

$$\mathrm{Hom}_R(P, \alpha) : \mathrm{Hom}_R(P, I) \to \mathrm{Hom}_R(P, J).$$

Since the mapping cone $\mathrm{Cone}(\alpha)$ is an acyclic complex of restricted injective R-modules,

$$\mathrm{Cone}(\mathrm{Hom}_R(P, \alpha)) \cong \mathrm{Hom}_R(P, \mathrm{Cone}(\alpha))$$

is acyclic by Lemma 2.6. Therefore, $\mathrm{Hom}_R(P, \alpha)$ is a quasi-isomorphism. Moreover, $\mathrm{Hom}_R(U, J) \simeq \mathrm{Hom}_R(P, I)$ and so $\mathbf{R}\mathrm{Hom}_R(U, X) \simeq \mathrm{Hom}_R(P, I)$.

Lemma 2.6 — Let $I \in \mathcal{D}(R)$ be a bounded above complex of restricted injective modules and $T \in \mathcal{P}_0^f(R)$. If $X \in \mathcal{D}(R)$ with $n \geq -\inf X$ and $X \simeq I$, then for any $i > 0$, we have

$$\mathrm{Ext}_R^i(T, \mathrm{Z}_{-n}(I)) = \mathrm{H}_{-(i+n)}(\mathbf{R}\mathrm{Hom}_R(T, X)).$$

PROOF : Since $\inf X = \inf I \geq -n$, we have ${}_{\sqsubset -n}I \simeq \Sigma^{-n}\,\mathrm{Z}_{-n}(I)$. Hence,

$$\Sigma^n\, {}_{\sqsubset -n}I \simeq \mathrm{Z}_{-n}(I).$$

For any $i > 0$, by Lemma 2.5, we have

$$\begin{aligned}
\operatorname{Ext}_R^i(T, \mathrm{Z}_{-n}(I)) &= \mathrm{H}_{-i}(\mathbf{R}\operatorname{Hom}_R(T, \mathrm{Z}_{-n}(I))) \\
&= \mathrm{H}_{-i}(\operatorname{Hom}_R(T, \Sigma^n {}_{\sqsubset -n}I)) \\
&= \mathrm{H}_{-i}(\Sigma^n \operatorname{Hom}_R(T, {}_{\sqsubset -n}I)) \\
&= \mathrm{H}_{-(i+n)}(\operatorname{Hom}_R(T, {}_{\sqsubset -n}I)) \\
&= \mathrm{H}_{-(i+n)}(\operatorname{Hom}_R(T, I)) \\
&= \mathrm{H}_{-(i+n)}(\mathbf{R}\operatorname{Hom}_R(T, X)) \,.
\end{aligned}$$

Now we prove Theorem A from the introduction.

Proof of Theorem A : $(i) \Rightarrow (iv)$: By [4, 5.2.1] we have $-\inf X \leq \operatorname{rid}_R X \leq n$. Let $T \in \mathcal{P}_0^f(R)$ and $X \simeq I$, where $I \in \mathcal{D}(R)$ is a bounded above complex of restricted injective modules. Since $\operatorname{rid}_R X \leq n$, for any $i > 0$ by Lemma 2.6 we have

$$\operatorname{Ext}_R^i(T, \mathrm{Z}_{-n}(I)) = \mathrm{H}_{-(i+n)}(\mathbf{R}\operatorname{Hom}_R(T, X)) = 0.$$

Hence $\mathrm{Z}_{-n}(I)$ is a restricted injective R-module.

$(iv) \Rightarrow (ii)$: Take a semi-injective resolution $X \to I$, where $I \in \mathcal{D}(R)$ is a bounded above complex of injective modules with $I_l = 0$ for any $l > \sup X$. Since $\inf I = \inf X \geq -n$, we have $X \simeq I \simeq I_{-n\supset}$, where

$$I_{-n\supset} = \cdots \to I_{-n+2} \to I_{-n+1} \to \mathrm{Z}_{-n}(I) \to 0.$$

Clearly, $I_{-n\supset}$ is a bounded complex of restricted injective R-modules.

$(ii) \Rightarrow (iii)$: Let $T \in \mathcal{P}_0^f(R)$. By Lemma 2.5,

$$\mathrm{H}_i(\mathbf{R}\operatorname{Hom}_R(T, X)) = \mathrm{H}_i(\operatorname{Hom}_R(T, I)).$$

Let $i < -n$. Note that $I_t = 0$ for any $t < -n$. So

$$\operatorname{Hom}_R(T, I)_i = \prod_{t \in \mathbb{Z}} \operatorname{Hom}_R(T_t, I_{t+i}) = 0,$$

and hence $\mathrm{H}_i(\mathbf{R}\operatorname{Hom}_R(T, X)) = 0$.

$(v) \Rightarrow (iii) \Rightarrow (i)$: Clearly.

$(i) \Rightarrow (v)$: Let $X \in \mathcal{D}(R)$ be a homologically bounded complex and $U \in \mathcal{P}^f(R)$ with $\mathrm{H}(U) \neq 0$. By [4, Propositon 5.3], we have

$$\begin{aligned}
-\inf \mathbf{R}\operatorname{Hom}_R(U, X) &\leq \operatorname{rid}_R X + \sup U \\
&\leq n + \sup U.
\end{aligned}$$

Definition 2.7 — An R-module M is *restricted flat* if $\operatorname{Tor}_i^R(T, M) = 0$ for any finite R-module T of finite projective dimension and $i > 0$.

By using a dual argument of that in the proof of Theorem A, one has the following result.

***Theorem* 2.8** — *Let $X \in \mathcal{D}(R)$ be a homologically bounded below complex and $n \in \mathbb{Z}$. Consider the following conditions:*

(i) $\mathrm{rfd}_R X \leq n$.

(ii) *X is equivalent to a bounded complex F of restricted flat R-modules with* $\sup\{i \in \mathbb{Z} | F_i \neq 0\} \leq n$.

(iii) $\mathrm{H}_i(T \otimes^{\mathbf{L}}_R X) = 0$ *for any $i > n$ and $T \in \mathcal{P}^f_0(R)$.*

(iv) *$n \geq \sup X$ and $\mathrm{C}_n(F)$ is a restricted flat R-module whenever $X \simeq F$, where $F \in \mathcal{D}(R)$ is a bounded below complex of restricted flat modules.*

(v) $n \geq -\sup U + \sup (U \otimes^{\mathbf{L}}_R X)$ *for any $U \in \mathcal{P}^f(R)$ with* $\mathrm{H}(U) \neq 0$.

Then we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftarrow$ (v). Moreover, if X is a homologically bounded complex, then all the above conditions are equivalent. When condition (ii) is satisfied, F can be chosen such that $F_l = 0$ for any $l < \inf X$.

By Theorem 2.8, we immediately obtain the following result.

Corollary 2.9 — [Theorem 2.11(a)]. If $X \in \mathcal{D}(R)$ is a homologically bounded complex, then

$$\mathrm{rfd}_R X = \sup \{\sup (U \otimes^{\mathbf{L}}_R X) - \sup U | U \in \mathcal{P}^f(R) \wedge \mathrm{H}(U) \neq 0\}.$$

Let $X \in \mathcal{D}(R)$ be a homologically bounded above complex. We say that I is a *restricted injective resolution* of X if I is a bounded above complex of restricted injective R-modules such that $X \simeq I$. A *restricted injective resolution* of an R-module is a sequence

$$0 \to I_0 \to I_{-1} \to \cdots \to I_l \to \cdots$$

of restricted injective R-modules which is exact at $I_i (i \neq 0)$ and satisfies

$$M \cong \mathrm{Ker}(I_0 \to I_{-1}).$$

That is, the sequence

$$0 \to M \to I_0 \to I_{-1} \to \cdots \to I_l \to \cdots$$

is exact.

The next two corollaries are immediate consequences of Theorem A.

Corollary 2.10 — Let $X \in \mathcal{D}(R)$ be a homologically bounded above complex. Then

$$\mathrm{rid}_R X = \inf \{\sup \{l \in \mathbb{Z} | I_{-l} \neq 0\} | I \text{ is a restricted injective resolution of } X\}.$$

Corollary 2.11 — [4, Proposition 5.3(b)]. Let $X \in \mathcal{D}(R)$ be a homologically bounded complex. Then

$$\mathrm{rid}_R X = \sup \{-\sup U - \inf \mathbf{R}\mathrm{Hom}_R(U, X) | U \in \mathcal{P}^f(R) \wedge \mathrm{H}(U) \neq 0\}.$$

In particular,

$$-\inf \mathbf{R}\mathrm{Hom}_R(U, X) \leq \mathrm{rid}_R X + \sup U$$

for any $U \in \mathcal{P}^f(R)$.

Lemma 2.12 — Let M be an R-module and let $I \in \mathcal{D}(R)$ be a bounded above complex of restricted injective modules such that $M \simeq I$. Then

$$_{\subset 0}I = 0 \to \mathrm{C}_0(I) \to I_{-1} \to \cdots \to I_l \to \cdots$$

is a restricted injective resolution of M.

Proof : Since $M \simeq I$, we have $\sup I = \sup M \leq 0$. Hence $M \simeq {}_{\subset 0}I$, i.e.,

$$M \cong \mathrm{Ker}(\mathrm{C}_0(I) \to I_{-1}).$$

There is an exact sequence

$$0 \to M \to \mathrm{C}_0(I) \to I_0 \to I_{-1} \to \cdots \to I_l \to \cdots$$

of R-modules. Next we show that $\mathrm{C}_0(I)$ is a restricted injective R-module. Let $u = \sup\{l \in \mathbb{Z} | I_l \neq 0\}$. Consider the following exact sequence

$$0 \to I_u \to I_{u-1} \to \cdots \to I_0 \to \mathrm{C}_0(I) \to 0.$$

Then $\mathrm{C}_0(I)$ is restricted injective by Lemma 2.3. □

Corollary 2.13 — Let M be an R-module and $n \in \mathbb{Z}$. Then the following conditions are equivalent.

(i) $\mathrm{rid}_R\, M \leq n$.

(ii) There is an exact sequence

$$0 \to M \to I_0 \to I_{-1} \to \cdots \to I_{-n} \to 0,$$

where each I_i is a restricted injective R-module.

(iii) $\mathrm{Ext}^i_R(T, M) = 0$ for any $i > n$ and $T \in \mathcal{P}^f_0(R)$.

(iv) For any restricted injective resolution $0 \to M \to I_0 \to I_{-1} \to \cdots$ of M, $W_{-n} = \mathrm{Coker}(I_{-n+2} \to I_{-n+1})$ is a restricted injective R-module.

Proof : It follows from Theorem A and Lemma 2.12. □

Dually, we have the following result by Theorem 2.14.

Corollary 2.14 — Let M be an R-module and $n \in \mathbb{Z}$. Then the following conditions are equivalent.

(i) $\mathrm{rfd}_R\, M \leq n$.

(ii) There is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where each F_i is a restricted flat R-module.

(iii) $\mathrm{Tor}^R_i(T, M) = 0$ for any $i > n$ and $T \in \mathcal{P}^f_0(R)$.

(iv) For any restricted injective resolution

$$\cdots \to F_l \to \cdots \to F_0 \to M \to 0$$

of M, $K_n = \mathrm{Ker}(F_{n-1} \to F_{n-2})$ is a restricted flat R-module.

3. Stability Results

Transfer of homological properties along ring homomorphisms is a classical field of study. It is an interesting question to find the relationships of the Gorenstein dimensions of R-complexes with those of S-complexes in the theory of Gorenstein dimensions. Notice that there are many stability results on Gorenstein projective, injective and flat dimensions; see for example [3, 6.4], Christensen and Holm [6], Khatami and Yassemi [11], Liu and Ren [12] and [14]. So it is natural to consider the behavior of the restricted homological dimensions under change of rings. Let $R \to S$ be a ring homomorphism. In this section, the restricted injective dimensions of R-complexes (or S-complexes) $X \otimes^{\mathbf{L}}_R Y$ and $\mathbf{R}\mathrm{Hom}_R(X, Y)$ are considered.

Proposition 3.1 — Let $\varphi : R \to S$ be a ring homomorphism, $X \in \mathcal{D}_{\Box}(S)$ and $Y \in \mathcal{D}_{\sqsupset}(R)$. Then the following inequalities hold.

(i) $\mathrm{rid}_R(X \otimes^{\mathbf{L}}_R Y) \leq \mathrm{rid}_R X - \inf Y$.
(ii) $\mathrm{rid}_S(X \otimes^{\mathbf{L}}_R Y) \leq \mathrm{rid}_S X - \inf Y$.

Proof : Choose $T \in \mathcal{P}^f_0(R)$. Then

$$\begin{aligned} \mathrm{rid}_R(X \otimes^{\mathbf{L}}_R Y) &\leq -\inf \mathbf{R}\mathrm{Hom}_R(T, X \otimes^{\mathbf{L}}_R Y) \\ &= -\inf (\mathbf{R}\mathrm{Hom}_R(T, X) \otimes^{\mathbf{L}}_R Y) \\ &\leq -\inf \mathbf{R}\mathrm{Hom}_R(T, X) - \inf Y \\ &\leq \mathrm{rid}_R X - \inf Y. \end{aligned}$$

The equality in the computation above follows from [3, A.4.23]Christensen2000 and the second inequality follows from [7, 1.5(1)].

Similarly, we have $\mathrm{rid}_S(X \otimes^{\mathbf{L}}_R Y) \leq \mathrm{rid}_S X - \inf Y$.

Now we prove Theorem B from the introduction.

Proof of Theorem B : (ii) Choose $T \in \mathcal{P}^f_0(S)$. Then

$$\mathrm{rid}_S(\mathbf{R}\mathrm{Hom}_R(X, Y)) \leq -\inf \mathbf{R}\mathrm{Hom}_S(T, \mathbf{R}\mathrm{Hom}_R(X, Y)).$$

Since $\dim R < \infty$, it follows from a well-known result of Gruson and Raynaud [8] and Jensen [9] that $\mathrm{fd}_R S < \infty$ implies $\mathrm{pd}_R S < \infty$. From this, one has $\mathrm{pd}_R(S \otimes^{\mathbf{L}}_R S) < \infty$. Hence, for all finitely generated projective S-modules P, Q we have $\mathrm{pd}_R(P \otimes^{\mathbf{L}}_R Q) < \infty$. It follows that $\mathrm{pd}_R(T \otimes^{\mathbf{L}}_R X) < \infty$ and so $T \otimes^{\mathbf{L}}_R X \in \mathcal{P}^f(R)$. By Corollary 2.11, we have

$$-\inf \mathbf{R}\mathrm{Hom}_R(T \otimes^{\mathbf{L}}_S X, Y) \leq \mathrm{rid}_R Y + \sup (T \otimes^{\mathbf{L}}_S X).$$

This explains the second inequality in the computation below.

$$\begin{aligned}\mathrm{rid}_S(\mathbf{R}\mathrm{Hom}_R(X,Y)) &\leq -\inf \mathbf{R}\mathrm{Hom}_S(T,\mathbf{R}\mathrm{Hom}_R(X,Y))\\ &= -\inf \mathbf{R}\mathrm{Hom}_R(T\otimes^{\mathbf{L}}_S X, Y)\\ &\leq \mathrm{rid}_R\, Y + \sup\,(T\otimes^{\mathbf{L}}_S X)\\ &\leq \mathrm{rid}_R\, Y + \mathrm{rfd}_S\, X.\end{aligned}$$

Similarly, one has $\mathrm{rid}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) \leq \mathrm{rfd}_R\, X + \mathrm{rid}_R\, Y$. □

Recall the *large restricted flat dimension* $\mathrm{Rfd}_R\, X$ is defined by

$$\mathrm{Rfd}_R X = \sup\{\sup(T\otimes^{\mathbf{L}}_R X) | T\in\mathcal{F}_0(R)\},$$

see [4, Definiton 2.1], and it is denoted by $\mathrm{Td}_R X$ in [3] and Sharif and Yassemi [13].

Proposition 3.2 — Let R be a Cohen-Macaulay local ring and $X,Y\in\mathcal{D}^f_\square(R)$. If X has finite projective dimension or Y has finite injective dimension, then the following equality holds,

$$\mathrm{Rid}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \mathrm{Rfd}_R\, X + \mathrm{Rid}_R\, Y.$$

Proof : Suppose Y has finite injective dimension. Then $\mathbf{R}\mathrm{Hom}_R(X,Y)\in\mathcal{D}_\square(R)$ by [3, A.5.2]. Hence $\mathbf{R}\mathrm{Hom}_R(X,Y)\in\mathcal{D}^f_\square(R)$ by [3, A.4.4]. Now the first equality in the computation below follows from (3.1), the second from Yassemi [16, Lemma 2.6] and the last from (3.1) and [4, Corollary 3.5].

$$\begin{aligned}\mathrm{Rid}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) &= \operatorname{depth} R - \mathrm{width}_R(\mathbf{R}\mathrm{Hom}_R(X,Y))\\ &= \operatorname{depth} R - \operatorname{depth}_R X + \operatorname{depth} R - \mathrm{width}_R\, Y\\ &= \mathrm{Rfd}_R\, X + \mathrm{Rid}_R\, Y.\end{aligned}$$

If X has finite projective dimension, then the assertion follows similarly from [4, Theorem 4.14].

Similarly, we have the following result by (3.2).

Proposition 3.3 — Let R be a local ring and $X,Y\in\mathcal{D}^f_\square(R)$. If X has finite projective dimension or Y has finite injective dimension, then the following equality holds,

$$\mathrm{rid}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \mathrm{rfd}_R\, X + \mathrm{rid}_R\, Y.$$

The next result follows from [4, Theorems 2.5 and 3.2]. However, we give here a direct proof.

Lemma 3.4 — For any homologically bounded below complex $X\in\mathcal{D}(R)$ there is an inequality

$$\mathrm{rfd}_R\, X \leq \mathrm{fd}_R\, X,$$

and equality holds if $\mathrm{fd}_R\, X<\infty$ and $\mathrm{cmd}\, R\leq 1$.

Proof : The inequality is obvious and equality holds if $X\simeq 0$. Now assume that $\mathrm{cmd}\, R\leq 1$ and $\mathrm{fd}_R\, X = n\in\mathbb{Z}$. By [1, Proposition 5.3.F], there exists a prime ideal $\mathfrak{p}\in\operatorname{Spec} R$ such that

$\mathrm{H}_n((k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} X_\mathfrak{p})) \neq 0$. By [4, Lemma 3.1] we may take a module $T \in \mathcal{P}^f_0(R)$ with $\mathfrak{p} \in \mathrm{Ass}_R T$. The short exact sequence

$$0 \to R/\mathfrak{p} \to T \to C \to 0$$

induces a long exact homology sequence which shows that $\mathrm{H}_n(T \otimes^{\mathbf{L}}_R X) \neq 0$. Thus,

$$\mathrm{rfd}_R X \geq \sup (T \otimes^{\mathbf{L}}_R X) \geq n.$$

Now we prove Theorem C from the introduction.

Proof of Theorem C : $(i) \Rightarrow (ii)$: By Lemma 3.4.

$(ii) \Rightarrow (iii)$: Clearly.

$(iii) \Rightarrow (i)$: If $\dim R = 0$, then there is nothing to prove. Now assume $\dim R > 0$. Take an R-module M with $\mathrm{fd}_R M = \dim R - 1$. In fact, let x be an R-sequence of length $\dim R - 1$. Then $\mathrm{fd}_R(R/xR) = \mathrm{pd}_R(R/xR) = \dim R - 1$; see Bruns and Herzog [2, 1.3.6]. Since R is local, one can choose a finite R-module T of finite projective dimension such that $\mathrm{rfd}_R M = \sup (T \otimes^{\mathbf{L}}_R M)$; see [4, 2.10]. Now [3m A.5.6.1] and the Auslander-Buchsbaum formula (see for example [4, 1.5]) yield

$$\dim R - 1 = \sup (T \otimes^{\mathbf{L}}_R M) \leq \mathrm{fd}_R T \leq \mathrm{pd}_R T \leq \mathrm{depth}\, R.$$

Acknowledgement

The research has been supported by National Natural Science Foundation of China (No. 11761047). The authors sincerely thank the referee for very carefully reading the manuscript and many helpful comments and valuable suggestions in shaping the paper into its present form.

References

1. Luchezar L. Avramov and Hans-Bjørn Foxby, Homological dimensions of unbounded complexes, *J. Pure Appl. Algebra*, **71** (1991), 129–155.

2. Winfried Bruns and H Jürgen Herzog, *Cohen-macaulay rings*, Cambridge university press, 1998.

3. Lars Winther Christensen, *Gorenstein dimensions*, Lecture Notes in Math. 1747. Springer, Berlin, 2000.

4. Lars Winther Christensen, Hans-Bjørn Foxby and Anders Frankild, Restricted homological dimensions and Cohen-Macaulayness, *J. Algebra*, **251** (2002), 479–502.

5. Lars Winther Christensen, Anders Frankild, and Henrik Holm, On Gorenstein projective, injective and flat dimensions—A functorial description with applications, *J. Algebra*, **302** (2006), 231–279.

6. Lars Winther Christensen and Henrik Holm, Ascent properties of Auslander categories, *Canad. J. Math.*, **61** (2009), 76–108.

7. Hans-Bjørn Foxby and Srikanth Iyengar, Depth and amplitude for unbounded complexes, Commutative algebra (Grenoble/Lyon, 2001) (Providence, RI), *Contemp. Math.*, vol. 331, *Amer. Math. Soc.*, 2003, 119–137.

8. Laurent Gruson and Michel Raynaud, Critères de platitude et de projectivité techniques de, *Invent. Math.*, **13** (1971), 1–89.

9. Christian U Jensen, On the vanishing of $\varprojlim^{(i)}$, *J. Algebra*, **15** (1970), 151–166.

10. Ming-Chang Kang, Almost Cohen-Macaulay modules, *Comm. Algebra*, **29** (2001) 781–787.

11. Leila Khatami and Siamak Yassemi, Gorenstein injective and gorenstein flat dimensions under base change, *Comm. Algebra*, **31** (2003) 991–1005.

12. Zhongkui Liu and Wei Ren, Transfer of Gorenstein dimensions of unbounded complexes along ring homomorphisms, *Comm. Algebra*, **42** (2014), 3325–3338.

13. Tirdad Sharif and Siamak Yassemi, Depth formulas restricted Tor-dimension under base change, *Rocky Mountain J. Math.*, **34** (2004) 1131–1146.

14. Dejun Wu, Gorenstein dimensions of unbounded complexes under base change, *Bull. Korean Math. Soc.* **53** (2016) 779–791.

15. Dejun Wu and Zhongkui Liu, On restricted injective dimensions of complexes, *Comm. Algebra*, **41** (2013) 462–470.

16. Siamak Yassemi, Width of complexes of modules, *Acta Math. Vietnam*, **23** (1998), 161–169.